ABSTRACT

Title of dissertation: GENERALIZED MULTIRESOLUTION ANALYSIS: CONSTRUCTION AND MEASURE THEORETIC CHARACTERIZATION

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In this dissertation, we first study the theory of frame multiresolution analysis (FMRA) and extend some of the most significant results to d - dimensional Euclidean spaces. A main feature of this theory is the fact that it was successfully applied to narrow band signals; however, the theory does have its limitations. Some orthonormal wavelets may not be obtained by the methods of FMRA. This is because non-MRA orthonormal wavelets have nonconstant dimension functions. This means that the number of scaling functions needed is more than one. The appropiate tools for non-MRA wavelets are the generalized multiresolution analyses (GFMRA, GMRA) theories developed by Manos Papadakis and Lawrence Baggett. At the end, we unify both theories by finding an explicit formula for an important map. Our approach also permits us to give a short and elegant proof of a classical result about a special type of decomposition in shift-invariant space theory.

Generalized Multiresolution Analysis: Construction and Measure Theoretic Characterization

by

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In memory of Juan Ramon Romero Sr. and Elena Ioana Cioranescu.

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Chapter 1

Introduction

1.1 Background and Motivation

A *dyadic wavelet* is a function $\psi \in L^2(\mathbb{R}^d)$ such that the family

$$\left\{\psi_{m,n}(x) := 2^{md/2}\psi\left(2^m x - n\right) : m \in \mathbb{Z},\ n \in \mathbb{Z}^d\right\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$.

The theory of dyadic multiresolution analysis (MRA), introduced by Mallat and Meyer, produces $2^d - 1$ wavelets $\{\psi_{(1)}, \ldots, \psi_{(2^d-1)}\}$ in $L^2(\mathbb{R}^d)$, in the sense that

$$\left\{\psi_{(1),m,n}, \ldots, \psi_{(2^d-1),l,p} : m, \ldots, l \in \mathbb{Z}, n, \ldots, p \in \mathbb{Z}^d\right\}$$

forms an orthonormal basis for $L^2(\mathbb{R}^d)$. This shows that the complexity of the Mallat-Meyer construction grows exponentially in d. Hence, a single dyadic wavelet in multidimensional Euclidean space must be a non-MRA wavelet.

In the search of finding single dyadic wavelets in multidimensional Euclidean space, Dai, Larson, and Speegle [20] defined multidimensional wavelet sets. A *wavelet set* is a measurable set $K \subset \mathbb{R}^d$ such that

$$\mathbb{1}_K^\vee = \psi,$$

is an orthonormal wavelet. Here, $\mathbb{1}_K^\vee$ denotes the inverse Fourier transform of $\mathbb{1}_K$, the characteristic function of the set K (see Section 1.3 for notation and definitions).

The first example of a one-dimensional non-MRA wavelet set was given by J. L. Journé (see [37]).

Wavelet sets have a beautiful geometric characterization, as shown in the following theorem [7].

Theorem 1.1. *A measurable set $K \subset \mathbb{R}^d$ is a wavelet set if and only if*

- $\left\{K + n : n \in \mathbb{Z}^d\right\}$ *tiles* $\mathbb{R}^d$.
- $\{2^m K : m \in \mathbb{Z}\}$ *tiles* $\mathbb{R}^d$.

Dai and Larson obtained wavelet sets by operator-theoretic methods. The first reaction from the mathematics community was of disbelief and lack of interest. Wavelet sets were seen as pathological counterexamples in wavelet theory, not only because they could be extremely complicated and hard to construct, but because of the lack of multiresolution structure.

Then, several generalized multiresolution theories appeared. First Benedetto and Li defined and developed the theory of frame multiresolution analysis (FMRA) [8, 9, 10, 11], [33, 31, 32]. This was followed by Papadakis' development of the generalized frame multiresolution analysis (GFMRA) [37, 38, 39], and the development of generalized multiresolution analysis (GMRA) introduced by Baggett et al. [4, 3, 2]. In the most general settings, GFMRA and GMRA are distinct theories. However, for the dyadic case the two theories coincide. FMRA is a special case of both GFMRA and GMRA.

The main success of the generalized multiresolution analysis theories is the fact that every orthonormal wavelet in any dimension is a GFMRA or GMRA wavelet

[37]. In particular, any wavelet produced from a wavelet set is a GFMRA wavelet. Due to this fact, Baggett, Medina, and Merrill developed a technique to construct all wavelet sets [2]. Their analysis involves using a complementary pair of maps satisfying some intertwining properties.

All multiresolution analysis theories have a feature in common: the core space V_0 is shift-invariant. Given a subspace $V \subset L^2(\mathbb{R}^d)$, we say that V is a *shift-invariant* subspace under a subgroup G of $\mathbb{R}^d$ if for all $f \in V$ and $g \in G$, $f(x-g) \in V$. The theory of shift-invariant subspaces was developed by C. de Boor, R. DeVore, A. Ron, and Z. Shen in [12, 13, 40] to tackle problems in approximation theory. There is a close connection between the theory of frames and the theory of shift-invariant subspaces. If the group G is discrete, then a subspace V is shift-invariant if and only if V has a frame consisting of translates of at most countably many functions of V.

The main object in GMRA theory is an integer-valued function, obtained from the Spectral Theorem, called the *multiplicity function*. Because V_0 is shift-invariant under $\mathbb{Z}^d$, operation of translating functions in V_0 by elements of $\mathbb{Z}^d$ is a unitary representation of $\mathbb{Z}^d$, and hence we can invoke Stone's theorem to get a spectral decomposition of this representation. The unique spectral measure in this decomposition uniquely determines the multiplicity function. In this setting, this multiplicity function plays the role of the scaling function. It was proved by Eric Weber [43] that the Auscher's dimension function [1] is in fact equal to the multiplicity function. The dimension function is defined as follows:

Definition 1.2. Given an orthonormal wavelet ψ in $L^2(\mathbb{R}^d)$, the dimension function associated to ψ is given by

$$D_\psi(\gamma) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \left| \widehat{\psi}\left(2^j(\gamma + k)\right) \right|^2 .$$

Observe that D_ψ is 1-periodic in each variable. The above definition can be extended to multiwavelets. It is well known that a multiwavelet is an MRA multiwavelet if and only if its associated dimension function equals 1 almost everywhere in $\mathbb{T}^d$. This dimension function was used by Papadakis to prove in [37] that every orthonormal wavelet in any dimension is a GFMRA, or GMRA, wavelet. The key observation he made was that the dimension function counts the number of scaling functions for a GFMRA associated to the wavelet. His method consisted of applying pointwise Gram-Schmidt to certain vectors, and by doing so one obtains the scaling functions. In [16], M. Bownik, Z. Rzeszotnik, and D. Speegle characterized the dimension functions of orthonormal wavelets, and in particular, they proved that the dimension function of a single orthonormal wavelet assumes all values between zero and its essential supremum. This fact was unknown during the time Manos Papadakis published [37].

In this thesis we study two related problems: the extension of the theory of FMRA to higher dimensions and the construction of the scaling functions for orthonormal wavelets in the space $L^2(\mathbb{R}^d)$, for $d \geq 1$. The first construction of such scaling functions was given by Manos Papadakis in [37]. Following Papadakis' ideas we present a new proof of the main theorem in [37]. This new proof reveals two important things: the scaling functions given by Papadakis are optimal, that is, the number of scaling functions is minimum, something that was not clear in [37], and connections between the theory of shift-invariant subspaces and the theory of GMRA (see [14], [4, 3, 2]) are established. This opens the path to construct an optimal GMRA for a given wavelet set, and in particular this can be done for the Benedetto-Leon-Sumetkijakan wavelet sets [6, 7]. Hence, we consider the reverse problem of the Mallat-Meyer theory: given an orthonormal wavelet, construct a GMRA for such a wavelet.

There are different techniques to construct wavelet frames (e.g., GMRA, Extension Principles, etc. [3, 4, 22, 34, 30, 41, 37, 38, 40]). In the first part of this thesis, we shall use the FMRA technique, which, although elementary and limited, allows us to accomplish our goal of constructing a multidimensional Mallat-Meyer algorithm for MRA frames by tensor products [5, 35, 21]. This theory depends on the measure theoretic properties of particular sets associated with natural periodizations. This measure theoretic point of view first appeared independently in [11] and [32].

These two problems (i.e., the construction of scaling functions and the extension of FMRA theory to $\mathbb{R}^d$) are related because the theory of FMRA, as the name suggests, is a special case of the theory of GFMRA. FMRAs are useful in signal processing because the perfect reconstruction filter banks associated to them can be narrow-band. Therefore, FMRA filter banks can achieve quantization noise reduction simultaneously with reconstruction of a given narrow-band signal. However, this theory itself is not sufficient to study all orthonormal wavelets as we shall see in the final chapter.

1.2 Results

In the first part of this thesis, in chapter 2, we generalize the main results of the theory of FMRA given in [10, 11], and provide an algorithm by means of tensor products to construct wavelet frames for $L^2(\mathbb{R}^d)$. The main results are Theorems 2.2, 2.3, 2.5, 2.6, 2.7 and the construction given in section 2.5 in chapter 2. Section 2.5 gives a formula by means of tensors similar to the Mallat-Meyer expression for the generators of W_0, provided that a certain subset of the d-dimensional torus has zero measure, i.e., the applicability of such an algorithm depends on the measure theoretic properties of a subset of the spectrum of the FMRA. Theorem 2.2 characterizes the generators of W_0 in terms of certain equations which we formulate. Theorem 2.3 gives a necessary and sufficient condition for a function ψ to belong to W_0 in terms of filters (see section 1.4 for the definition of a filter). Theorem 2.5 is the dual version of Theorem 2.2, i.e., the equations in Theorem 2.5 are obtained by taking the Fourier transform of the ones given in Theorem 2.2. The equations in Theorem 2.3 and 2.5 are a system of linear equations that needs to be solved pointwise in order to obtain frame generators for W_0.

In the second part of this thesis, in chapter 3, we analyze the generalized multiresolution schemes of Papadakis and Baggett, and construct, for any given orthonormal wavelet, a GFMRA. This is Theorem 3.14. This construction is particularly easy to implement when the wavelet has a compactly supported Fourier transform. In Theorem 3.17, we unify the generalized multiresolution theories of Papadakis and Baggett by providing an explicit formula of an important unitary map given in [3, 19]. In general, this map cannot be formulated explicitly. As a consequence, the scaling functions obtained by our methods are equivalent to the ones given in [3, 19]. Another consequence of our approach is an alternative proof of a classical theorem given in [14] involving a special type of decomposition of shift-invariant subspaces. This is Theorem 3.19, and, in fact, it can be obtained as a corollary from the existence of the unitary map mentioned above.

1.3 Notation and Definitions

Definition 1.3. The *Fourier transform* $\widehat{f} : \widehat{\mathbb{R}}^d \longrightarrow \mathbb{C}$ of $f \in L^1(\mathbb{R}^d)$ is defined by

$$\forall \gamma \in \widehat{\mathbb{R}}^d,\ \widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\gamma} dx.$$

This function $\widehat{f}$ is uniformly continuous and vanishes at infinity. $\widehat{\mathbb{R}}^d$ is $\mathbb{R}^d$ considered as the spectral domain of the Fourier transform, and $x \cdot \gamma$ denotes the standard inner product on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The map $f \rightarrow \widehat{f}$ restricted to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ extends to a unitary map on $L^2(\mathbb{R}^d)$. The *inverse Fourier transform* is formally defined by

$$f^{\vee}(x) = \int_{\mathbb{R}^d} f(\gamma)\, e^{2\pi i x\cdot\gamma} d\gamma.$$

The term "inverse" is justified by the following fact: if $f \in L^1(\mathbb{R}^d)$ and $\widehat{f} \in L^1(\widehat{\mathbb{R}}^d)$, then

$$\forall x \in \mathbb{R}^d,\ f(x) = \int_{\mathbb{R}^d} \widehat{f}(\gamma)\, e^{2\pi i x\cdot\gamma} d\gamma.$$

Now we state the definition of a frame for a separable Hilbert space H:

Definition 1.4. Given a separable Hilbert space H (i.e., it has a countable orthonormal basis). A *frame* for H is a sequence $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ of vectors, where $\mathbb{I}$ is a countable index set, for which there are constants $0 < A \leq B < \infty$ such that

$$\forall f \in H,\ A||f||^2 \leq \sum_{i \in \mathbb{I}} |\langle f, f_i \rangle|^2 \leq B||f||^2.$$

If $A = B$, we say that the frame is *tight*, and if $A = B = 1$, we call it a *Parseval* frame.

Frames can be seen as overcomplete bases. More precisely, every element $f \in H$ can be represented as

$$f = \sum_{i \in \mathbb{I}} a_i f_i, \tag{1.1}$$

where $(a_i)_{i \in \mathbb{I}} \in l^2(\mathbb{I}) = \left\{(c_i)_{i \in \mathbb{I}} : \sum_{i \in \mathbb{I}} |c_i|^2 < \infty\right\}$, and the convergence of (1.1) is independent of the order of summation. The main difference between a frame and a basis of any kind is that the coefficients in (1.1) are not unique, and in the case of a basis they are unique. There are many definitions for bases, but we are interested in Riesz bases because of the stability they provide:

Definition 1.5. A *Riesz basis* (or exact frame) is a sequence $\{f_i\}_{i \in \mathbb{I}} \subseteq H$ of vectors for which there are constants $A, B > 0$ such that

$$A \sum_i |c_i|^2 \leq \left\| \sum_i c_i f_i \right\|^2 \leq B \sum_i |c_i|^2,$$

for all sequences $\{c_i\}$ with finite number of nonzero entries.

If the constants in the definition of a Riesz basis are $A = B = 1$, it can be shown that the Riesz basis is in fact an orthonormal basis, i.e.,

$$\forall i, j \in \mathbb{I},\ \ \langle f_i, f_j \rangle = \delta_{ij}$$

where $\delta_{i,j}$ is the Kronecker delta. There is the following characterization of Riesz bases in terms of continuous bijective operators defined on a Hilbert space and orthonormal bases:

Theorem 1.6. *A sequence of vectors $\{f_i\}_{i\in\mathbb{I}} \subseteq H$ is a Riesz basis if and only if there is a bounded bijective operator $T : H \to H$ and an orthonormal basis $\{e_i\}_{i\in\mathbb{I}}$ such that*

$$\forall i \in \mathbb{I},\ \ f_i = Te_i.$$

For a given $y \in \mathbb{R}^d$, τ_y is the translation operator defined formally by $\tau_y f\,(x) = f\,(x-y)$, for a function f defined on $\mathbb{R}^d$. The dilation operator D is defined formally by $Df\,(x) = 2^{d/2} f\,(2x)$. It turns out that these operators are isometries on $L^2(\mathbb{R}^d)$, by the invariance of the Lebesgue measure and the change of variable formula, respectively.

We define the map $\mathcal{X}$ in the following way: for $f \in L^2(\mathbb{R}^d)$,

$$\mathcal{X}\,(f)\,(\gamma) = \left\{\widehat{f}\,(\gamma+k)\right\}_{k\in\mathbb{Z}^d}.$$

Then $\mathcal{X}\,(f)\,(\gamma) \in l^2(\mathbb{Z}^d)$ for almost every γ in $\widehat{\mathbb{R}}^d$. The periodization of $|\widehat{\varphi}|^2$, for $\varphi \in L^2(\mathbb{R}^d)$, is defined as $\Phi\,(\gamma) = \|\mathcal{X}\,(\varphi)\,(\gamma)\|^2_{l^2(\mathbb{Z}^d)}$. It is clear that if $\varphi \in L^2(\mathbb{R}^d)$, then $\Phi{\in}L^1(\mathbb{T}^d)$; and $\|\Phi\|_{L^1(\mathbb{T}^d)} = \|\varphi\|^2_{L^2(\mathbb{R}^d)}$ by the Parseval-Plancherel theorem. $A'(\mathbb{Z}^d)$ is defined to be the set of all Fourier coefficients of bounded periodic functions, and the space $\mathcal{L}^\infty$ is defined as

$$\mathcal{L}^\infty = \left\{f : \mathbb{R}^d \longrightarrow \mathbb{C} : \exists B > 0, \sum_{n\in\mathbb{Z}^d} |f\,(x-n)| \leq B \text{ a.e.}\right\}.$$

For $\varphi \in L^2(\mathbb{R}^d)$, let $V_0 = \overline{\text{span}}\{\tau_n\varphi : n{\in}\mathbb{Z}^d\}$ be the closed linear span of the sequence $\{\tau_n\varphi\}_{n\in\mathbb{Z}^d}$. Then it is elementary to prove that $\{\tau_n\varphi\}$ is an exact frame (or a Riesz basis) for its closed linear span if and only if there exist constants A, B, with $0 < A \leq B < \infty$ for which

$$A \leq \Phi \leq B \text{ a.e.}$$

A similar result is true for frames of translates [8, 11, 17, 18, 40]. In order to state this result we use the "pullback" notation $[f > 0]$ to designate the set of points x in the domain of f for which f is positive. Then $\{\tau_n \varphi\}_{n \in \mathbb{Z}^d}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \leq B < \infty$ for which

$$A \leq \Phi \leq B \text{ a.e. on } [\Phi > 0].$$

This result can be generalized in the case of several $\varphi_i s$ in terms of the *Gramian matrix*

$$G_X(\gamma) = \left(\sum_{n \in \mathbb{Z}^d} \widehat{\varphi_k}(\gamma - n) \overline{\widehat{\varphi_j}(\gamma - n)} \right)_{1 \leq k, j \leq N} = \left(\langle \mathcal{X}(\varphi_k)(\gamma), \mathcal{X}(\varphi_j)(\gamma) \rangle \right)_{1 \leq k, j \leq N},$$

where $X = \{\varphi_i\}_{i=1}^N$. Let $m(\gamma), m^+(\gamma)$, and $M(\gamma)$ be the smallest, smallest positive, and largest eigenvalues of $G_X(\gamma)$, respectively. Then $\left\{\tau_n \varphi_i : n \in \mathbb{Z}^d, 1 \leq i \leq N\right\}$ is a frame for its closed linear span if and only if there exist constants A, B, with $0 < A \leq B < \infty$, for which

$$A \leq m^+(\gamma) \leq M(\gamma) \leq B \text{ a.e.}$$

holds in a particular set called the *spectrum* of $\overline{\operatorname{span}}_{k \in \mathbb{Z}^d} \{\tau_k \varphi : \varphi \in X\}$ [12, 13, 14, 15]. The spectrum is defined in the next section. In the case the translations of elements of X form a Riesz basis, $m^+(\gamma)$ can be replaced by $m(\gamma)$, and the inequalities hold a.e.

1.4 Shift Invariant Subspaces

We shall use the shift-invariant approach in section 2.3 to prove Theorem 2.7. Here are the main concepts and definitions needed for Theorem 2.7. For more details about the shift-invariant subspace theory see [12, 13, 14, 15, 29].

If $W \subset L^2(\mathbb{R}^d)$ and $\gamma \in \widehat{\mathbb{R}}^d$, we set

$$\mathcal{X}(W)(\gamma) = \{(\mathcal{X}(f))(\gamma) : f \in W\},$$

and hence $\mathcal{X}(W)(\gamma) \subset l^2(\mathbb{Z}^d)$ for almost all γ. If $W \subset L^2(\mathbb{R}^d)$ is a linear subspace of $L^2(\mathbb{R}^d)$, then, by the linearity of $\mathcal{X}$, $\mathcal{X}(W)(\gamma)$ is a linear subspace of $l^2(\mathbb{Z}^d)$. $Sp_{\mathcal{X},\gamma}(W)$ is defined to be

$$Sp_{\mathcal{X},\gamma}(W) = \overline{\mathrm{span}}\,\{(\mathcal{X}(f))(\gamma) : f \in W\}.$$

For $W \subset L^2\left(\mathbb{R}^d\right)$, we define $S(W)$ as

$$S(W) = \overline{\mathrm{span}}_{k \in \mathbb{Z}^d}\,\{\tau_k f : f \in W\},$$

the *shift-invariant space generated by* W. If W is a finite set, we say that $S = S(W)$ is a *finitely generated shift-invariant space* (FSI). The *length* of a shift-invariant subspace S is defined to be len $S = \min \mathrm{card}\,\{W : S = S(W)\}$. For S, a shift-invariant subspace of $L^2\left(\mathbb{R}^d\right)$, the *spectrum of* S, $\sigma(S)$, is defined by

$$\sigma(S) = \left\{\gamma \in \mathbb{T}^d : \dim Sp_{\mathcal{X},\gamma}(W) > 0\right\},$$

where dim indicates dimension. In the case of an FMRA (defined in the next chapter), $\sigma(V_0) = [\Phi > 0] = \left\{\gamma \in \mathbb{T}^d : \Phi(\gamma) > 0\right\}$.

Let H be a Hilbert space and let F be a linear subspace of H. Then $F^{\perp}$, the *orthogonal complement of* F *in* H, is defined as

$$F^{\perp} = \{x \in H : \forall f \in F, \ \ \langle x, f \rangle = 0\} .$$

The continuity of the inner product implies that $F^{\perp}$ is a closed linear subspace of H. We now state some results which we shall need in section 2.3 in chapter 2.

Theorem 1.7. *Let S be a finitely generated shift-invariant space and let T be a shift-invariant subspace of S. Then $T^{\perp}$ is also shift-invariant and, for almost every $\gamma \in \widehat{\mathbb{R}}^d$,*

$$\mathcal{X}(S)(\gamma) = \mathcal{X}(T)(\gamma) \bigoplus \mathcal{X}\left(T^{\perp}\right)(\gamma) .$$

Theorem 1.8. *Given any FSI S, there is a finite subset $W \subset L^2(\mathbb{R}^d)$, for which the multi-integer translates of W are a frame for S.*

Theorem 1.9. *For a shift-invariant subspace $S \subset L^2(\mathbb{R}^d)$, we have*

$$\operatorname{len} S = \textit{ess-sup}\left\{\dim \mathcal{X}(S)(\gamma), \gamma \in \mathbb{T}^d\right\} .$$

The map $D_S(\gamma) = \dim \mathcal{X}(S)(\gamma)$ is the dimension function *of the subspace S.*

These theorems, together with their proofs, can be found in [12], [40], and [13], respectively.

1.5 The Haar Multiresolution

The first example of a wavelet was given by Alfred Haar in 1910. This type of decomposition is what we call a *time-scale* decomposition. Time-scale analysis is better suited for spaces where the Fourier transform or Fourier series are not well behaved. The Haar system is elegant, beautiful, and simple. It is also an excellent example of a multiresolution analysis which we define below

The *Haar wavelet* is

$$\psi(t) = \begin{cases} 1 & \text{if } t \in \left[0, \frac{1}{2}\right) \\ -1 & \text{if } t \in \left[\frac{1}{2}, 1\right] \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

and set

$$\psi_{j,k}(t) = 2^{\frac{j}{2}} \psi\left(2^j t - k\right), \ j, k \in \mathbb{Z}. \tag{1.3}$$

An interval of the form $[k2^{-j}, (k+1)2^{-j}]$, $j, k \in \mathbb{Z}$, is called a *dyadic interval.* Notice that $[k2^{-j}, (k+1)2^{-j}]$ is the support of $\psi_{j,k}$. The j-th level consists of those intervals whose length is 2^{-j}; and for a fixed j, distinct dyadic intervals are disjoint or intersect at most in one point. More precisely,

Proposition 1.10. *Given* $[k2^{-j}, (k+1)2^{-j}]$ *and* $\left[n2^{-l}, (n+1)2^{-l}\right]$, $n, l, j, k \in \mathbb{Z}$. *Then the intersection* $[k2^{-j}, (k+1)2^{-j}] \cap \left[n2^{-l}, (n+1)2^{-l}\right]$ *is either:*

(i) a singleton,

(ii) $\left[n2^{-l}, (n+1)2^{-l}\right]$, *if the intervals are equal, or*

(iii) one is contained either in the right half or in the left half of the other.

As a consequence of the previous proposition, we can conclude that the Haar system is an orthonormal system for $L^2(\mathbb{R})$:

Theorem 1.11. $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ *is an orthonormal set in* $L^2(\mathbb{R})$.

Proof. Assuming that $j \leq l$ we obtain, by setting $p = l - j, m = n - 2^p k$, and $x = 2^j t - k$, that

$$\langle \psi_{j,k}, \psi_{l,n} \rangle = \int_{\mathbb{R}} \psi_{j,k}(t)\, \psi_{l,n}(t)\, dt = \int_{\mathbb{R}} \psi(x)\, \psi_{p,m}(x)\, dt. \tag{1.4}$$

Note that for every $j, k \in \mathbb{Z}$,

$$\int_{\mathbb{R}} \psi_{j,k}(t)\, dt = 0 \text{ and } \int_{\mathbb{R}} |\psi_{j,k}(t)|^2\, dt = 1.$$

by the definition of ψ. Hence, in any of the three cases in the above proposition, equation (1.4) is zero or one (in the case that $m = 0$ and $p = 1$). This proves the orthonormality of the system $\{\psi_{j,k}\}_{j,k \in \mathbb{Z}}$. $\square$

The next step is to prove that this set is in fact an orthonormal basis for $L^2(\mathbb{R})$. In order to do this we consider the following two families of closed subspaces of $L^2(\mathbb{R})$:

$$V_n := \overline{span}\, \{\psi_{j,k} : j < n\ ; k \in \mathbb{Z}\} \tag{1.5}$$

and

$$V_n' := \text{functions in } L^2(\mathbb{R}) \text{ constant in } \left[k2^{-n}, (k+1)\, 2^{-n}\right] \text{ for all } k \in \mathbb{Z}. \tag{1.6}$$

The next properties are shared by both families (1.5) and (1.6):

$$... \subset V_n \subset V_{n+1} \subset ... \tag{1.7}$$

$$f(t) \in V_n \Longleftrightarrow f(2t) \in V_{n+1} \tag{1.8}$$

$$f(t) \in V_0 \Longleftrightarrow f(t+k) \in V_0. \tag{1.9}$$

Moreover,

Lemma 1.12. *For every $n \in \mathbb{Z}$, $V_n = V_n'$.*

Proof. In light of (1.8), which is valid for both families, it suffices to show that $V_0 = V_0'$. Every $\psi_{j,k}$ for $j < 0$ is constant in intervals of the form $[m, m+1]$ for $m \in \mathbb{Z}$, so it is clear that $V_0 \subset V_0'$. To show the other inclusion, observe that every $f \in V_0'$ can be written as

$$f = \sum\nolimits_{m \in \mathbb{Z}} \alpha(m) \mathbb{1}_{[m,m+1]} \text{ with convergence in } L^2(\mathbb{R}). \tag{1.10}$$

Hence, by (1.9), it is suffices to show $\mathbb{1}_{[0,1]} \in V_0$. The key observation here is to look at the following series:

$$\sum\nolimits_{j<0} 2^{\frac{j}{2}} \psi_{j,0}(t) = \sum\nolimits_{j<0} 2^j \psi\left(2^j t\right).$$

Since $\|2^j \psi(2^j t)\|_{L^2(\mathbb{R})} = 2^j$ and $j < 0$, this is an absolutely convergent series in $L^2(\mathbb{R})$. Now, considering (1.2), we obtain

$$\sum\nolimits_{j<0} 2^{\frac{j}{2}} \psi_{j,0}(t) = 0 \text{ if } t \leq 0,$$

and

$$\sum\nolimits_{j<0} 2^{\frac{j}{2}} \psi_{j,0}(t) = \sum\nolimits_{j<0} 2^j = 1 \text{ if } 0 < t < 1.$$

Moreover, for $t \in (2^p, 2^{p+1})$ for $p = 0, 1, 2, ...$ we have

$$\sum\nolimits_{j<0} 2^{\frac{j}{2}} \psi_{j,0}(t) = -2^{-p-1} + \sum\nolimits_{j=p+2}^{\infty} 2^{-j} = 0.$$

Hence,

$$\sum\nolimits_{j<0} 2^{\frac{j}{2}} \psi_{j,0} = \mathbb{1}_{[0,1]} \text{ a.e. in } L^2(\mathbb{R})$$

so that $\mathbb{1}_{[0,1]} \in V_0$ as desired. $\square$

Using the density in $L^2(\mathbb{R})$ of the set $\bigcup_{n=-\infty}^{\infty} V'_n$, as well as the previous theorem and lemma, we obtain:

Theorem 1.13 (Haar)**.** *The system $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

This means that every $f \in L^2(\mathbb{R})$ has a unique decomposition of the form

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}.$$

Since every $\psi_{j,k}$ belongs to $L^p(\mathbb{R})$ for $p \geq 1$, every $f \in L^p(\mathbb{R})$ can be represented as above with convergence in $L^p(\mathbb{R})$ [36], but the study of the convergence of this series for every $f \in L^p(\mathbb{R})$ is beyond the scope of this thesis.

Note that the following is satisfied for the above family $\{V_j\}$ of closed subspaces of $L^2(\mathbb{R})$:

1. $\forall j \in \mathbb{Z},\ V_j \subset V_{j+1}$,
2. $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$,
3. $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$,
4. $f(t) \in V_j \Longleftrightarrow f(2^{-j}t) \in V_0$
5. $\forall k \in \mathbb{Z},\ f(t) \in V_0 \Longleftrightarrow f(t+k) \in V_0$
6. The integer translates of $\varphi = \mathbb{1}_{[0,1]}$ form an orthonormal basis for V_0.

A family $\{V_j\}$ with a function φ (not necessarily $\varphi = \mathbb{1}_{[0,1]}$) satisfying (1)-(6) is called a *multiresolution analysis* (MRA). φ is a *scaling function.* Mallat and Meyer [35, 36] developed an algorithm to construct a wavelet for a given MRA, and this is the tool used by Daubechies [21] to construct arbitrary smooth compactly supported wavelets. We outline the Mallat-Meyer algorithm:

First, note that $\varphi(t) \in V_1$, hence, $\varphi\left(\frac{t}{2}\right) \in V_0$. Therefore, the following equation holds:

$$\varphi\left(\frac{t}{2}\right) = \sum_{n\in\mathbb{Z}} a(n)\, \varphi(t-n), \tag{1.11}$$

so that, by change of variables,

$$\varphi(t) = \sum_{n\in\mathbb{Z}} a(n)\, \varphi(2t-n). \tag{1.12}$$

Equivalently,

$$\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}\right), \tag{1.13}$$

or

$$\widehat{\varphi}(2\gamma) = H_0(\gamma)\, \widehat{\varphi}(\gamma), \tag{1.14}$$

where

$$H_0(\gamma) = \sum_{n\in\mathbb{Z}} a(n)\, e^{-2\pi i n\gamma}.$$

Since $\left\|\varphi\left(\frac{t}{2}\right)\right\|_{L^2(\mathbb{R})} = \sqrt{2}$, we obtain that $\sum_{n\in\mathbb{Z}} |a(n)|^2 = 2$, because the translates $\varphi(t-n)$ form an orthonormal basis for V_0, and so

$$\|H_0\|_{L^2(\mathbb{T})} = \sqrt{2}.$$

The equivalent equations (1.11) - (1.14) are called scaling equations. The function H_0 is called a *filter*. This function H_0 satisfies

$$|H_0(\gamma)|^2 + |H_0(\gamma+1/2)|^2 = 1, \text{ for almost every } \gamma \in \mathbb{R}. \tag{1.15}$$

Observe that, in the case that $\widehat{\varphi}$ is continuous at zero, $|\widehat{\varphi}(0)| = 1$. Now, iterating equation (1.13) N times we obtain

$$\widehat{\varphi}(\gamma) = \prod_{j=1}^{N} H_0\left(\frac{\gamma}{2^j}\right)\widehat{\varphi}\left(\frac{\gamma}{2^j}\right),$$

and hence,

$$\widehat{\varphi}(\gamma) = \prod_{j=1}^{\infty} H_0\left(\frac{\gamma}{2^j}\right). \tag{1.16}$$

It is clear that only very special sequences $\{a(n)\}_{n\in\mathbb{Z}}$ defining H_0 could give rise to scaling function by means of (1.16). Hence, certain filters produce MRAs. These filters are called MRA filters. Such filters were used by Daubechies to construct compactly supported wavelets with arbitrary degree of smoothness, as mentioned above.

A function f belongs to V_0 if and only if

$$f = K_0\widehat{\varphi}$$

for some unique periodic $L^2(\mathbb{T})$ function K_0. The coefficients of the Fourier expansion of K_0 are also the coefficients of the frame expansion of f. The same f belongs to V_1 if and only if $f(x) = g(2x)$, for some $g \in V_0$. This is equivalent to

$$\widehat{f}(\gamma) = M_f\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right),$$

for some $L^2(\mathbb{T})$ function M_f. If we define W_0 to be the orthogonal complement of V_0 in V_1, then $W_0 \bigoplus V_0 = V_1$. In the case $f = \varphi$, $M_f = H_0$. The next proposition characterizes the space W_0 :

Proposition 1.14. *A function f belongs to W_0 if and only if*

$$\widehat{f}(\gamma) = e^{\pi i\gamma}\upsilon_f(\gamma)\overline{H_0\left(\frac{\gamma}{2}+\frac{1}{2}\right)}\widehat{\varphi}\left(\frac{\gamma}{2}\right), \tag{1.17}$$

for some $L^2(\mathbb{T})$ function $\upsilon(\gamma)$ and for H_0 defined by (1.13). Moreover,

$$\|f\|_{L^2(\mathbb{R})} = \|\upsilon\|_{L^2(\mathbb{T})}.$$

Proof. f belongs to W_0 if and only if f belongs to V_1 and $f(x) \perp \varphi(x-k)$ for every k in $\mathbb{Z}$, i.e.,

$$0 = \langle f, \tau_k \varphi \rangle = \left\langle \widehat{f}, e_k \widehat{\varphi} \right\rangle = \int_{\mathbb{T}} e_k(\gamma) \sum_{k\in\mathbb{Z}} \widehat{f}(\gamma+k) \overline{\widehat{\varphi}(\gamma+k)} d\gamma. \tag{1.18}$$

Here,

$$e_k(\gamma) = e^{2\pi i k \gamma}.$$

Note that

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{T}} \left|\widehat{f}(\gamma+k)\right| \left|\overline{\widehat{\varphi}(\gamma+k)}\right| d\gamma = \int_{\mathbb{R}} \left|\widehat{f}(\gamma+k)\right| \left|\overline{\widehat{\varphi}(\gamma+k)}\right| d\gamma \leq \|f\|_2 \|\varphi\|_2,$$

and, hence, $\sum_{k\in\mathbb{Z}} \widehat{f}(\gamma+k) \overline{\widehat{\varphi}(\gamma+k)}$ represents an integrable function on the torus. The right side of the last equality of (1.18) is the k-th Fourier coefficient of

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\gamma+k) \overline{\widehat{\varphi}(\gamma+k)}$$

Thus, by the uniqueness of Fourier series,

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\gamma+k) \overline{\widehat{\varphi}(\gamma+k)} = 0 \text{ a.e.}$$

Now $\widehat{f}(\gamma) = M_f\left(\frac{\gamma}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}\right)$ and $\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}\right)$ a.e., and, hence,

$$\sum_{k\in\mathbb{Z}} M_f\left(\frac{\gamma}{2}+\frac{k}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}+\frac{k}{2}\right) \overline{H_0\left(\frac{\gamma}{2}+\frac{k}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}+\frac{k}{2}\right)} = 0 \text{ a.e.}$$

Making the substitution $\xi = \frac{\gamma}{2}$, this sum splits as follows:

$$
\begin{aligned}
0 &= \sum\nolimits_{k\in\mathbb{Z}} M_f\left(\xi + \frac{k}{2}\right)\left|\widehat{\varphi}\left(\xi + \frac{k}{2}\right)\right|^2 \overline{H_0\left(\xi + \frac{k}{2}\right)} \\
&= \sum\nolimits_{k\in\mathbb{Z}} M_f\left(\xi + k\right)\left|\widehat{\varphi}\left(\xi + k\right)\right|^2 \overline{H_0\left(\xi + k\right)} \\
&+ \sum\nolimits_{k\in\mathbb{Z}} M_f\left(\xi + k + \frac{1}{2}\right)\left|\widehat{\varphi}\left(\xi + k + \frac{1}{2}\right)\right|^2 \overline{H_0\left(\xi + k + \frac{1}{2}\right)}.
\end{aligned}
$$

Using the periodicity of M_f and H_0 this implies that

$$
\begin{aligned}
0 &= M_f\left(\xi\right)\overline{H_0\left(\xi\right)} \sum\nolimits_{k\in\mathbb{Z}} \left|\widehat{\varphi}\left(\xi + k\right)\right|^2 \\
&+ \overline{H_0\left(\xi + \frac{1}{2}\right)} M_f\left(\xi + \frac{1}{2}\right) \sum\nolimits_{k\in\mathbb{Z}} \left|\widehat{\varphi}\left(\xi + k + \frac{1}{2}\right)\right|^2 \\
&= M_f\left(\xi\right)\overline{H_0\left(\xi\right)} + M_f\left(\xi + \frac{1}{2}\right) H_0\left(\xi + \frac{1}{2}\right).
\end{aligned}
\tag{1.19}
$$

Each of these steps are reversible, and, hence, f belongs to W_0 if and only if f belongs to V_1 and (1.19) holds. On the other hand, (1.19) is the orthogonality (pointwise) of the vectors $\left(M_f\left(\xi\right), M_f\left(\xi + \frac{1}{2}\right)\right)$ and $\left(H_0\left(\xi\right), H_0\left(\xi + \frac{1}{2}\right)\right)$, so that

$$
\left(M_f\left(\xi\right), M_f\left(\xi + \frac{1}{2}\right)\right) = c\left(\xi\right)\left(\overline{H_0\left(\xi + \frac{1}{2}\right)}, -\overline{H_0\left(\xi\right)}\right) \tag{1.20}
$$

for some 1-periodic complex valued function c. Using the periodicity of the functions in (1.20) we obtain

$$
\left(M_f\left(\xi + \frac{1}{2}\right), M_f\left(\xi\right)\right) = c\left(\xi + \frac{1}{2}\right)\left(\overline{H_0\left(\xi\right)}, -\overline{H_0\left(\xi + \frac{1}{2}\right)}\right)
$$

so that $M_f\left(\xi\right) = c\left(\xi\right)\overline{H_0\left(\xi + \frac{1}{2}\right)}$ and $c\left(\xi\right) = -c\left(\xi + \frac{1}{2}\right)$. Summarizing, $f \in W_0$ if and only if

$$
\widehat{f}\left(\gamma\right) = M_f\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right) \text{ and } M_f\left(\gamma\right) = c\left(\gamma\right)\overline{H_0\left(\gamma + \frac{1}{2}\right)},
$$

where c is a 1-periodic function satisfying $c(\gamma) = -c\left(\gamma + \frac{1}{2}\right)$. This condition is equivalent to $b(\gamma) = e^{-2\pi i\gamma}c(\gamma)$ being one half periodic. Setting $\upsilon_f(\gamma) = b\left(\frac{\gamma}{2}\right)$, υ_f is 1-periodic and

$$\widehat{f}(\gamma) = e^{\pi i\gamma}\upsilon_f(\gamma)\,\overline{H_0\left(\frac{\gamma}{2} + \frac{1}{2}\right)}\widehat{\varphi}\left(\frac{\gamma}{2}\right)$$

as claimed. $\|f\|_{L^2(\mathbb{R})} = \|\upsilon\|_{L^2(\mathbb{T})}$ can be easily proved using (1.15). □

The system $\{\psi(x-k)\}_{k\in\mathbb{Z}} \subset W_0$ is an orthonormal system if and only if $\sum_k \left|\widehat{\psi}(\gamma - k)\right|^2 = 1$ a.e. More precisely,

Lemma 1.15. *The system $\{\psi(x-k)\}_{k\in\mathbb{Z}} \subset W_0$ is an orthonormal basis for W_0 if and only if $|\upsilon_\psi(\gamma)| = 1$ a.e., where υ_ψ a 1-periodic function given by (1.17) and ψ satisfies (1.17).*

Proof.

$$\begin{aligned}
1 &= \textstyle\sum_k \left|\widehat{\psi}(\gamma - k)\right|^2 \\
&= |\upsilon_\psi(\gamma)|^2 \left[\left|M_\psi\left(\frac{\gamma}{2} + \frac{1}{2}\right)\right|^2 \textstyle\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi}\left(\frac{\gamma}{2} + k\right)\right|^2 + \left|M_\psi\left(\frac{\gamma}{2}\right)\right|^2 \textstyle\sum_{k\in\mathbb{Z}} \left|\widehat{\varphi}\left(\frac{\gamma}{2} + \frac{1}{2} + k\right)\right|^2\right] \\
&= |\upsilon_\psi(\gamma)|^2 \left[\left|M_\psi\left(\frac{\gamma}{2} + \frac{1}{2}\right)\right|^2 + \left|M_\psi\left(\frac{\gamma}{2}\right)\right|^2\right] = |\upsilon_\psi(\gamma)|^2.
\end{aligned}$$

Hence, $\{\psi(x-k)\}_{k\in\mathbb{Z}} \subset W_0$ is an orthonormal system if and only if $|\upsilon(\gamma)| = 1$ a.e. To prove that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ is complete in W_0, let $f \in W_0$. Then

$$\widehat{f}(\gamma) = e^{\pi i\gamma}\upsilon_f(\gamma)\,\overline{H_0\left(\frac{\gamma}{2} + \frac{1}{2}\right)}\widehat{\varphi}\left(\frac{\gamma}{2}\right),$$

so that

$$\begin{aligned}
\widehat{f}(\gamma) &= e^{\pi i\gamma}\upsilon_f(\gamma)\,\overline{H_0\left(\frac{\gamma}{2} + \frac{1}{2}\right)}\widehat{\varphi}\left(\frac{\gamma}{2}\right) = \upsilon_\psi^{-1}(\gamma)\,\upsilon_f(\gamma)\left[e^{\pi i\gamma}\upsilon_\psi(\gamma)\,\overline{H_0\left(\frac{\gamma}{2} + \frac{1}{2}\right)}\widehat{\varphi}\left(\frac{\gamma}{2}\right)\right] \\
&= \upsilon_\psi^{-1}(\gamma)\,\upsilon_f(\gamma)\,\widehat{\psi}(\gamma) = K_f(\gamma)\,\widehat{\psi}(\gamma).
\end{aligned}$$

Here, $K_f(\gamma) = \upsilon_\psi^{-1}(\gamma)\,\upsilon_f(\gamma)$. This K_f is an $L^2(\mathbb{T})$ function since $\upsilon_\psi(\gamma)$ is unimodular a.e. Taking the inverse Fourier transform we see that every function in W_0 is a series of the form

$$f(x) = \textstyle\sum_{k\in\mathbb{Z}} k_f(k)\,\psi(x-k) \text{ in } L^2(\mathbb{R}),$$

where $\{k_f(n)\}_{n\in\mathbb{Z}}$ is in $l^2(\mathbb{Z})$. This completes the proof. $\square$

Now,

$$V_j \bigoplus W_j = V_{j+1},$$

$$D^j(V_0 \bigoplus W_0) = D^j(V_0) \bigoplus D^j(W_0) = V_j \bigoplus D^j(W_0),$$

and hence,

$$D^j(W_0) = W_j.$$

Moreover, since

$$\forall j \in \mathbb{Z}, V_j \subset V_{j+1}, \overline{\textstyle\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$$

and

$$\bigcap\nolimits_{j=-\infty}^{\infty} V_j = \{0\},$$

we get that

$$\overline{\text{span}}\,\{\psi_{j,k}\}_{j\in\mathbb{Z},k<p} = V_p$$

and

$$L^2(\mathbb{R}) = \textstyle\bigoplus_{j\in\mathbb{Z}} W_j.$$

Chapter 2

Frame Multiresolution Analysis (FMRA)

2.1 Overview

Frames were introduced in the 1950s to deal with problems in nonharmonic Fourier series [25]. They are an appropiate tool to deal with problems where redundancy, robustness, oversampling, and /or nonuniform sampling play a role.

A feature that makes FMRAs potentially useful in signal processing is the fact that the perfect reconstruction filter banks associated to them can be narrow band, whence FMRA filter banks can achieve quantization noise reduction simultaneously with reconstruction of a given narrow-band signal [8, 9, 10, 11].

In sections 2.2 and 2.3 we generalize the main results of the theory of FMRA proved in [10, 11] to $\mathbb{R}^d$. Our main results are Theorems 2.2, 2.3, 2.5, 2.6 in section 2.2 and Theorems 2.7 in section 2.3. Sections 2.4 and 2.5 are devoted to the construction of wavelet frames for $L^2\left(\mathbb{R}^d\right)$ in the spirit of Mallat-Meyer algorithm.

Theorems 2.2, 2.3, 2.5, and 2.6 provide the equations necessary to state quantitative sufficient conditions in order that an FMRA should give rise to a wavelet frame for $L^2\left(\mathbb{R}^d\right)$. In fact, Theorem 2.6, which summarizes Theorems 2.2, 2.3, and 2.5 gives sufficient conditions for translates of a given finite set of functions to be a wavelet frame for a basic subspace W_0 of $L^2\left(\mathbb{R}^d\right)$.

Theorem 2.7 was proved independently by Benedetto and Treiber [11], and

Kim et al. [31, 32]. It states that a neccesary and sufficient condition to obtain wavelets by a generalized Mallat-Meyer algorithm is that a set related to the spectrum of the core subspace V_0 of the FMRA should have measure zero.

2.2 Frame Multiresolution Analysis

Definition 2.1. A frame multiresolution analysis (FMRA) $(V_j, \varphi)_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ is an increasing sequence of closed linear subspaces $V_j \subset L^2(\mathbb{R}^d)$ and an element $\varphi \in V_0$ for which the following hold:

1. $\overline{\cup_j V_j} = L^2(\mathbb{R}^d)$ and $\cap V_j = \{0\}$,
2. $f \in V_j \Longleftrightarrow Df \in V_{j+1}$, where $Df(x) = 2^{d/2} f(2x)$,
3. $\forall k \in \mathbb{Z}^d, f \in V_0 \Longleftrightarrow \tau_k f \in V_0$,
4. $\{\tau_k \varphi : k \in \mathbb{Z}^d\}$ is a frame for V_0.

The following results are well known for the case $d = 1$, see [10, 11]. The equations in these results are the key for the construction of FMRA frames.

Theorem 2.2. *Let (V_j, φ) be an FMRA of $L^2(\mathbb{R}^d)$, let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, the orthogonal complement of V_0 in V_1, and set $\psi_0 = \varphi$.*

1. *If $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \{\tau_k\psi_p : 1 \le p \le m; k \in \mathbb{Z}^d\}$ defines W_0, i.e., $\overline{span}\left(\cup_{k\in\mathbb{Z}^d}\tau_k\omega\right) = W_0$, then there are $g_0, ..., g_m \in l^2\left(\mathbb{Z}^d\right)$ such that*

$$\forall n\in\mathbb{Z}^d,\ \varphi\left(2x-n\right) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p\left(2k-n\right)\psi_p\left(x-k\right)\ \ in\ L^2(\mathbb{R}^d). \tag{2.1}$$

2. *If there are $g_0, ..., g_m \in A'\left(\mathbb{Z}^d\right)$ such that (2.1) is valid, and if $\|\mathcal{X}(\psi_p)(\gamma)\|^2_{l^2\left(\mathbb{Z}^d\right)}$ is essentially bounded for each $1 \le p \le m$, i.e., each $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Proof. Any $f \in V_1$ can be written uniquely as $f_0 + k_0$, with $f_0 \in V_0$ and $k_0 \in W_0$. For each $m \in \mathbb{Z}^d$ and for each $u\in\{0,1\}^d$, $\varphi(2x - 2m - u)$ is an element of V_1. Since $\cup_{k\in\mathbb{Z}^d}\tau_k\omega = \{\tau_k\psi_p : 1 \le p \le m;\ k \in \mathbb{Z}^d\}$ generates W_0, and since $\cup_{k\in\mathbb{Z}^d}\tau_k\varphi$ is a frame for V_0, there exists a set $\left\{g_{i,u} \in l^2\left(\mathbb{Z}^d\right) : 0 \le i \le m; u\in\{0,1\}^d\right\}$ such that for each $m \ge 0$ and each $u\in\{0,1\}^d$, we have

$$\varphi(2x - 2m - u) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}(k - m)\,\psi_p(x - k)\,.$$

Now, define $\{g_p\}_{p=0}^m$ by means of the formula

$$g_p(2k + u) = g_{p,u}(k)\,,\ \ 0 \le p \le m,\ \ u\in\{0,1\}^d,\ \ k \in \mathbb{Z}^d.$$

If $n = 2l + u$, $n, l \in \mathbb{Z}^d, u\in\{0,1\}^d$, then

$$\begin{aligned}\varphi(2x - n) = \varphi(2x - 2l - u) &= \sum_{i=0}^{m}\sum_{k\in\mathbb{Z}^d} g_{p,u}(k - l)\,\psi_p(x - k)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k - 2l - u)\,\psi_p(x - k)\\ &= \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p(2k - n)\,\psi_p(x - k)\ \text{ in } L^2(\mathbb{R}^d).\end{aligned}$$

Thus, the proof of (1) is complete. Next, assume that the hypotheses of part (2) hold. For each $f \in W_0 \subset V_1$, there is $\{c(n)\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$ such that

$$\begin{aligned}f(x) &= \sum_{n\in\mathbb{Z}^d} c(n)\,\varphi(2x - n)\\ &= \sum_{u\in\{0,1\}^d}\sum_{k\in\mathbb{Z}^d} c(2k + u)\,\varphi(2x - 2k - u)\,.\end{aligned}$$

The previous equation is equivalent to

$$\begin{aligned}
\widehat{f}(\gamma) &= \sum_{k\in\mathbb{Z}^d} 2^{-d} \sum_{u\in\{0,1\}^d} c\left(2k+u\right)\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-2\pi ik\cdot\gamma}e^{-\pi iu\cdot\gamma} \\
&= \sum_{u\in\{0,1\}^d} 2^{-d}\left[\sum_{k\in\mathbb{Z}^d} c\left(2k+u\right)e^{-2\pi ik\cdot\gamma}\right]e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right) \qquad (2.2)\\
&= \sum_{u\in\{0,1\}^d} e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)C_u\left(\gamma\right) \text{ in } L^2(\widehat{\mathbb{R}}^d),
\end{aligned}$$

where $C_u\left(\gamma\right) = 2^{-d}\sum_{k\in\mathbb{Z}^d} c\left(2k+u\right)e^{-2\pi ik\cdot\gamma} \in L^2(\mathbb{T}^d)$, and where the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the C_u by the Parseval-Plancherel theorem. If we take the Fourier transform of (2.1), we obtain for $n = 2l+u$, $u\in\{0,1\}^d$, and $l \in \mathbb{Z}^d$, that

$$\varphi\left(2x-2l-u\right) = \sum_{p=0}^{m}\sum_{k\in\mathbb{Z}^d} g_p\left(2\left(k-l\right)-u\right)\psi_p\left(x-k\right)$$

if and only if

$$2^{-d}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-2\pi il\cdot\gamma}e^{-\pi iu\cdot\gamma} = \sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d} g_p\left(2\left(k-l\right)-u\right)e^{-2\pi i\mathbf{k}\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right).$$

Hence,

$$\begin{aligned}
\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)e^{-\pi iu\cdot\gamma} &= 2^d\sum_{p=0}^{m}\left[\sum_{k\in\mathbb{Z}^d} g_p\left(2\left(k-l\right)-u\right)e^{-2\pi i(k-l)\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right) \\
&= \sum_{p=0}^{m} G_{p,u}\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),
\end{aligned}$$

where $G_{p,u}\left(\gamma\right) = 2^d\sum_{k\in\mathbb{Z}^d} g_p\left(2k-u\right)e^{-2\pi ik\cdot\gamma} \in L^2(\mathbb{T}^d)$, for all p, u, and the convergence in $L^2(\widehat{\mathbb{R}}^d)$ is in terms of the partial sums of the $G_{p,u}$s. Substituting into equation (2.2), we have

$$\begin{aligned}
\widehat{f}\left(\gamma\right) &= \sum_{u\in\{0,1\}^d}\left(e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\tfrac{\gamma}{2}\right)\right)C_u\left(\gamma\right)\\
&= \sum_{u\in\{0,1\}^d}\left(\sum_{j=0}^{m}G_{p,u}\left(\gamma\right)\widehat{\psi_j}\left(\gamma\right)\right)C_u\left(\gamma\right)\\
&= \sum_{p=0}^{m}\left(\sum_{u\in\{0,1\}^d}G_{p,u}\left(\gamma\right)C_u\left(\gamma\right)\right)\widehat{\psi_j}\left(\gamma\right)\\
&= \sum_{p=0}^{m}E_p\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),
\end{aligned}$$

where $E_p\left(\gamma\right)=\sum_{u\in\{0,1\}^d}G_{p,u}\left(\gamma\right)C_u\left(\gamma\right)$. Using the hypothesis that $g_p\in A'\left(\mathbb{Z}^d\right), 0\leq p\leq m$, we see that $E_p\in L^2(\mathbb{T}^d)$. Now, for $N\in\mathbb{N}$, we define

$$\widehat{f}_N\left(\gamma\right)=\sum_{p=0}^{m}\left[\sum_{|n|\leq N}E_p^{\vee}\left(n\right)e^{-2\pi in\cdot\gamma}\right]\widehat{\psi_p}\left(\gamma\right)=\sum_{p=0}^{m}E_p^N\left(\gamma\right)\widehat{\psi_p}\left(\gamma\right),$$

where $E_p^N\left(\gamma\right)=\sum_{|n|\leq N}E_p^{\vee}\left(n\right)e^{-2\pi in\cdot\gamma}$ (the N-th symmetric partial sum of $E_p\left(\gamma\right)$), $n=\left(n_1,...,n_d\right)\in\mathbb{Z}^d$, and $|n|=\sum_{i=1}^{d}|n_i|$. Clearly, $\widehat{f}_N\in L^2(\widehat{\mathbb{R}}^d)$ and

$$\lim_{N\to\infty}||\widehat{f_N}-\widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)}=0,$$

since each $\widehat{\psi_p}^2\in\mathcal{L}^\infty$ and since

$$\begin{aligned}
||\widehat{f_N}-\widehat{f}||_{L^2(\widehat{\mathbb{R}}^d)} &\leq \sum_{p=0}^{m}||\widehat{\psi_p}\left(E_p-E_p^N\right)||_{L^2(\widehat{\mathbb{R}}^d)}\\
&= \sum_{p=0}^{m}\sum_{n\in\mathbb{Z}^d}\left(\int_{[0,1]^d}\tau_n|\widehat{\psi_p}|^2\left|E_p-E_p^N\right|^2\right)^{\frac{1}{2}}\\
&= \sum_{p=0}^{m}||\sqrt{\Phi_p}\left(E_p-E_p^N\right)||_{L^2(\mathbb{T}^d)}\\
&\leq \sum_{p=0}^{m}||\sqrt{\Phi_p}||_{L^\infty(\mathbb{T}^d)}||\left(E_p-E_p^N\right)||_{L^2(\mathbb{T}^d)},
\end{aligned}$$

where $\Phi_p = P\left(|\widehat{\psi_p}|^2\right)$, $0 \le p \le m$. Hence, the Parseval-Plancherel theorem implies that

$$f = \sum_{p=0}^{m} \sum_{n \in \mathbb{Z}^d} E_p^{\vee}(n)\, \tau_n \psi_p \text{ in } L^2(\mathbb{R}^d).$$

Because $f \in W_0$, $\sum_{n\in\mathbb{Z}^d} E_0^{\vee}(n)\, \tau_n \psi_0 = \sum_{n\in\mathbb{Z}^d} E_0^{\vee}(n)\, \tau_n \varphi = 0$. Hence,

$$f = \sum_{p=1}^{m} \sum_{n \in \mathbb{Z}^d} E_p^{\vee}(n)\, \tau_n \psi_p \text{ in } L^2(\mathbb{R}^d),$$

where $\left\{E_p^{\vee}(n)\right\}_{n\in\mathbb{Z}^d} \in l^2\left(\mathbb{Z}^d\right)$, for $p \in \{1, ..., m\}$. So far we have proved that $\cup_{k\in\mathbb{Z}^d} \tau_k \omega$ generates W_0. This mean that the linear operator $T_{W_0}^* : l^2\left(\mathbb{Z}^d\right) \to W_0$, $T_\omega^*\left(\{c_k\}\right) = \sum_{p=1}^{m} \sum_k c_k \tau_k \psi_p$, is a surjection onto W_0. Its adjoint, T_{W_0}, is bounded since $\widehat{\psi_p}^2 \in \mathcal{L}^\infty$, $1 \le p \le m$. This implies that $T_{W_0}^*$ is also bounded. The open mapping theorem guarantees that $T_{W_0}^*$ is also bounded below on $\mathcal{N}\left(T_{W_0}^*\right)^\perp$, since the restriction of this map to $\mathcal{N}\left(T_{W_0}^*\right)^\perp$ is an invertible operator. It follows that $\cup_{k\in\mathbb{Z}^d} \tau_k \omega$ is a frame for W_0 (see Proposition 3.4 in [11]). $\square$

Theorem 2.3. *Let $H_0 \in L^\infty\left(\mathbb{T}^d\right)$, and let (V_j, φ) be an FMRA, where H_0 and φ satisfy*

$$\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}\right) \text{ a.e. in } L^2(\widehat{\mathbb{R}}^d). \tag{2.3}$$

Define W_0 as the orthogonal complement of V_0 in V_1. Further, for $\{h_1[n]\} \in l^2\left(\mathbb{Z}^d\right)$, let $\widehat{h_1} = H_1 \in L^2\left(\mathbb{T}^d\right)$, and let $\psi \in V_1$ be defined as

$$\widehat{\psi}(\gamma) = H_1\left(\frac{\gamma}{2}\right) \widehat{\varphi}\left(\frac{\gamma}{2}\right) \text{ a.e. in } L^2(\widehat{\mathbb{R}}^d). \tag{2.4}$$

Then $\psi \in W_0$ if and only if

$$\sum_{u\in\{0,1\}^d} \tau_{\frac{u}{2}}\left(H_1 \overline{H_0} \Phi\right) = 0 \text{ a.e. in } L^2(\mathbb{T}^d). \tag{2.5}$$

Proof. Set $e_k(\gamma) = e^{-2\pi i k\cdot\gamma}$. Then $\psi \in V_0^{\perp}$ (*in* V_1)

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \left\langle \widehat{\psi}, e_k\widehat{\varphi} \right\rangle = \langle \psi, \tau_k\varphi \rangle = 0$$

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \left\langle H_1\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right), e_k H_0\left(\tfrac{\cdot}{2}\right)\widehat{\varphi}\left(\tfrac{\cdot}{2}\right) \right\rangle = 0$$

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \langle H_1\widehat{\varphi}, e_{2k}H_0\widehat{\varphi} \rangle = 0$$

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \int_{\mathbb{R}^d} H_1\overline{H_0}|\widehat{\varphi}|^2 e_{2k} = 0$$

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^d} H_1\overline{H_0}\Phi e_{2k} = 0$$

$$\Longleftrightarrow \forall k \in \mathbb{Z}^d, \sum_{u\in\{0,1\}^d} \int_{\left[0,\frac{1}{2}\right]^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right) e_{2k} = 0$$

$$\Longleftrightarrow \quad \int_{\left[0,\frac{1}{2}\right]^d} \sum_{u\in\{0,1\}^d} \tau_{\frac{u}{2}}\left(H_1\overline{H_0}\Phi\right) e_{2k} = 0.$$

The result follows by the L^1- uniqueness theorem for Fourier series. $\square$

Remark 2.4. *After periodizing the modulus squared of (2.3), equation (2.3) is equivalent to the following equation:*

$$\Phi = \sum_{u\in\{0,1\}^d} \tau_{-\frac{u}{2}}\left[\left(|H_0|^2\,\Phi\right)\left(\frac{\cdot}{2}\right)\right]. \tag{2.6}$$

This equation will be needed for a remark after Theorem 2.7.

Theorem 2.5. *Let (V_j, φ) be an FMRA, let $H_0 \in L^\infty(\mathbb{T}^d)$ satisfy $\widehat{\varphi}(\gamma) = H_0\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$, and let $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$ and $H_p \in L^2(\mathbb{T}^d)$, $1 \le p \le m$, satisfy $\widehat{\psi_p}(\gamma) = H_p\left(\frac{\gamma}{2}\right)\widehat{\varphi}\left(\frac{\gamma}{2}\right)$ in $L^2(\widehat{\mathbb{R}}^d)$. Suppose that $\widehat{g_p} = G_p \in L^\infty(\mathbb{T}^d)$, $0 \le p \le m$. Then (2.1) holds if and only if*

$$\forall u \in \{0,1\}^d, \ \sum_{p=0}^{m} \tau_{-\frac{1}{2}u}(H_p\Phi)G_p = \Phi\delta(0,u). \tag{2.7}$$

Here, $\delta(0,u)$ is the Kronecker delta.

Proof. Equation (2.1) is equivalent to the following equation:

$$\begin{aligned}\forall\ n \in \mathbb{Z}^d,\ \widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-n\right) e^{-2\pi i(2k-n)\cdot\gamma} \widehat{\psi_p}\left(2\gamma\right) \\ &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-n\right) e^{-2\pi i(2k-n)\cdot\gamma} H_p\left(\gamma\right) \widehat{\varphi}(\gamma). \end{aligned} \tag{2.8}$$

Adding these equations over all $v \in \{0,1\}^d$ we obtain

$$\begin{aligned} 2^d\widehat{\varphi}(\gamma) &= 2^d \sum_{p=0}^{m} \sum_{v\in\{0,1\}^d} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i(2k-v)\cdot\gamma} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{q\in\mathbb{Z}^d} g_p\left(q\right) e^{-2\pi i q\cdot\gamma} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} G_p\left(\gamma\right) H_p(\gamma)\widehat{\varphi}(\gamma). \end{aligned}$$

Here, $G_p\left(\gamma\right) = \sum_{q\in\mathbb{Z}^d} g_p\left(q\right) e^{-2\pi i q\cdot\gamma}$, $0 \le p \le m$. The second equality follows from the fact that any $q \in \mathbb{Z}^d$ can be written as $2k - v$, where $k \in \mathbb{Z}^d$ and $v \in \{0,1\}^d$ are uniquely determined. Now, let $v \in \{0,1\}^d$ and $u \in \{0,1\}^d \setminus \{0\}$ be fixed. Multiplying (2.8) by $e^{-\pi i v\cdot u}$, we have

$$\begin{aligned} \widehat{\varphi}(\gamma)e^{-\pi i v\cdot u} &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i(2k-v)\cdot\gamma} e^{-\pi i v\cdot u} H_p\left(\gamma\right) \widehat{\varphi}(\gamma) \\ &= 2^d \sum_{p=0}^{m} \sum_{k\in\mathbb{Z}^d} g_p\left(2k-v\right) e^{-2\pi i(2k-v)\cdot\left(\gamma-\frac{u}{2}\right)} H_p\left(\gamma\right) \widehat{\varphi}(\gamma). \end{aligned}$$

On the other hand, $\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u} = 0$ for a fixed $u \in \{0,1\}^d \setminus \{0\}$ since $e^{-\pi i v\cdot u} = (-1)^{v\cdot u}$. In fact, $\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u} = \sum_{v\in\{0,1\}^d} (-1)^{v\cdot u}$; and so, for a fixed $u \in \{0,1\}^d \setminus \{0\}$, half of the $\{v\cdot u\}_{v\in\{0,1\}^d}$ are even and half are odd, and so the original sum of exponentials is zero.

Hence,

$$\begin{aligned}
0 &= (\sum_{v\in\{0,1\}^d} e^{-\pi i v\cdot u})\widehat{\varphi}(\gamma)\\
&= 2^d \sum_{p=0}^{m} \sum_{v\in\{0,1\}^d} \sum_{k\in\mathbb{Z}^d} g_p\,(2k-v)\, e^{-2\pi i(2k-v)\cdot(\gamma-\frac{u}{2})} H_p\,(\gamma)\,\widehat{\varphi}(\gamma)\\
&= 2^d \sum_{p=0}^{m} \sum_{q\in\mathbb{Z}^d} g_p\,(q)\, e^{-2\pi i q\cdot(\gamma-\frac{u}{2})} H_p\,(\gamma)\,\widehat{\varphi}(\gamma)\\
&= 2^d \sum_{p=0}^{m} G_p(\gamma-\frac{u}{2})H_p(\gamma)\widehat{\varphi}(\gamma)\\
&= 2^d \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})\widehat{\varphi}(\xi+\frac{u}{2}).
\end{aligned}$$

Summarizing the previous calculations, we have verified that

$$\widehat{\phi}(\xi) = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi)\widehat{\varphi}(\xi),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d \tag{2.9}$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})\widehat{\varphi}(\xi+\frac{u}{2}),\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d \tag{2.10}$$

are equivalent to (2.1).

To prove the theorem, assume (2.1), i.e., assume (2.9) and (2.10). Multiplying (2.9) and (2.10) by $\overline{\widehat{\varphi}(\xi)}$ and $\overline{\widehat{\varphi}(\xi+\frac{u}{2})}$, respectively, we obtain

$$|\widehat{\phi}(\xi)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\, H_{pj}(\xi)|\widehat{\varphi}(\xi)|^2 \text{ a.e. } \xi\in\widehat{\mathbb{R}}^d$$

and

$$0 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+\frac{u}{2})|^2,\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d.$$

By the periodicity of the $G_p s$ and the $H_p s$, the change of variable $\xi\to\xi+k$ produces

$$|\widehat{\varphi}(\xi+k)|^2 = \sum_{p=0}^{m} G_p\,(\xi)\, H_p(\xi)|\widehat{\varphi}(\xi+k)|^2,\ \text{a.e. } \xi\in\widehat{\mathbb{R}}^d$$

and

$$0=\sum_{p=0}^{m}G_p\left(\xi\right)H_p(\xi+\frac{u}{2})|\widehat{\varphi}(\xi+k+\frac{u}{2})|^2,\ \text{a.e. }\ \xi\in\widehat{\mathbb{R}}^d.$$

Summing over all k in $\mathbb{Z}^d$, we have

$$\Phi(\xi)=\sum_{p=0}^{m}G_p\left(\xi\right)H_p(\xi)\Phi(\xi),\ \text{a.e. }\ \xi\in\widehat{\mathbb{R}}^d$$

and

$$0=\sum_{p=0}^{m}G_p\left(\xi\right)H_p(\xi+\frac{u}{2})\Phi(\xi+\frac{u}{2}),\ \text{a.e. }\ \xi\in\widehat{\mathbb{R}}^d,$$

i.e.,

$$\sum_{p=0}^{m}\tau_{-\frac{1}{2}u}(H_p\Phi)G_p=\Phi\delta\left(0,u\right),\ \text{a.e. on }\mathbb{T}^d.$$

This proves the"only if" part.

Conversely, if equation (2.7) has a solution $G_p\in L^\infty(\mathbb{T}^d)$, $0\le p\le m$, then

$$1=\sum_{p=0}^{m}G_p\left(\xi\right)H_p(\xi),\ \text{a.e. }\ \xi\in[\Phi>0]$$

and

$$0=\sum_{p=0}^{m}G_p\left(\xi\right)H_p(\xi+\frac{u}{2}),\ \text{a.e. }\ \xi\in\left[\tau_{-\frac{u}{2}}\Phi>0\right].$$

Clearly, $\widehat{\varphi}(\xi)\neq 0$ implies that $\Phi(\xi)\neq 0$; and therefore $[\widehat{\varphi}\neq 0]\subseteq[\Phi>0]$ and $\left[\tau_{-\frac{u}{2}}\varphi\neq 0\right]\subseteq\left[\tau_{-\frac{u}{2}}\Phi>0\right]$. This implies that the previous two equations hold a.e. on $[\Phi{>}0]$ and on $\left[\tau_{-\frac{u}{2}}\Phi>0\right]$, respectively. This yields (2.9) and (2.10), and, hence, the proof is complete. $\square$

Combining Theorems 2.3 and 2.5 we obtain the following result:

Theorem 2.6. *Let (V_j,φ) be an FMRA, let $\omega=\{\psi_1,...,\psi_m\}\subset W_0$, and let H_p, $0\le p\le m$, be as above. Assume that $\widehat{\psi_p}^2\in\mathcal{L}^\infty$, $1\le p\le m$. If there are $\widehat{g}_p=G_p\in L^\infty(\mathbb{T}^d)$, $0\le p\le m$, such that (2.7) holds, then $\cup_{k\in\mathbb{Z}^d}\tau_k\omega$ is a frame for W_0.*

Once we have a frame for W_0, standard methods can be used to construct an FMRA frame for all of $L^2(\widehat{\mathbb{R}}^d)$, e.g., [8, 9, 10, 11, 18, 22, 33, 31, 32, 40].

2.3 Measure-Theoretic Criterion for Wavelets

The main idea of FMRAs is to apply the ideas of classical multiresolution analysis to contruct wavelet frames by means of a generalized Mallat-Meyer algorithm. Theorem 2.6 is a characterization of when such a construction is possible. As will be seen, this construction is not always guaranteed, but depends solely on the measure properties of a certain set which is intimately related to the spectrum of V_0. On the other hand, for the one dimensional case, H. O. Kim et al. construct two wavelets generating $L^2(\mathbb{R})$ independently of the measure of the aforementioned set [32].

In [10], Benedetto and Li applied the theory of FMRAs to the analysis of narrow band signals. Then, in [11], Benedetto and Treiber presented the main results of the theory of FMRAs from a functional analytic perspective. The proof of the main result in [11] gives a recipe for constructing wavelet frames when a natural measure theoretic criterion is satisfied, see Theorem 2.7. In this case, the construction in [11] can be extended to $\mathbb{R}^d$, for $d > 1$, by tensor products. We shall make this construction in section 2.5.

Theorem 2.7. *Suppose (V_j, φ) is an FMRA of $L^2(\mathbb{R}^d)$, and let $H_0 \in L^\infty(\mathbb{T}^d)$ have the property that $\widehat{\varphi}(2\gamma) = H_0(\gamma)\,\widehat{\varphi}(\gamma)$ a.e. Set*

$$\Gamma = \left\{\gamma \in \mathbb{T}^d : \Phi(2\gamma) = 0,\ \Phi\left(\gamma + \frac{u}{2}\right) > 0,\ u \in \{0,1\}^d\right\}.$$

Then, there is a set of wavelet functions $\omega = \{\psi_1, ..., \psi_m\} \subset W_0$, $m \leq 2^d - 1$, for *which the translations of ω are a frame for W_0 if and only if $|\Gamma| = 0$.*

Proof. (H. O. Kim, R. Y. Kim, J. K. Lim)

(a) We shall show that len $V_1 \leq 2^d$. We know that $V_0 = \overline{\operatorname{span}}\left\{\tau_k \varphi : k \in \mathbb{Z}^d\right\}$. Now, $V_1 = DV_0$ which implies that $V_1 = \overline{\operatorname{span}}\left\{D\tau_k \varphi : k \in \mathbb{Z}^d\right\}$. The relations $\forall k \in \mathbb{Z}^d$, $D\tau_{2k} = \tau_k D$ give us

$$D\tau_{2k+u}\varphi = D\tau_{2k}\tau_u\varphi = \tau_k D\tau_u\varphi = \tau_k\varphi_u,\ k \in \mathbb{Z}^d,\ u \in \{0,1\}^d .$$

Here we are using the fact that every $n \in \mathbb{Z}^d$ can be written uniquely in the form $2k + u$, with $k \in \mathbb{Z}^d$ and $u \in \{0,1\}^d$. Also, $\varphi_u = D\tau_u\varphi$. Therefore,

$$V_1 = \overline{\operatorname{span}}\left\{\tau_k\varphi_u : k \in \mathbb{Z}^d,\ u \in \{0,1\}^d\right\}.$$

since card $\{0,1\}^d$ is 2^d, len $V_1 \leq 2^d$.

(b) We shall show that len $V_0 \leq 1$. From $\varphi(2\gamma) = H_0(\gamma)\varphi(\gamma)$, we obtain

$$\mathcal{X}(V_0)(\gamma) = \operatorname{span}\left\{ \left(H_0\left(\frac{1}{2}(\gamma-k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d}\right\}$$

which implies that $\mathcal{X}(V_0)(\gamma)$ is at most one dimensional :

(c) The $-k$ th component of $\mathcal{X}(\varphi_u)(\gamma)$ is given by $2^{-\frac{d}{2}}e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)$ so that

$$\mathcal{X}(V_1)(\gamma) = \operatorname{span}\left\{ \left(e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right)\right)_{k\in\mathbb{Z}^d} : u \in \{0,1\}^d\right\}.$$

If we compute,

$$\begin{aligned}(\mathcal{X}(\varphi_u)(\gamma))_{-k} &= \widehat{\varphi}_u(\gamma-k) = \widehat{D\tau_u\varphi}(\gamma-k) = \int 2^{\frac{d}{2}}\varphi(2x-u)\,e^{-2\pi i x\cdot(\gamma-k)}dx \\ &= \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}(y+u)\cdot(\gamma-k)}dy = \int 2^{-\frac{d}{2}}\varphi(y)\,e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}e^{-2\pi i\frac{1}{2}y\cdot(\gamma-k)}dy \\ &= 2^{-\frac{d}{2}}e^{-2\pi i\frac{1}{2}u\cdot(\gamma-k)}\int\varphi(y)\,e^{-2\pi i y\cdot\frac{1}{2}(\gamma-k)}dy = 2^{-\frac{d}{2}}e^{-2\pi i u\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}(\gamma-k)\right).\end{aligned}$$

(d) We now compute $\sigma\left(V_1\right)$. For $\mathbf{c}\in l^2\left(\mathbb{Z}^d\right)$ define

$$\mathbf{c}_u\left(k\right)=\begin{cases}\mathbf{c}\left(k\right) & \text{if} \quad k=2m+u,\ m\in\mathbb{Z}^d,\\ \\ 0 & \qquad\qquad \text{otherwise.}\end{cases}$$

Two multi-integers m,n are congruent mod $\{0,1\}^d$ if they have the same multi-remainder $u\in\{0,1\}^d$, i.e., if $m=2k_1+u$ and $n=2k_2+u$ where $k_1,k_2\in\mathbb{Z}^d$ and $u\in\{0,1\}^d$. It follows that $\mathbf{c}=\sum\limits_{u\in\{0,1\}^d}\mathbf{c}_u$ and

$$\left\langle\mathbf{c}_u,\mathbf{c}_v\right\rangle=\delta\left(u,v\right)\left\|\mathbf{c}_u\right\|^2,$$

where δ is the Kronecker delta. Hence,

$$\begin{aligned}\left(e^{-2\pi iu\cdot\frac{1}{2}(\gamma-k)}\widehat{\varphi}\left(\frac{1}{2}\left(\gamma-k\right)\right)\right)_{k\in\mathbb{Z}^d}&=\left(e^{\pi iu\cdot k}e^{-\pi iu\cdot\gamma}\widehat{\varphi}\left(\frac{1}{2}\left(\gamma-k\right)\right)\right)_{k\in\mathbb{Z}^d}\\&=e^{-\pi iu\cdot\gamma}\sum_{v\in\{0,1\}^d}e^{\pi iu\cdot v}\mathbf{c}_{v,\gamma},\end{aligned}$$

where $\mathbf{c}_\gamma\left(k\right)=\widehat{\varphi}\left(\frac{1}{2}\left(\gamma-k\right)\right)$ and $\mathbf{c}_{v,\gamma}$ is defined as above (only the v-congruent entries survive). Therefore, $\mathcal{X}\left(V_1\right)\left(\gamma\right)=\operatorname{span}\left\{\ \mathbf{c}_{u,\gamma}:u\in\{0,1\}^d\right\}$ and

$$\sigma\left(V_1\right)=\left\{\gamma\in\mathbb{T}^d:\dim\mathcal{X}\left(V_1\right)\left(\gamma\right)>0\right\}=\left\{\gamma\in\mathbb{T}^d:\ \text{at least one }\mathbf{c}_{u,\gamma}\neq0\right\}.$$

On the other hand $\mathbf{c}_{u,\gamma}\neq0$ implies that $\left\|\mathbf{c}_\gamma\right\|^2=\sum\limits_{u\in\{0,1\}^d}\left\|\mathbf{c}_{u,\gamma}\right\|^2>0$ by the Pythagorean theorem. We compute

$$\begin{aligned}\sigma\left(V_1\right)&=\left\{\gamma\in\mathbb{T}^d:\left\|\mathbf{c}_\gamma\right\|^2>0\right\}\\&=\left\{\gamma\in\mathbb{T}^d:\sum_{k\in\mathbb{Z}^d}\left|\widehat{\varphi}\left(\frac{\gamma}{2}-\frac{2k+u}{2}\right)\right|^2>0,\ \text{for some }u\right\}\\&=\left\{\gamma\in\mathbb{T}^d:\sum_{k\in\mathbb{Z}^d}\left|\widehat{\varphi}\left(\frac{\gamma}{2}-\frac{u}{2}+k\right)\right|^2>0,\ \text{for some }u\right\}\\&=\left\{\gamma\in\mathbb{T}^d:\Phi\left(\frac{\gamma}{2}-\frac{u}{2}\right)>0,\ \text{for some }u\right\}\\&=\left\{\gamma\in\mathbb{T}^d:\Phi\left(\frac{\gamma}{2}+\frac{u}{2}\right)>0,\ \text{for some }u\right\}.\end{aligned}$$

If we now define $H_{0,u} = H_0\left(\frac{\gamma}{2} - \frac{u}{2}\right)$, then,

$$\left(H_0\left(\frac{1}{2}(\gamma - k)\right)\widehat{\varphi}\left(\frac{1}{2}(\gamma - k)\right)\right)_{k\in\mathbb{Z}^d} = \sum_{u\in\{0,1\}^d} H_{0,u}\mathbf{c}_{u,\gamma}$$

because H_0 is 1-periodic in each variable. Hence,

$$\mathcal{X}(V_0)(\gamma) = \operatorname{span}\left\{\sum_{u\in\{0,1\}^d} H_{0,u}\mathbf{c}_{u,\gamma}\right\} \subset \operatorname{span}\left\{\mathbf{c}_{u,\gamma} : u \in \{0,1\}^d\right\} = \mathcal{X}(V_1)(\gamma).$$

(e) $\Gamma = \frac{1}{2}\left\{\gamma \in \mathbb{T}^d : \dim \mathcal{X}(V_1)(\gamma) = 2^d \text{ and } \forall u\in\{0,1\}^d,\ H_{0,u} = 0\right\}$. If we define $\Gamma_j := \left\{\gamma \in \mathbb{T}^d : D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = j.\right\}, 1 \leq j \leq 2^d$, then,

$$\sigma(V_1) = \bigcup_j \Gamma_j,$$

a disjoint union. If $\gamma \in \Gamma_{2^d}$ and $H_{0,u} = 0$ for every $u\in\{0,1\}^d$, then we have that $\mathcal{X}(V_0)(\gamma) = \{0\}$ and $\mathcal{X}(W_0)(\gamma) = \mathcal{X}(V_1)(\gamma)$. Hence, $D_{W_0}(\gamma) = \dim \mathcal{X}(W_0)(\gamma) = 2^d$, since $D_{V_1}(\gamma) = \dim \mathcal{X}(V_1)(\gamma) = 2^d$ $(\gamma \in \Gamma_{2^d})$.

Now,

$$\begin{aligned}
\Theta &= \left\{\gamma \in \Gamma_{2^d} : \forall u\in\{0,1\}^d,\ H_{0,u} = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u\in\{0,1\}^d,\ H_0\left(\frac{\gamma}{2} - \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \Gamma_{2^d} : \forall u\in\{0,1\}^d,\ H_0\left(\frac{\gamma}{2} + \frac{u}{2}\right) = 0\right\} \\
&= \left\{\gamma \in \mathbb{T}^d : \forall u\in\{0,1\}^d,\ \Phi(\gamma) = 0, \Phi\left(\frac{\gamma}{2} + \frac{u}{2}\right) > 0\right\} \\
&= \left\{2\lambda \in \mathbb{T}^d : \forall u\in\{0,1\}^d,\ \Phi(2\lambda) = 0, \Phi\left(\lambda + \frac{u}{2}\right) > 0\right\} = 2\Gamma.
\end{aligned}$$

Hence, $|\Theta| = 2^d|\Gamma|$, which implies that $|\Theta| > 0$ if and only if $|\Gamma| > 0$. It is now clear that if $|\Gamma| > 0$, then $D_{W_0}(\gamma) = 2^d$ in a subset of $\mathbb{T}^d$ with positive measure. Further, because of the way Θ is defined, *at least* 2^d *wavelets are necessary*, since in this case, len $W_0 = 2^d$. Now, applyng Theorems 1.7, 1.8, and 1.9 concerning shift-invariant subspaces, we obtain the result. □

Remark 2.8. *The equation (2.6),*

$$\Phi(2\gamma) = \sum_{u \in \{0,1\}^d} |H_0|^2 \left(\gamma + \frac{u}{2}\right) \Phi\left(\gamma + \frac{u}{2}\right),$$

is true a.e. The left side of this equation is zero in Γ, *and* $\Phi\left(\gamma + \frac{u}{2}\right) > 0$ *in* Γ. *Hence,* $H_0\left(\gamma + \frac{u}{2}\right) = 0$, *i.e.,*

$$\Gamma \subset \left\{\gamma \in \mathbb{T}^d : \forall u \in \{0,1\}^d,\ H_0\left(\gamma + \frac{u}{2}\right) = 0\right\}.$$

In other words, Γ *can be seen as a subset of the zero set of the low pass filter* H_0 *with the additional geometric condition that its elements* γ *also have the property that* $\gamma + \frac{u}{2}$ *is a zero of* H_0. *This observation can also be obtained from the proof of the previous theorem (see part (e) in the proof of Theorem 2.7).*

Because of the conclusions of Theorem 2.6 and Theorem 2.7, it seems reasonable to point out the distinction between these two results. Theorem 2.6 provides sufficient conditions in terms of equations (2.1) and (2.7) developed in section 2.2, that a finite sequence of elements is a frame of W_0. Theorem 2.7 provides necessary and sufficient conditions for the existence of such generators by a measure theoretic criterion. Theorem 2.7 is an existence theorem and and Theorem 2.6 can be view as part of a recipe to give explicitly the generators.

2.4 Tensors

The following calculation allows one to construct a frame of translates in higher dimensions by means of tensor products. It is a special case of a result found in [27].

Lemma 2.9. *Let* $\{\tau_k\varphi\}_{k\in\mathbb{Z}}$ *and* $\{\tau_k\varphi'\}_{k\in\mathbb{Z}}$ *be two frames for* $L^2(\mathbb{R})$ *with frame bounds* $0 < A \leq B < \infty$ *and* $0 < A' \leq B' < \infty$ *respectively, then* $\{\tau_m(\varphi\otimes\varphi')\}_{m\in\mathbb{Z}^2}$ *is a frame for* $L^2(\mathbb{R}^2)$ *with frame bounds* AA' *and* BB'*.*

Proof. Consider a function of the form $f = \sum_{i=1}^{n}\alpha_i\mathbb{1}_{A_i\times B_i}$, where the $A_i\times B_i$ are disjoint measurable rectangles. The set of functions of this form is dense in $L^2(\mathbb{R}^2)$. Now, lets compute:

$$\sum_{m\in\mathbb{Z}^2}|\langle f,\tau_m(\varphi\otimes\varphi')\rangle|^2 =$$

$$\sum_{k\in\mathbb{Z}^2}\left|\iint_{\mathbb{R}^2} f(x,y)\,\tau_k(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2 =$$

$$\sum_{\mathbf{k}\in\mathbb{Z}^2}\left|\iint_{\mathbb{R}^2}\sum_{i=1}^{n}\alpha_i\mathbf{1}_{A_i\times B_i}(x,y)\,\tau_m(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2 =$$

$$\sum_{m\in\mathbb{Z}^2}\sum_{i=1}^{n}\left|\iint_{A_i\times B_i}\alpha_i\mathbf{1}_{A_i\times B_i}(x,y)\,\tau_m(\varphi\otimes\varphi')(x,y)\,dxdy\right|^2 =$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^{n}\left|\iint_{A_i\times B_i}\alpha_i\mathbf{1}_{A_i}(x)\,\mathbf{1}_{B_i}(y)\,\varphi(x-k_1)\,\varphi'(y-k_2)\,dxdy\right|^2 =$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^{n}|\alpha_i|^2\left|\iint_{A_i\times B_i}\mathbf{1}_{A_i}(x)\,\mathbf{1}_{B_i}(y)\,\varphi(x-k_1)\,\varphi'(y-k_2)\,dxdy\right|^2 =$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^{n}|\alpha_i|^2\left|\int_{B_i}\mathbf{1}_{B_i}(y)\,\varphi'(y-k_2)\,dy\int_{A_i}\mathbf{1}_{A_i}(x)\,\varphi(x-k_1)\,dx\right|^2 =$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^{n}|\alpha_i|^2\left|\int_{B_i}\mathbf{1}_{B_i}(y)\,\varphi'(y-k_2)\,dy\right|^2\left|\int_{A_i}\mathbf{1}_{A_i}(x)\,\varphi(x-k_1)\,dx\right|^2 =$$

$$\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\sum_{i=1}^{n}|\alpha_i|^2\,|\langle\mathbf{1}_{B_i},\tau_{k_2}\varphi'\rangle|^2\,|\langle\mathbf{1}_{A_i},\tau_{k_1}\varphi\rangle|^2 =$$

$$\sum_{i=1}^{n} |\alpha_i|^2 \sum_{k_2\in\mathbb{Z}} |\langle \mathbf{1}_{B_i}, \tau_{k_2}\varphi' \rangle|^2 \sum_{k_1\in\mathbb{Z}} |\langle \mathbf{1}_{A_i}, \tau_{k_1}\varphi \rangle|^2$$

Now, $A' |B_i| \leq \sum_{k_2\in\mathbb{Z}} |\langle \mathbb{1}_{B_i}, \tau_{k_2}\varphi' \rangle|^2 \leq B' |B_i|$ and $A |A_i| \leq \sum_{k_1\in\mathbb{Z}} |\langle \mathbb{1}_{B_i}, \tau_{k_2}\varphi \rangle|^2 \leq B |A_i|$; hence,

$$AA' \sum_{i=1}^{n} |\alpha_i|^2 |A_i| |B_i| \leq \sum_{i=1}^{n} |\alpha_i|^2 \left[\sum_{k_2\in\mathbb{Z}} |\langle \mathbb{1}_{B_i}, \tau_{k_2}\varphi' \rangle|^2 \sum_{k_1\in\mathbb{Z}} |\langle \mathbb{1}_{A_i}, \tau_{k_1}\varphi \rangle|^2 \right] \leq BB' \sum_{i=1}^{n} |\alpha_i|^2 |A_i| |B_i|$$

that is,

$$AA' \|f\|^2 \leq \sum_{m\in\mathbb{Z}^2} |\langle f, \tau_m (\varphi \otimes \varphi') \rangle|^2 \leq BB' \|f\|^2 .$$

Hence, the result follows from the fact that these inequalities are satisfied on a dense subset of $L^2(\mathbb{R}^2)$. By induction, we can extend this argument to $\mathbb{R}^d$, $d \geq 2$. □

2.5 The Algorithm

Following Lemma 2.9, we shall construct FMRA wavelets by tensor products in the same way it is done for the classical MRA case. First, assume that (V_j, φ) is an FMRA of $L^2(\mathbb{R})$. Assume the set Γ defined by

$$\Gamma = \left\{ \gamma\in\mathbb{T} : \Phi(2\gamma) = 0,\ \Phi(\gamma) > 0,\ \Phi\left(\gamma + \frac{1}{2}\right) > 0 \right\}$$

has measure zero, and the wavelet ψ is given by

$$\widehat{\psi}(2\gamma) = H_1(\gamma)\,\widehat{\varphi}(\gamma),$$

where $H_1(\gamma)$ is defined by

$$H_1(\gamma) = \begin{cases} e^{-2\pi i\gamma}\overline{H_0}\left(\gamma + \frac{1}{2}\right)\Phi\left(\gamma + \frac{1}{2}\right) & \text{if} \quad \gamma \in \Delta_2, \\ 1 & \text{if} \quad \gamma \in \Delta_3 \text{ and } H_0(\gamma) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The sets Δ_j, $j = 1, 2, 3, 4$, are

$$\Delta_1 = \left\{\gamma{\in}\mathbb{T} : \Phi\left(\gamma\right) = 0,\ \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_2 = \left\{\gamma{\in}\mathbb{T} : \Phi\left(\gamma\right) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$
$$\Delta_3 = \left\{\gamma{\in}\mathbb{T} : \Phi\left(\gamma\right) > 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) = 0\right\},$$
$$\Delta_4 = \left\{\gamma{\in}\mathbb{T} : \Phi\left(\gamma\right) = 0 \text{ and } \Phi\left(\gamma + \frac{1}{2}\right) > 0\right\},$$

and they form a partition of $\mathbb{T}$, see [11, 18]. We now define the d-fold tensor product $V_1^{(d)} = \bigotimes_d V_1$ associated with the given FMRA of $L^2\left(\mathbb{R}\right)$. Recall that

$$V_1 = V_0 \bigoplus W_0 \subset L^2\left(\mathbb{R}\right).$$

Hence, if we set $X_0 = V_0$, $X_1 = W_0$, and

$$X_\nu = X_{v_1} \bigotimes X_{v_2} \bigotimes ... \bigotimes X_{\nu_1},$$

for $\nu = (\nu_1, ..., \nu_d) \in \{0, 1\}^d$, we have

$$\bigotimes_d V_1 = \bigoplus_{\nu\in\{0,1\}^d} X_\nu.$$

With the convention that $\psi_0 = \varphi$, denote

$$\psi_\nu\left(x_1, ..., x_d\right) = \psi_{\nu_1}\left(x_1\right)\psi_{\nu_2}\left(x_2\right)...\psi_{\nu_d}\left(x_d\right).$$

By Lemma 2.9, $\{\tau_k\psi_\nu\}_{k\in\mathbb{Z}^d}$ is a frame for X_ν, so that $\{\tau_k\psi_\nu\}_{k\in\mathbb{Z}^d,\nu\in\{0,1\}^d}$ is a frame for $V_1^{(d)}$.

In order to write these wavelets as a Mallat-Meyer algorithm, we compute $\Phi^{(d)}$

obtained by the periodization of the square of the modulus of $\widehat{\varphi}(\gamma_1)\ldots\widehat{\varphi}(\gamma_d)$:

$$\begin{aligned}
\Phi^{(d)}(\gamma_1,\ldots,\gamma_d) &= \sum_{(k_1,\ldots,k_d)\in\mathbb{Z}^d} \left|\widehat{\varphi}(\gamma_1+k_1)\ldots\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\
&= \sum_{(k_1,\ldots,k_d)\in\mathbb{Z}^d} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\ldots\left|\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\
&= \sum_{k_1\in\mathbb{Z}}\ldots\sum_{k_d\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\ldots\left|\widehat{\varphi}(\gamma_d+k_d)\right|^2 \\
&= \left[\sum_{k_1\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_1+k_1)\right|^2\right]\ldots\left[\sum_{k_d\in\mathbb{Z}} \left|\widehat{\varphi}(\gamma_d+k_d)\right|^2\right] \\
&= \Phi(\gamma_1)\ldots\Phi(\gamma_d).
\end{aligned}$$

We then define Φ_ν by $\Phi_\nu(\gamma_1,\ldots,\gamma_d)=\Phi_{\nu_1}(\gamma_1)\ldots\Phi_{\nu_d}(\gamma_d)$, where

$$\Phi_{\nu_j}(\gamma_j)=\begin{cases} 1 & \text{if} \quad \nu_j=0 \\ \Phi(\gamma_j) & \text{if} \quad \nu_j=1, \end{cases}$$

and H^ν by $H^\nu(\gamma_1,\ldots,\gamma_d)=H^{\nu_1}(\gamma_1)\ldots H^{\nu_d}(\gamma_d)$, where

$$H^{\nu_j}(\gamma_j)=\begin{cases} \overline{H_0}(\gamma_j) & \text{if} \quad \nu_j=1 \\ H_0(\gamma_j) & \text{if} \quad \nu_j=0. \end{cases}$$

The sets $\Delta_j^{(d)}$, $j=1,2,3,4$, for the tensor product, take the form

$$\begin{aligned}
\Delta_1^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \forall u\in\{0,1\}^d,\ \Phi_u\left(\gamma+\frac{u}{2}\right)=0\right\}, \\
\Delta_2^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)>0 \text{ and } \exists u\in\{0,1\}^d\setminus\{0\},\ \Phi_u\left(\gamma+\frac{u}{2}\right)>0\right\}, \\
\Delta_3^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)>0 \text{ and } \Phi\left(\gamma_1+\frac{1}{2}\right)=\ldots=\Phi\left(\gamma_d+\frac{1}{2}\right)=0\right\}, \\
\Delta_4^{(d)} &= \left\{\gamma\in\mathbb{T}^d : \Phi^{(d)}(\gamma)=0 \text{ and } \exists u\in\{0,1\}^d\setminus\{0\} \text{ with } \Phi_u\left(\gamma+\frac{u}{2}\right)>0\right\}.
\end{aligned}$$

Moreover, the filters $H_\nu(\gamma_1,\ldots,\gamma_d)=H_{\nu_1}(\gamma_1)\ldots H_{\nu_d}(\gamma_d)$, for $\nu\neq(0,\ldots,0)$ are given by

$$H_\nu(\gamma) = \begin{cases} e^{-2\pi i\gamma\cdot\nu} H^\nu\left(\gamma+\frac{\nu}{2}\right)\Phi_\nu\left(\gamma+\frac{\nu}{2}\right) & \text{if} \quad \gamma\in\Delta_2, \\ \prod\limits_{\nu_p=0} H_0(\gamma_p) & \text{if} \quad \gamma\in\Delta_3,\ H^{\nu_j}(\gamma_j)=0, \nu_j=1, \\ 0 & \text{otherwise,} \end{cases}$$

where, by convention, the product $\prod_{\nu_p=0} H_0(\gamma_p) = 1$ in the case none of the ν_p is 0, i.e., $\nu = (1, ..., 1)$. H_0 is the low pass filter on $\mathbb{T}$. Thus, the FMRA wavelets, ψ_ν, $\nu \neq (0, ..., 0)$, defined above can now be formulated as in the Mallat-Meyer algorithm as follows:

$$\widehat{\psi_\nu}(2\gamma) = H_\nu(\gamma)\,\widehat{\varphi^{(d)}}(\gamma),$$

where $\varphi^{(d)}(x_1, ..., x_d) = \varphi(x_1)...\varphi(x_d)$.

Remark 2.10. *This argument can be generalized to the case* $(V_j(1), \varphi_1), ..., (V_j(d), \varphi_d)$ *of d distinct FMRAs of $L^2(\mathbb{R})$. The idea is the same, but beginning with*

$$\Phi^{(d)}(\gamma_1, ..., \gamma_d) = \Phi_1(\gamma_1)...\Phi_d(\gamma_d).$$

However, the notation becomes cumbersome. If one of the sets

$$\Gamma_j = \left\{\gamma\in\mathbb{T} : \Phi_j(2\gamma)=0,\ \Phi_j(\gamma)>0,\ \Phi_j\left(\gamma+\frac{1}{2}\right)>0\right\}$$

has measure zero, then Theorem 2.7 says that a set of FMRA wavelets for $L^2\left(\mathbb{R}^d\right)$, with cardinality less than or equal 2^d-1, exists. Moreover, if Γ_k, $k\neq j$, has positive measure, then it is impossible to construct via tensor products a set of wavelet generators with cardinality less than or equal 2^d-1, since we need at least two wavelet generators for $W_0(k) = V_0(k)^\perp \cap V_1(k)$ by Theorem 2.7.

Example 2.11. *Let* $\widehat{\varphi}(\gamma) = \mathbb{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma)$*. Then*

$$H_0(\gamma) = \mathbb{1}_{\left[-\frac{1}{8},\frac{1}{8}\right)}(\gamma), \ \Phi(\gamma) = \mathbb{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma), \text{ and } H_1(\gamma) = \mathbb{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma) - \mathbb{1}_{\left[-\frac{1}{8},\frac{1}{8}\right)}(\gamma).$$

This gives

$$\widehat{\psi} = \mathbb{1}_{\left[-\frac{1}{2},\frac{1}{2}\right)}(\gamma) - \mathbb{1}_{\left[-\frac{1}{4},\frac{1}{4}\right)}(\gamma).$$

Example 2.12. *Example 1 can be extended to any dimension. Let* $Q = \left[-\frac{1}{4},\frac{1}{4}\right)^d$*, the cube of volume* $\left(\frac{1}{2}\right)^d$ *center at the origin, and define*

$$\widehat{\varphi}(\gamma) = \mathbb{1}_Q(\gamma).$$

Then

$$H_0(\gamma) = \mathbb{1}_{\frac{1}{2}Q}(\gamma), \ H_1(\gamma) = \mathbb{1}_Q(\gamma) - \mathbb{1}_{\frac{1}{2}Q}(\gamma),$$

and

$$\widehat{\psi}(\gamma) = \mathbb{1}_{2Q}(\gamma) - \mathbb{1}_Q(\gamma).$$

Note that in this case Δ_2 *has measure zero. In particular, if the support of* $\widehat{\varphi}$ *lies inside the cube centered at the origin and with volume* $\left(\frac{1}{2}\right)^d$*, the algorithm produces one wavelet. This is because the translations by the vectors* $\frac{u}{2}$ *of this cube intersect the others by either a vertex, an edge, or a face. All of these latter sets have measure zero. For perspective, in the classical MRA theory the required number of wavelet functions is* $2^d - 1$*.*

Chapter 3

General Multiresolution Analysis Structures

3.1 Overview

A collection of closed subspaces $\{V_j\}_{j=-\infty}^{\infty}$ of $L^2\left(\mathbb{R}^d\right)$ is called a *Generalized Multiresolution Analysis* (GMRA) if the following is satisfied:

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(b) $D\left(V_j\right) = V_{j+1}$,

(c) $\overline{\cup_j V_j} = L^2(\mathbb{R}^d)$ and $\cap_j V_j = \{0\}$,

(d) V_0 is invariant under the lattice $\mathbb{Z}^d$, i.e. $\tau_n V_0 \subset V_0$ for all $n \in \mathbb{Z}^d$, or V_0 is *shift invariant.*

If instead of (d) we have that,

(d') There exist a sequence $\{\varphi_i : i \in \mathbb{I}\}$ of functions, possibly finite such that $\{\varphi_i\left(x-k\right) : k \in \mathbb{Z}^d, i \in \mathbb{I}\}$ is a frame for V_0.

then, $\{V_j\}_{j=-\infty}^{\infty}$ is called a *Generalized Frame Multiresolution Analysis* (GFMRA).

For the dyadic case, there is no difference between these definitions, since every space invariant by the unitary operators τ_n, $n \in \mathbb{Z}^d$ contains a set of functions $\{f_i : i \in \mathbb{I}\}$ for which $\{f_i\left(x-k\right) : k \in \mathbb{Z}^d, i \in \mathbb{I}\}$ is a frame, and any space generated by the translates of functions $\{f_i : i \in \mathbb{I}\}$ is automatically invariant under the lattice $\mathbb{Z}^d$.

Possibly the main tool in the theory of generalized multiresolution analysis is the dimension function defined in the introduction. It is well known that the dimension function and the multiplicity function coincides for the dyadic case [43]. Hence, its importance, as shown in [2, 4], relies in the fact that the dimension function gives as much information as a scaling set of functions. In fact, the essential supremum of the dimension function is the minimum number of scaling functions needed to generate the core subspace V_0. Also, this function can be used to construct a generalized frame multiresolution analysis for a given orthonormal wavelet [37].

The dimension function can be obtained from the Spectral Theorem [28], and these approach reveals a beautiful connection between the theory of wavelets and abstract harmonic analysis. A well known structural theorem about the core subspace V_0 is revealed by this approach [14].

The first part of this chapter is devoted to the proof of the Spectral Theorem and how dimension functions characterize spectral measures. We shall use the approach of Henry Helson [28]. The discusssion and presentation is informal and intuitive. The standard way to present Spectral theory is by using Banach Algebras techniques, and Henry Helson accomplish his goal avoiding such technicalities. The presentation is beautiful and the reading is relatively easy.

Then, using the the dimension function, we construct an optimal scaling set of functions. This is theorem 3.14. After that, we give an explicit formula for an important unitary isomorphism mentioned in [4, 19]. This is theorem 3.17.

Theorem 3.17 establishes a deeper connection between GMRAs and GFMRAs in their apparently distincts approaches. In general, it was claimed in [4, 19] that an explicit formula for this map cannot be given.

We finish with a new proof of a structural theorem for shift invariant subspaces [14]. This result is theorem 3.19 and can be seen as a corollary of the existence of the unitary isomorphism mentioned above.

3.2 The Spectral Theorem

The finite dimensional spectral theorem says that every normal operator T can be diagonalized, i.e., there is a basis of orthonormal eigenvectors for T. This result, as stated, is false in the infinite dimensional case. We shall reformulate this result so that it generalizes to linear operators on general separable Hilbert spaces.

Let T be a normal operator on a finite dimensional Hilbert space. If $\sigma(T)$ is the spectrum of T, and $\lambda \in \sigma(T)$, denote by P_λ as the orthogonal projection onto the eigenspace of λ, that is, the eigenspace of λ the set of vectors v satisfying

$$Tv = \lambda v.$$

It follows from the finite dimensional spectral theorem that,

$$\begin{aligned} P_{\lambda_1} P_{\lambda_2} &= 0 \text{ if } \lambda_1 \neq \lambda_2, \\ I &= \sum_{\lambda \in \sigma(T)} P_\lambda, \\ T &= \sum_{\lambda \in \sigma(T)} \lambda P_\lambda. \end{aligned}$$

The third equation is the *spectral decomposition* of T. If T is unitary, it can be shown that $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$ and any $\lambda \in \sigma(T)$ has the form $\lambda = e^{2\pi i x_\lambda}$, for some unique $x_\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. The spectral decomposition becomes $T = \sum_{\lambda \in \sigma(T)} e^{2\pi i x_\lambda} P_\lambda$. This expression is also valid for a cyclic group of unitary operators, i.e., operators of the form $U = T^n$ for some integer n. Specifically, we have,

$$\forall n \in \mathbb{Z},\ T^n = \sum_{\lambda \in \sigma(T)} e^{2\pi i n x_\lambda} P_\lambda.$$

This is the form of the spectral theorem that we want to extend to arbitrary separable Hilbert spaces. In the infinite dimensional setting the sum is replaced by an integral and the projections are replaced by a projection-valued measure.

Our goal is to decompose all unitary operators as a "continuous sum" or an "integral" using a projection-valued measure. When we have a family of unitary operators, we decompose each member of the family using the same projection-valued measure, but each member is decomposed with a different integrand. Moreover, the spectral theorem asserts that there is a one-to-one correspondence between projection-valued measures and unitary groups of operators.

Definition 3.1. Let $(\Omega, \mathbf{B})$ be a Borel space, that is, a topological space with a sigma algebra generated by the open sets. Let H a separable complex Hilbert space, and denote by $\mathcal{L}(H)$ as the space of continuous linear operators on H. A *projection-valued measure*, sometimes called a *spectral measure* or a *resolution of the identity*, is a map $P : \mathbf{B} \to \mathcal{L}(H)$ such that:

- For all $E \in \mathbf{B}$, $P(E)$ is an orthogonal projection,

- $P(\emptyset) = 0$ and $P(\Omega) = I_H$, the identity map in H.

- $P(E \cap F) = P(E)\,P(F)$, where $P(E)\,P(F)$ denotes composition,

- $P(\cup E_j) = \sum P(E_j)$ for all countable families $\{E_j\}$ of disjoint Borel sets.

Example 3.2. *Let E be a Borel set on $\mathbb{R}^d$, $H = L^2(\mathbb{R}^d)$, and define, for all $f \in H$, $P(E)\,f = \mathbb{1}_E f$. Then P is a projection-valued measure. It is called the* canonical projection-valued measure.

For $u, v \in H$ and $E \in B$, the map $E \mapsto \langle P(E)\,u, v\rangle$ defines a scalar bounded complex measure. In particular, the previous map is a positive bounded measure when $u = v$.

Theorem 3.3. *Every projection-valued measure, up to unitary equivalence, is completely determined by a measure class and a multiplicity function defined a.e. with respect to this class.*

In order to prove this theorem, we have to define the terminology needed and state several well known results. The term measure class and multiplicity function are explained later in the next discussion.

Let H be a Hilbert space, $(\Omega, \mathbf{B})$ a Borel space and define, for $1 \leq p < \infty$, $L^p(\Omega, \mu, H)$ to be the space of all weakly measurable vector functions $F : \Omega \to H$ (that is, $\forall u \in H$, $f_u(x) = \langle F(x), u\rangle$ is measurable) such that

$$\|F\|_{L^p(\Omega,\mu,H)} = \left[\int \|F(x)\|_H^p \, d\mu(x)\right]^{\frac{1}{p}} < \infty.$$

Note that $L^p(\Omega, \mu, H)$ is a Banach space. $L^\infty(\Omega, \mu, H)$ is the space of all bounded vector functions, i.e.,

$$\|F\|_{L^\infty(\Omega,\mu,H)} = \operatorname*{ess\,sup}_{x\in\Omega} \|F(x)\|_H < \infty.$$

In the case $p = 2$, for any F, G in $L^2(\Omega, \mu, H)$, we can define an inner product

$$\langle F, G\rangle_{L^2(\Omega,\mu,H)} = \int \langle F(x), G(x)\rangle_H \, d\mu(x).$$

If $\Omega = \mathbb{T}^d$, and $p = 1$, the relation

$$F(x) \sim \sum_{n\in\mathbf{Z}^d} c_n(F)\, e^{2\pi i n\cdot x}, c_n(F) \in H$$

implies that, for all $v \in H$, the Fourier series for $\langle F(x), v\rangle$ is given by

$$\sum_{n\in\mathbf{Z}^d} \langle c_n(F), v\rangle\, e^{2\pi i n\cdot x}$$

where

$$\langle c_n(F), v\rangle = \int \langle F(x), v\rangle\, e^{-2\pi i n\cdot x} dx.$$

The right side of the last equality is bounded by $\|F\|_{L^1(\Omega,\mu,H)} \|v\|_H$, so $\langle c_n(F), \cdot\rangle$ defines a bounded conjugate-linear functional on H. This unique element $c_n(F)$ in H is what we call the n-th vector coefficient for the vector Fourier series of F. A *range function* J is a mapping from Ω to the collection of closed subspaces of H. Let $P_{J(x)}$ be the orthogonal projection onto $J(x)$. We say that J is *measurable* if $\forall u \in H$, $P_{J(x)}(u)$ is a weakly measurable operator function.

For each measurable range function J, define M_J to be the set of all vector functions $F \in L^2(\Omega, \mu, H)$ such that $F(x) \in J(x)$ for almost every $x \in \Omega$. Then, it follows that M_J is a closed subspace of $L^2(\Omega, \mu, H)$. Moreover, M_J behaves like an "ideal ring" in the sense that $hM_J \subset M_J$ for every bounded measurable scalar function $h \in L^\infty(\Omega)$. This fact characterizes the subspaces M_J.

Theorem 3.4. *Let M be a closed subspace of $L^2(\mathbb{T}^d, dx, l^2(\mathbb{Z}^d))$ such that M is* *invariant under multiplication by bounded measurable scalar functions, that is,*

$$\forall h \in L^{\infty}\left(\mathbb{T}^{d}\right), hM \subseteq M. \tag{3.1}$$

Then, $M = M_J$ *for some measurable range function* J*.*

For the proof of this theorem, see [28]. The following result motivates the definition of the dimension function, which is used with the spectral theorem in the implementation of GMRAs.

Theorem 3.5. *Let* J *be a measurable range function. Then there is a sequence* $\{G_k\}_{k\in\mathbb{N}} \subset L^2(\Omega, \mu, H)$ *such that* $\{G_k(x)\}_{k\in\mathbb{N}}$ *forms an orthonormal basis for* $J(x)$ *at the points where* $J(x)$ *is infinite dimensional,* $\{G_k(x)\}_{k\leq n}$ *is an orthonormal basis for* $J(x)$ *when this space has dimension* n*, and* $G_k(x)$ *vanishes if* $k > n$*.* $D_J(x) = \dim J(x)$ *is called the* dimension function *of* J*.*

The proof of this result can be found in [28].

Example 3.6. *Assume that* H *is infinite dimensional and let* $\{e_1, e_2, ...\}$ *be an orthonormal basis for* H*. Let* D *be any measurable function taking values in* $\{0, 1, 2, ..., \infty\}$*. Define* $J(x)$ *to be subspace of* H *generated by* $\{e_1, ..., e_n\}$*, if* $D(x) = n < \infty$ *and* $J(x) = H$ *if* $D(x) = \infty$*. Then, by the measurability of* D*,* J *is a measurable range function. We call* J *the* standard measurable range function *with dimension function* D*.*

Two range functions J and K are *unitarily equivalent* if there are unitary isomorphisms, depending on x, $U(x) : M_{J(x)} \longrightarrow M_{K(x)}$ such that they commute with all bounded measurable scalar functions. The following two theorems give a criteria to tell if two range functions are equivalent. We omit the proofs of these results. They can be found in [28].

Theorem 3.7. *Let μ and μ' be two mutually absolutely continuous $\sigma-$ finite measures on Ω. $M_J \subset L^2(\Omega, \mu, H)$ and $M_K \subset L^2(\Omega, \mu', H)$ are equivalent if and only if $D_J = D_K$ a.e.*

Theorem 3.8. *If μ and μ' are not mutually absolutely continuous, then J and K cannot be equivalent.*

Given a spectral measure P defined on a Borel space $(\Omega, \mathbf{B})$. For fixed $v, u \in H$,

$$\forall E \in \mathbf{B},\, m_{u,v}(E) = \langle P(E)\, u, v\rangle$$

defines a complex finite measure on $\mathbf{B}$. Hence, if h is a bounded measurable function on Ω, the integral

$$\int_{\Omega} h(x)\, dm_{u,v}(x)$$

is finite for every $u, v \in H$. Moreover, the following is satisfied:

Theorem 3.9. *For every bounded measurable function h on Ω, there is a unique normal operator T such that*

$$\langle Tu, v\rangle = \int_{\Omega} h(x)\, dm_{u,v}(x) \tag{3.2}$$

for every $v, u \in H$, and $\|T\| \leq \|h\|_{\infty}$. The map κ that sends h to T is an algebra $*$-homomorphism, i.e., $\kappa(\overline{h}) = T^*$ where $\overline{h}$ denotes the complex conjugate of h. It follows that if h is real, then T is self-adjoint. Also, $\kappa(\mathbb{1}_{\Omega}) = I_H$.

We abbreviate (3.2) by $T = \int_{\Omega} h(x)\, dP(x)$.

Proof. For a bounded function h, set

$$\forall v, u \in H,\ F(u,v) = \int_\Omega h(x)\, dm_{u,v}(x).$$

Then F is a conjugate linear functional in the second variable. Also,

$$|F(u,v)| = \left| \int_\Omega h(x)\, dm_{u,v}(x) \right| \leq \|h\|_\infty |m_{u,v}|(\Omega).$$

$|m_{u,v}|(\Omega)$ can be proved to be less than or equal $\|u\| \|v\|$, by the definition of $m_{u,v}$. Hence, $\|F(u,\cdot)\| \leq \|u\| \|h\|_\infty$. Define Tu to be the unique element in H for which

$$\forall v \in H,\ F(u,v) = \langle Tu, v \rangle.$$

This is exactly (3.2). By above, the norm of T is at most $\|h\|_\infty$.

The linearity of κ, $\kappa\left(\overline{h}\right) = T^*$, and the fact that if h is real implies that T is self-adjoint are trivial to show. Note that $\kappa(\mathbb{1}_E) = P(E)$. To prove that κ is multiplicative, we compute on characteristic functions:

$$\kappa(\mathbb{1}_E \mathbb{1}_F) = \kappa(\mathbb{1}_{E\cap F}) = P(E \cap F) = P(E)\, P(F) = \kappa(\mathbb{1}_E)\, \kappa(\mathbb{1}_F).$$

By the linearity of κ, it follows that κ is multiplicative on simple functions and κ is a contraction. Since every bounded function is the uniform limit of simple functions, it follows that κ preserves multiplication on bounded measurable functions. Now, the range of κ is a commutative subalgebra of linear operators on H, since bounded functions commute. Also, every h can be written as $h = h_{\mathrm{Re}} + ih_{\mathrm{Im}}$ where h_{Re} and h_{Im} are real-valued bounded measurable functions. Setting $\kappa(h_{\mathrm{Re}}) = T_{\mathrm{Re}}$ and $\kappa(h_{\mathrm{Im}}) = T_{\mathrm{Im}}$, we have that both T_{Re} and T_{Im} are self adjoint commuting operators and $T = T_{\mathrm{Re}} + iT_{\mathrm{Im}}$. Hence, T is normal. $\square$

The next theorem states that any spectral measure can be uniquely determined, up to unitary equivalence, by a measure class and a particular dimension function that we shall call the **multiplicity function**, i.e., we are ready to prove theorem 3.3.

Theorem 3.10. *Every spectral measure on* $(\Omega, \mathbf{B})$ *in* H *is unitarily equivalent to the standard spectral measure associated with some finite measure* m *on* $\mathbf{B}$ *and some dimension function* D.

Proof. For each $u \in H$, let m_u be defined for all $E \in \mathbf{B}$ by

$$m_u(E) = \langle P(E)\, u, u\rangle = \|P(E)\, u\|^2 .$$

From the definition of the spectral measure we obtain that $m_u(\Omega - E) = 0$ if and only if $P(E)\, u = u$. Denote by H_u the smallest closed subspace of H containing $P(E)\, u$ for every $E \in \mathbf{B}$. Define $\Lambda(P(E)\, u) = \mathbb{1}_E$. This map can be linearly extended to all finite linear combinations of vectors $P(E)\, u$ for $E \in \mathbf{B}$. For $E, F \in \mathbf{B}$,

$$\begin{aligned} \langle P(E)\, u, P(F)\, u\rangle &= \langle P(E)\, P(F)\, u, u\rangle = \langle P(E \cap F)\, u, u\rangle \\ &= m_u(E \cap F) = \int \mathbb{1}_{E\cap F}\, dm_u = \langle \mathbb{1}_E, \mathbb{1}_F\rangle \\ &= \langle \Lambda(P(E)\, u), \Lambda(P(F)\, u)\rangle . \end{aligned}$$

The above expression shows that the map Λ can be extended to all of H_u and this extension is an isometry of H_u onto $L^2(\Omega, m_u)$. Moreover,

$$\Lambda(P(E)\, v) = \mathbb{1}_E \Lambda(v)$$

holds for all elements v of the form $v = P(F)\, u$. The measures m_u and the spaces H_u for $u \in H$ satisfy:

(a) If $m_u \perp m_v$, then $H_u \perp H_v$,

(b) there is an element $u_0 \in H$ such that $m_u \ll m_{u_0}$ for all $u \in H$.

(a) follows from the Radon-Nikodym theorem and the fact that $m_u(\Omega - E) = 0$ if and only if $P(E)u = u$. For (b) let $\mathcal{F}$ be the family of subsets Q of H such that all elements of Q have norm 1 and $H_u \perp H_v$ for distinct elements u and v. Singletons $\{u\}$ with norm 1 belongs to $\mathcal{F}$, and $\mathcal{F}$ is ordered by inclusion. Zorn's lemma implies the existence of a maximal element Q_0. This element Q_0 is countable since H is separable. List the elements of Q_0 by $v_1, v_2, \ldots$. We claim that

$$H = \bigoplus_{i=1}^{\infty} H_{v_i}.$$

Suppose not. Then we can find a unit norm vector w orthogonal to all the spaces H_{v_i}. Also, $P(E) H_{v_i} \subset H_{v_i}$ for all i and for all E. Hence $P(E) w$ is orthogonal to all H_{v_i} for all E. This contradicts the maximality of Q_0. Define

$$u_0 = \sum_{i=1}^{\infty} \frac{v_i}{i^2}.$$

Then

$$m_{u_0}(E) = \sum_{i=1}^{\infty} \frac{m_{v_i}(E)}{i^4}$$

and

$$m_{v_i} \ll m_{u_0}, \ \forall i.$$

By the Radon-Nikodym theorem, there is a natural isometry T from $L^2(\Omega, m_{v_i})$ into $L^2(\Omega, m_{u_0})$ given by $Tf = f\sqrt{dm_{v_i}/dm_{u_0}}$ that commutes with multiplication of bounded functions. Therefore,

$$T\Lambda(P(E)v) = T(\mathbb{1}_E \Lambda(v)) = \mathbb{1}_E T(\Lambda(v)) \tag{3.3}$$

for all v in H_{v_i}, $i \in \mathbb{Z}$. So far we have defined an isometric mapping commuting with multiplication of bounded functions on each H_{v_i} onto the subspaces of $L^2(\Omega, m_{v_i})$ consisting of functions supported on the support of dm_{v_i}/dm_{u_0}. If we denote by Ω_i the support of dm_{v_i}/dm_{u_0}, then this isometrically maps each H_{v_i} onto $L^2(\Omega_i, m_{v_i})$. Thus elements u of H are associated to sequences $(f_1, f_2, ...)$ with $f_i \in L^2(\Omega_i, m_{v_i})$ and

$$\|u\|^2 = \sum_{i=1}^{\infty} \int_{\Omega_i} |f_i|^2 \, dm_{v_i}.$$

From (3.3) we see that in this representation, $P(E)$ is just multiplication by $\mathbb{1}_E$ on each component. Let K be a new separable Hilbert space with orthonormal basis $\{e_1, e_2, ...\}$. To $(f_1, f_2, ...)$ we associate

$$F = \sum_{i=1}^{\infty} f_i e_i.$$

Such space of functions is invariant under multiplication by scalar functions, that is, the product of a scalar bounded measurable function by any F defined as above is also in this space. Hence, this space is of the form M_J for some range function J and the given spectral measure is unitarily equivalent to the spectral measure given by multiplication of $\mathbb{1}_E$ in M_J, contained in $L^2(\Omega, m_{u_0}, K)$. J can be taken to be a standard range function. Up to unitary equivalence, a spectral measure is determined by the measure class $\{m_{v_i}\}_i$ and a dimension function defined a.e. with respect to this measure class. $\square$

Remark 3.11. *The dimension function of J given at the end of theorem 3.10 is called the multiplicity function of the spectral measure P.*

Now we are ready to state the spectral theorem.

Theorem 3.12 (Stone)**.** *Let $\{U_x\}_{x\in\mathbb{R}^d}$ be a weakly continuous parametrized group of unitary operators defined on a Hilbert space H. Then there is a unique spectral measure P on $\mathbb{R}^d$ such that*

$$U_x = \int_{\widehat{\mathbb{R}^d}} e^{2\pi i x\cdot\gamma} dP(\gamma).$$

Theorem 3.12 still true if the parameter group is an arbitrary locally compact abelian group, that is, the theorem is true for unitary representations of arbitrary locally compact abelian groups.

If τ denotes the induced unitary representation of the group $\mathbb{Z}^d$ given by its action on V_0, then the Spectral Theorem says that τ has a unique continuous spectral decomposition, and can be recovered by the Fourier transform

$$\tau_n = \int_{\mathbb{T}^d} e^{2\pi i\langle n,\gamma\rangle} dP(\gamma),$$

of the spectral measure P defined on $\mathbb{T}^d$, the dual of $\mathbb{Z}^d$, and this spectral measure is in turn uniquely determined by a multiplicity function $m : \mathbb{T}^d \longrightarrow \{\infty, 0, 1, ...\}$ and a measure class on $\mathbb{T}^d$, for details, see [28]. In [2], it was shown that this measure class associated with a GMRA in $\mathbb{R}^d$ must be absolutely continuous with respect to the Lebesgue measure. Hence, the multiplicity function, which basically counts the number of times each character γ in $\mathbb{T}^d$ occurs in τ, completely characterizes τ. If the GMRA is just an MRA, translates of the scaling function form an ONB for V_0, and, hence, τ is equivalent to the regular representation of $\mathbb{Z}^d$, which acts by translation on $l^2(\mathbb{Z}^d)$. The regular representation contains every character exactly once so that $m \equiv 1$ in this case. It is also known that the dimension function and the multiplicity function are the same in our setting, that is dilation by 2 and translation by the group $\mathbb{Z}^d$, and moreover, the condition $m \equiv 1$ characterizes all MRA wavelets [42].

3.3 The Construction of the Scaling Set

Recall that $L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)$ is the Hilbert space of all square integrable l^2- valued functions over the d-dimensional torus $\mathbb{T}^d$ with the following inner product:

$$\forall F, G \in L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right), \ \ \langle F, G\rangle_{L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)} = \int_{\mathbb{T}^d} \langle F(\gamma), G(\gamma)\rangle_{l^2} \, d\gamma. \tag{3.4}$$

The operator $\mathcal{X}$ can now be seen as a map $\mathcal{X} : L^2(\mathbb{R}^d) \longrightarrow L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)$, where, again

$$\mathcal{X}(f)(\gamma) = \left\{\widehat{f}(\gamma + k)\right\}_{k \in \mathbb{Z}^d}. \tag{3.5}$$

This operator $\mathcal{X}$ is an isometry:

$$\begin{aligned}
\|\mathcal{X}(f)\|^2_{L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)} &= \int_{\mathbb{T}^d} \langle \mathcal{X}(f)(\gamma), \mathcal{X}(f)(\gamma)\rangle_{l^2} \, d\gamma \qquad (3.6)\\
&= \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} \left|\widehat{f}(\gamma + k)\right|^2 d\gamma = \int_{[0,1)^d} \sum_{k \in \mathbb{Z}^d} \left|\widehat{f}(\gamma + k)\right|^2 d\gamma \\
&= \sum_{k \in \mathbb{Z}^d} \int_{[0,1)^d + k} \left|\widehat{f}(\gamma)\right|^2 d\gamma = \int_{\widehat{\mathbb{R}}^d} \left|\widehat{f}(\gamma)\right|^2 d\gamma \\
&= \int_{\mathbb{R}^d} |f(x)|^2 \, dx = \|f\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

From now on, ψ will denote an orthonormal wavelet for $L^2(\mathbb{R}^d)$. The *dimension function associated to* ψ is given by the following formula:

$$D_\psi(\gamma) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} \left|\widehat{\psi}\left(2^j(\gamma + k)\right)\right|^2.$$

A simple calculation shows that

$$\left\{\widehat{\psi}\left(2^{j}\left(\gamma+k\right)\right)\right\}_{k\in\mathbb{Z}^{d}}=\mathcal{X}\left(2^{\frac{-jd}{2}}D^{-j}\psi\right)\left(\gamma\right).$$

If we set $\Psi_{j}=\mathcal{X}\left(2^{\frac{-jd}{2}}D^{-j}\psi\right)$, then the following facts are true [16]:

1. $D_{\psi}\left(\gamma\right)=\dim\overline{span}\left\{\Psi_{j}\left(\gamma\right):j\geq 1\right\}=\sum_{j=1}^{\infty}\left\|\Psi_{j}\left(\gamma\right)\right\|_{l^{2}}^{2}$;

2. $\int_{\mathbb{T}^{d}}D_{\psi}\left(\gamma\right)d\gamma=\frac{1}{2^{d}-1}$;

3. $\liminf_{n\longrightarrow\infty}D_{\psi}\left(2^{-n}\gamma\right)\geq 1$;

4. $\sum_{u\in\{0,1\}^{d}}D_{\psi}\left(\gamma+\frac{u}{2}\right)=D_{\psi}\left(2\gamma\right)+1$ (the *consistency equation*);

5. $\sum_{k}\mathbb{1}_{\Delta}\left(\gamma+k\right)\geq D_{\psi}\left(\gamma\right)$ a.e. where $\Delta=\left\{\gamma\in\mathbb{T}^{d}:D_{\psi}\left(2^{-j}\gamma\right)\geq 1,j\in\mathbb{N}\right\}$;

6. D_{ψ} assumes all integer values between its essential supremum and 0;

7. If $\widehat{\psi}$ has compact support, then D_{ψ} is essentially bounded.

Remark 3.13. *Using the reproducing formula and Ausher's geometrical lemma, see [1], we can prove statement (1); in [2],[3] Baggett et.al. proved (4); (2) follows from the fact that ψ is an orthonormal wavelet; Bownik, Rzeszotnik, and Speegle [16] proved (5),(6) and (7). These properties characterize dimension functions. See [16, 3]. Moreover, (7) implies that the set of scaling functions for a GFMRA associated to a bounded wavelet set is finite, in particular for the ones obtained by the neighborhood-mapping construction given in [6, 7]. For more about wavelet sets, see [2, 6, 7, 16].*

Let $J_\psi(\gamma) = \overline{span}\{\Psi_j(\gamma) : j \geq 1\}$. Then J_ψ is a range function. In our particular case, $J_\psi(\gamma)$ is a closed subspace of $l^2(\mathbb{Z}^d)$. Also, $D_\psi(\gamma) = \dim J_\psi(\gamma)$ for almost every γ in $\mathbb{T}^d$. Thus, in light of Theorem 3.3,

$$\forall h \in L^\infty\left(\mathbb{T}^d\right), hM_{J_\psi} \subseteq M_{J_\psi}.$$

Suppose that we can obtain a set $\{\Phi_j\}_{j=1}^\infty \subseteq M_{J_\psi}$ with the following properties:

1. $\{\Phi_j(\gamma)\}_{j=1}^N$ is an ONB for $J_\psi(\gamma)$ a.e. at points γ where $D_\psi(\gamma) = \dim J_\psi(\gamma) = N$ and $\Phi_k(\gamma) = 0$ (the zero sequence) if $k > N$.

2. $\{\Phi_j(\gamma)\}_{j=1}^\infty$ is an ONB for $J_\psi(\gamma)$ a.e. at points γ where $D_\psi(\gamma) = \dim J_\psi(\gamma) = \infty$.

Then every function F in $M_{J_\psi} \subset L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)$ can be written as

$$F = \sum_j f_j \Phi_j, \; f_j \in L^2\left(\mathbb{T}^d\right), \tag{3.7}$$

and hence,

$$\|F\|^2_{L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)} = \sum_j \|f_j\|^2_{L^2\left(\mathbb{T}^d\right)}. \tag{3.8}$$

This set $\{\Phi_j\}_{j=1}^\infty \subseteq M_{J_\psi}$ is what we call a *canonical basis* for M_{J_ψ}, and $\varphi_j = \mathcal{X}^{-1}(\Phi_j)$ will be our scaling set. It is clear that any basis for M_{J_ψ} need not be like this: F and G can be orthogonal (at M_{J_ψ}) without being pointwise orthogonal (at $J_\psi(\gamma) \subseteq l^2\left(\mathbb{Z}^d\right)$) . Now we shall state the main theorem.

Theorem 3.14. *For M_{J_ψ}, a canonical basis $\{\Phi_j\}_{j=1}^\infty$ exists and the functions $\varphi_j = \mathcal{X}^{-1}(\Phi_j)$, $1 \leq j$ are the scaling functions for $\mathcal{X}^{-1}\left(M_{J_\psi}\right) = \bigoplus_{j\geq 1} W_{-j}$, and this set is optimal in the sense that the minimum number of scaling functions required for V_0 is exactly N, the essential supremum of D_ψ.*

Proof. We shall prove the case where D_ψ is essentially bounded. The same argument can be used to prove the general case.

(*a*) Set $\Omega_p = D_\psi^{-1}(\{p\})$, $1 \leq p$ and let $N = \operatorname{ess\,sup} D_\psi$. Since $J_\psi(\gamma) = \overline{span_{j\geq 1}}\{\Psi_j(\gamma)\}$ a.e., the idea is to apply Gram-Schmidt pointwise to a subset of $\{\Psi_j(\gamma) : j \geq 1\}$. For $\gamma \in \Omega_p$, since $\dim \overline{span_j}\{\Psi_j(\gamma)\}$ is p, we can find indexes $j_1(\gamma,p) < j_2(\gamma,p) < ... < j_p(\gamma,p)$ such that $\{\Psi_{j_k}(\gamma)\}_{k=1}^p$ is a basis for $J_\psi(\gamma)$ and $j_1(\gamma,p)$ is the first index for which $\Psi_{j_1}(\gamma) \neq 0$, $j_2(\gamma,p)$ is the first index bigger than $j_1(\gamma,p)$ such that $\Psi_{j_1}(\gamma)$ and $\Psi_{j_2}(\gamma)$ are linearly independent, $j_3(\gamma,p)$ is the first index bigger than $j_2(\gamma,p)$ such that $\Psi_{j_1}(\gamma)$, $\Psi_{j_2}(\gamma)$ and $\Psi_{j_3}(\gamma)$ are linearly independent, and so on. Then, $\{\Psi_{j_k}(\gamma)\}_{k=1}^p$ will be an ordered basis for $J_\psi(\gamma)$. The Gram-Schmidt algorithm produces $\{\Phi_j(\gamma)\}_{j=1}^p$, an orthonormal basis for $J_\psi(\gamma)$ for almost every point γ in Ω_p. We repeat this proccess at every level p. Note that Φ_j vanishes outside $S_j = \cup_{k\geq j}\Omega_k = \left\{\gamma \in \mathbb{T}^d : D_\psi(\gamma) \geq j\right\}$. The following grid illustrates what our orthonormal basis looks like:

$$\begin{array}{cccccccc}
\Omega_N \longrightarrow & \Phi_1 & \Phi_2 & \Phi_3 & ... & \Phi_{N-1} & \Phi_N \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Omega_2 \longrightarrow & \Phi_1 & \Phi_2 & 0 & ... & 0 & 0 \\
\Omega_1 \longrightarrow & \Phi_1 & 0 & 0 & ... & 0 & 0 \\
\Omega_0 \longrightarrow & 0 & 0 & 0 & ... & 0 & 0
\end{array}$$

This set $\{\Phi_j\}_{j=1}^N$ has the desired properties. Moreover, these functions obtained in the way described above are measurable (see [28]).

(*b*) Note that $\bigoplus_{j\geq 1} W_{-j} = \overline{span_{j\geq 1}}\left\{D^{-j}\tau_k\psi : k \in \mathbb{Z}^d\right\}$, hence $\mathcal{X}\left(\bigoplus_{j\geq 1} W_{-j}\right) =$

M_{J_ψ}, because $(D^{-j}\tau_k\psi)^\wedge = e_{-2^jk}D^j\widehat{\psi}$, producing the following equation:

$$\mathcal{X}\left(D^{-j}\tau_k\psi\right) = 2^{\frac{jd}{2}}e_{-2^jk}\Psi_j.$$

Since $2^{\frac{jd}{2}}e_{-2^jk}$ is periodic and bounded, by the characterization of M_{J_ψ}, $\mathcal{X}\left(D^{-j}\tau_k\psi\right) \in M_{J_\psi}$ for every $j \geq 1$. In fact, by the definition of $J_\psi(\gamma)$,

$$\begin{aligned} J_\psi(\gamma) &= \overline{span_{j\geq 1}}\left\{\Psi_j(\gamma)\right\} = \overline{span_{j\geq 1}}\left\{2^{\frac{jd}{2}}e_{-2^jk}(\gamma)\,\Psi_j(\gamma)\right\} \\ &= \overline{span}\left\{\Phi_j(\gamma) : 1 \leq j \leq N\right\}. \end{aligned}$$

(c) The set $\left\{e_n\Phi_j : 1 \leq j \leq N, n \in \mathbb{Z}^d\right\}$ is a Parseval frame: If $T_p : L^2(S_p) \longrightarrow L^2(S_p)\cdot\Phi_p \hookrightarrow L^2\left(\mathbb{T}^d, l^2\left(\mathbb{Z}^d\right)\right)$ is defined by the following equation:

$$T_p u = u\Phi_p \tag{3.9}$$

then it follows that this operator is clearly a co-isometry ($T_p^*(v\Phi_p) = v$ is also an isometry)

$$\begin{aligned} \|T_p u\|^2_{L^2\left(\mathbb{T}^d,l^2\left(\mathbb{Z}^d\right)\right)} &= \int_{\mathbb{T}^d}\langle u(\gamma)\,\Phi_p(\gamma), u(\gamma)\,\Phi_p(\gamma)\rangle_{l^2}\,d\gamma \\ &= \int_{\mathbb{T}^d}|u(\gamma)|^2\langle\Phi_p(\gamma), \Phi_p(\gamma)\rangle_{l^2}\,d\gamma \\ &= \int_{\mathbb{T}^d}|u(\gamma)|^2\,\mathbb{1}_{S_p}(\gamma)\,d\gamma \\ &= \int_{S_p}|u(\gamma)|^2\,d\gamma = \|u\|^2_{L^2(S_p)}. \end{aligned}$$

Now, the set $\left\{e_n\mathbb{1}_{S_p} : n \in \mathbb{Z}^d\right\}$ is a Parseval frame for $L^2(S_p)$, and therefore we have that $\left\{e_n\Phi_p : n \in \mathbb{Z}^d\right\}$ is a Parseval frame for its closed span (see [26]). Using the fact that

$$\langle\Phi_p(\gamma), \Phi_q(\gamma)\rangle_{l^2} = \delta_{pq}\mathbb{1}_{S_p\cap S_q}(\gamma)$$

we obtain that $\left\{e_n\Phi_p : n \in \mathbb{Z}^d, 1 \leq p \leq N\right\}$ is a Parseval frame for M_{J_ψ}. Hence, the functions $\varphi_j = \mathcal{X}^{-1}(\Phi_j), 1 \leq j \leq N$ are the desired scaling functions. These scaling functions have an interesting property: using equation (3.4) and the fact that $\mathcal{X}$ is unitary, we easily obtain that

$$\varphi_j \perp \varphi_l \text{ if } j \neq l. \tag{3.10}$$

This completes the proof. □

Note that the previous construction depends only on the *invariance* of the space V_0: if V is shift-invariant, a set of generators $\{\varphi_l\}$ exist. Then M_J is spanned by $\{\mathcal{X}(\varphi_l)\}$ and we can apply the Gram-Schmidt algorithm to obtain a set of generators with exactly the same properties as the ones above exhibited. The problem is that $\{\varphi_l\}$ might be hard to find just from the invariance assumption of V.

Remark 3.15. *Setting $\Phi_{p,i} = \Phi_i \mathbb{1}_{\Omega_p}$, we obtain the scaling functions provided by Papadakis in [37]. We claim that:*

$$\overline{span}\left\{e_n\Phi_p : n \in \mathbb{Z}^d, 1 \leq p \leq N\right\} = \overline{span}\left\{e_n\Phi_{p,i} : n \in \mathbb{Z}^d, 1 \leq p \leq N, 1 \leq i \leq p\right\}.$$

Any function f in $\overline{span}\left\{e_n\Phi_p : n \in \mathbb{Z}^d, 1 \leq p \leq N\right\}$ is of the form

$$f = \sum_{n\in\mathbf{Z}^d}\sum_{p=1}^{N} h_{p,n} e_n \Phi_p, \;\; h_{p,n} \in L^2(S_p)$$

Hence, $e_n\Phi_{p,i} = e_n\Phi_i\mathbb{1}_{\Omega_p} = \mathbb{1}_{\Omega_p} e_n \Phi_i$ belongs to

$$\overline{span}\left\{e_n\Phi_p : n \in \mathbb{Z}^d, 1 \leq p \leq N\right\}.$$

For the other inclusion observe the following:

$$e_n\Phi_p = \sum_{i=p}^{N} e_n\Phi_{i,p}$$

so that $e_n\Phi_p \in \overline{span}\left\{e_n\Phi_{p,i} : n \in \mathbb{Z}^d, 1 \leq p \leq N, 1 \leq i \leq p\right\}$.

3.4 Connection with Some Other Construction of Scaling Sets

3.4.1 A Scaling Set for a Generalized Multiresolution Analysis

In order to obtain the information that the multiplicity function contains about τ, we form the direct sum

$$\bigoplus_{j=1}^{\infty} L^2\left(S_j\right)$$

where $S_j = \left\{\gamma \in \mathbb{T}^d : m\left(\gamma\right) \geq j\right\}$. Denote by η the representation of $\mathbb{Z}^d$ on $\bigoplus_{j=1}^{\infty} L^2\left(S_j\right)$ given by

$$\eta_k\left(\sum\nolimits_{j\geq 1} f_j\right) = \sum\nolimits_{j\geq 1} e_{-k} f_j,\ \ f_j \in L^2\left(S_j\right).$$

Now, from the information about m, we can obtain a unitary map

$$J : V_0 \longrightarrow \bigoplus_{j=1}^{\infty} L^2\left(S_j\right),$$

such that

$$\forall k \in \mathbb{Z}^d,\ \ J \circ \tau_k = e_{-k} \cdot J,$$

or equivalently,

$$\forall k \in \mathbb{Z}^d,\ \ J \circ \tau_k = \eta_k \circ J.$$

For more details, see [19]. This J map provides the scaling set for V_0.

Proposition 3.16. *Let* $\phi_j = J^{-1}\left(\mathbb{1}_{S_j}\right)$ *or* 0 *if* $S_j = \emptyset$. *Then the set* $\left\{\tau_k\phi_j : k \in \mathbb{Z}^d, j \geq 1\right\}$ *is a Parseval frame for* V_0.

Proof. Let $f \in V_0$, then

$$\begin{aligned}
\textstyle\sum_k \sum_j \left|\langle f, \tau_k \phi_j \rangle\right|^2 &= \textstyle\sum_k \sum_j \left|\langle J(f), J(\tau_k \phi_j)\rangle\right|^2 \\
&= \textstyle\sum_k \sum_j \left|\langle J(f), (J \circ \tau_k)(\phi_j)\rangle\right|^2 \\
&= \textstyle\sum_k \sum_j \left|\langle J(f), (\eta_k \circ J)(\phi_j)\rangle\right|^2 \\
&= \textstyle\sum_k \sum_j \left|\left\langle J(f), e_{-k} \mathbb{1}_{S_j}\right\rangle\right|^2 \\
&= \textstyle\sum_k \sum_j \left|c_k\left(J(f)_j \mathbb{1}_{S_j}\right)\right|^2 \\
&= \|J(f)\|^2 = \|f\|^2,
\end{aligned}$$

where $J(f)_j$ is the j-th component of $J(f)$ in $\bigoplus_i L^2(S_i)$ and $c_k\left(J(f)_j \mathbb{1}_{S_j}\right)$ is the k-th Fourier coefficient of $J(f)_j \mathbb{1}_{S_j}$. $\square$

Usually, it is hard to write down an explicit expression for the map J, but in our setting this map has an explicit form.

Theorem 3.17. *The map* $\left(\bigoplus_{p=1}^N T_p^{-1}\right) \circ \mathcal{X}$, *where* $\mathcal{X}$ *is given by (3.5) and* T_p *is given by (3.9) is a unitary isomorphism from* V_0 *onto* $\bigoplus_{p=1}^{N} L^2(S_p)$, *where* $S_p = \left\{\gamma \in \mathbb{T}^d : m(\gamma) = D_\psi(\gamma) \geq p\right\}$, *and this intertwines translation with modulation. In other words,* $\left(\bigoplus_{p=1}^N T_p^{-1}\right) \circ \mathcal{X}$ *has all the properties of the map* J *given by the previous proposition.*

Proof. We have the following unitary isomorphisms, since the set $\{\Phi_p\}_{p=1}^N$ is a canonical basis for M_{J_ψ} and Φ_p vanishes exactly outside $S_p = \left\{\gamma \in \mathbb{T}^d : D_\psi(\gamma) \geq p\right\}$:

$$V_0 := \bigoplus_{j \geq 1} W_{-j} \overset{\mathcal{X}}{\simeq} M_{J_\psi} \simeq \bigoplus_{p=1}^{N} L^2(S_p) \cdot \Phi_p \overset{\bigoplus T_p^{-1}}{\simeq} \bigoplus_{p=1}^{N} L^2(S_p),$$

and $\mathcal{X}$ intertwines translation with modulation. The map $\bigoplus_{p=1}^{N} T_p^{-1}$ acts like the identity map on $\bigoplus_{p=1}^{N} L^2(S_p)$. Hence, $\left(\bigoplus_{p=1}^{N} T_p^{-1}\right) \circ \mathcal{X}$ intertwines translation with modulation and

$$\left\{\mathcal{X}^{-1}(e_n \Phi_j) : 1 \leq j \leq N, n \in \mathbb{Z}^d\right\} = \left\{\tau_n \varphi_j : 1 \leq j \leq N, n \in \mathbb{Z}^d\right\}$$

is the Parseval frame for $V_0 = \bigoplus_{j \geq 1} W_{-j}$ similar to the one given in the previous proposition. $\square$

Remark 3.18. *There is an important difference between our situation and the situation in [19]: We start without knowing the GMRA, but we know the wavelet, while in [19] they start knowing the GMRA, but without having any information about the wavelet. In the latter situation, it is more difficult to construct the unitary isomorphism J. Also, our map depends on the construction given in Theorem 3.14, which may not be easily constructed as well. But, for wavelet sets with essentially bounded dimension function, at least, the construction given in Theorem 3.14 is relatively easy: (i) the entries of the vectors $\Psi_j(\gamma)$ are either zero or one, (ii) only finitely many entries are one and (iii) only finitely many of these vectors, and no more than N, are nonzero for almost every γ.*

3.4.2 Decomposition of V_0 into Quasi-Regular Spaces Generated by Quasi-Orthogonal Generators

- It is well known that any shift-invariant subspace $V \subset L^2(\mathbb{R}^d)$ contains a (countable or finite) set $\{\varphi_j : j \in \mathbb{N}\}$ such that the translates of these functions form a frame for V. The *spectrum* of V is denoted and defined by

$\sigma(V) := \left\{\gamma \in \mathbb{T}^d : J(\gamma) \neq \{0\}\right\}$ where $J(\gamma) = \overline{span}\{\mathcal{X}(\varphi_j)(\gamma) : j \in \mathbb{N}\}$. In other words,

$$\sigma(V) = \left\{\gamma \in \mathbb{T}^d : \dim J(\gamma) \geq 1\right\}.$$

- Given $\varphi \in L^2(\mathbb{R}^d)$, $S(\varphi) := \overline{span}\left\{\tau_k\varphi : k \in \mathbb{Z}^d\right\}$ is called a *principal shift invariant space* (PSI) generated by φ.

- A *quasi-regular* space S is a shift-invariant space with dimension function $D = c\mathbb{1}_{\sigma(S)}$, for some measurable $\sigma(S) \subset \mathbb{T}^d$ and some constant integer c. It is clear that any PSI is trivially a quasi-regular space: If we set

$$J(\gamma) = span\{\mathcal{X}(\varphi)(\gamma)\},$$

then $D(\gamma) = \dim J(\gamma)$ is either zero, when $\mathcal{X}(\varphi)(\gamma)$ is the zero sequence, or 1, when $\mathcal{X}(\varphi)(\gamma)_k \neq 0$, for some $k \in \mathbb{Z}^d$.

- A *quasi-orthogonal* generator is a $\varphi \in L^2(\mathbb{R}^d)$ such that the translates of φ form a Parseval frame for $S(\varphi)$.

For more details about the theory of shift-invariant subspaces see [14, 23]. Now, we state and give a simpler proof of Theorem 3.3 in [14]:

Theorem 3.19. *Any shift-invariant subspace $V \subset L^2(\mathbb{R}^d)$ can be decomposed as*

$$V = \bigoplus_{j=1}^{\infty} S(\varphi_j),$$

where each element φ_j in the decomposition is a quasi-orthogonal generator, $\sigma(S(\varphi_{j+1})) \subset \sigma(S(\varphi_j))$, and $D_V = \bigoplus_{j=1}^{\infty} D_{S(\varphi_j)}$. In other words, any shift-invariant subspace $V \subset L^2(\mathbb{R}^d)$ contains a set $\{\varphi_j : j \geq 1\}$ with the following properties:

(i) $\left\{\tau_k\varphi_j : j \geq 1, k \in \mathbb{Z}^d\right\}$ *is a Parseval frame for* V*,*

(ii) the dimension function of V *is the sum of the dimension functions of the quasi-orthogonal components, and*

(iii) $\varphi_j \perp \varphi_l$ *if* $j \neq l$ *and* $\forall k, n \in \mathbb{Z}^d$*,* $\tau_k\varphi_j \perp \tau_n\varphi_l$ *if* $j \neq l$*.*

Proof. Let τ be the representation induced by the action of $\mathbb{Z}^d$ on V. By the Spectral Theorem, we can find a unitary map $J : V \longrightarrow \bigoplus_{j=1}^{\infty} L^2(S_j)$, where $S_j = \left\{\gamma \in \mathbb{T}^d : D(\gamma) \geq j\right\}$ (it was shown in [2] that the measure class associated to the representation induced by the action of $\mathbb{Z}^d$ on $L^2(\mathbb{R}^d)$ is absolutely continuous with respect to the Haar measure. Hence, the multiplicity function completely characterizes this representation and therefore any subrepresentation. This implies the existence of the unitary map J given above). Setting $\varphi_j = J^{-1}\left(\mathbb{1}_{S_j}\right)$, and $S(\varphi_j) = J^{-1}\left(L^2(S_j)\right)$, we obtain the desired decomposition. $\square$

Remark 3.20. *By the previous theorem and applying Theorem 3.14,*

$$V_0 = \bigoplus_{j=1}^{N} S(\varphi_j).$$

The dimension functions for the components $S(\varphi_j)$ *are given by* $D_j(\gamma) = \overline{span}\{\Phi_j(\gamma)\} = \mathbb{1}_{S_j}(\gamma)$*. Then,* φ_j*,* $1 \leq j \leq N$ *are the quasi-orthogonal generator. Moreover,*

$$S_{j+1} = \sigma(S(\varphi_{j+1})) \subset \sigma(S(\varphi_j)) = S_j.$$

In other words, the scalings obtained in Theorem 3.14 are quasi-orthogonal generators.

Remark 3.21. *The proof in [14] is constructive, while the one presented here is not. An explicit formula for the map J given above is, again, in general hard to obtain.*

Remark 3.22. *The construction used in the proof of the above theorem in [14] can be seen as a generalization of Theorem 3.14, since we limited our case to GMRAs and orthonormal wavelets. In the case that this decomposition has a finite number of nontrivial components, the number of nontrivial components corresponds to the essential supremum of the dimension function and this number is the minimal number of components, attaining the optimality.*

Remark 3.23. *If we begin by knowing the GMRA instead, the construction in [14] allows you to get the scalings* ***a priori****. Then we can apply the generalized conjugate mirror filter algorithm given in [19] to get the wavelets. Using the fact that the space V_1 is also shif-invariant, with essential supremum of the dimension function at most equal to $N2^d$ (the relation $\forall n \in \mathbb{Z}^d,\ D\tau_{2n} = \tau_n D$ implies that the functions*

$$D\tau_{2k+u}\varphi_j = D\tau_{2k}\tau_u\varphi_j = \tau_k D\tau_u\varphi_j = \tau_k\varphi_{j,u},\ k \in \mathbb{Z}^d,\ u \in \{0,1\}^d, 1 \le j \le N$$

where

$$D\tau_u\varphi_j = \varphi_{j,u}\ ,$$

forms a Parseval frame for $V_1 = DV_0$ so that $\left\{\varphi_{j,u} :\ u \in \{0,1\}^d, 1 \le j \le N\right\}$ is a set (not neccessarily optimal) of generators for the shift-invariant space V_1), the information about the spectrum of V_1 and the fact that $V_1 = V_0 \bigoplus W_0$, gives us an estimate about the the number of wavelets we can obtain. The latest progress in this matter can be found in the recent work of Baggett et.al. [4].

3.5 Conclusion and Future Research

3.5.1 Conclusion

In this thesis, the theoretical aspects of frame multiresolution analysis have been extended to the multidimensional case. To find the wavelet generators we need to solve pointwise a system of linear equations. When the set Γ has measure zero, the number of generators do not exceed $2^d - 1$, and we have a Mallat-Meyer type algorithm. If Γ has positive measure, then 2^d is a lower bound for the numbers of generators and no algorithm is known in this case.

The theory of frame multiresolution analysis extends the classical theory of multiresolution analysis, every MRA is a FMRA, but the theory itself has limitations. For example, non-MRA orthonormal wavelets cannot be produced by FMRA. The reason is that non-MRA orthonormal wavelets need multiscaling MRA schemes, i.e., GFMRAs. Theorem 3.14 showed how to construct a GFMRA for a given non-MRA orthonormal wavelet by constructing an optimal scaling set. We also showed how our construction linked GFMRAs with GMRAs by finding an explicit formula for an important unitary map in the GMRA theory. In general, it is hard to give an explicit formula for this map. The main drawback of any multiresolution scheme is the fact that the frames produced are semi-orthogonal, i.e., the wavelets are orthogonal at different resolutions.

3.5.2 Future Research

I plan to implement the construction in Theorem 3.14 to the wavelet sets obtained by the neighborhood-mapping algorithm developed in [6, 7]. Each iteration in the neigborhood-mapping method produces a tight frame. We can think about this in terms of a convergent sequence of GMRAs. Also, there is an underlying convergence of dimension functions. We want to give a meaning to all these statements.

BIBLIOGRAPHY

[1] P. Auscher, Solution of two problems in wavelets, *J. Geom. Anal.* **5** (1995), 181-236.

[2] L. W. Baggett, H. A. Medina, K. D. Merrill, Generalized multiresolution analyses and a construction procedure of all wavelet sets in $\mathbb{R}^n$, *J. Fourier Anal. Appl.* **5** (1999), 563-573.

[3] L. W. Baggett and K. D. Merrill, Abstract harmonic analysis and wavelets in $\mathbb{R}^n$, in *The functional and harmonic analysis of wavelets and frames* (San Antonio, TX, 1999), Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999, 17-27.

[4] L. W. Baggett, J. E. Courter, and K. D. Merrill, The construction of wavelets from generalized conjugate mirror filters in $L^2(\mathbb{R}^n)$, *Appl. Comput. Harm. Anal.* **13** (2002), 201-223.

[5] J. J. Benedetto and M. Frazier (eds.). Wavelets: Mathematics and Applications, CRC Press, Boca Raton, 1993.

[6] J. J. Benedetto and M. Leon, The construction of multiple dyadic minimally supported frequency wavelets in $\mathbb{R}^d$, in *The functional and harmonic analysis of wavelets and frames* (San Antonio, TX, 1999), Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999, 43-74.

[7] J. J. Benedetto and M. Leon, The construction of single wavelets in d - dimensions, *J. Geom. Anal.* **11** (2001), 1-15.

[8] J. J. Benedetto and S. Li, Multiresolution analysis frames with applications in *ICASSP'93*, Minneapolis, III, April 26-30, 1993, 304-307.

[9] J. J. Benedetto and S. Li, Subband coding and noise reduction in multiresolution analysis frames in *Proccedings of SPIE Conference on Mathematical Imaging*, San Diego, July (1994).

[10] J. J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, *Appl. Comp. Harm. Anal.* **5** (1998), 389-427.

[11] J. J. Benedetto and O. M. Treiber, Wavelet frames: multiresolution analysis and extension principles, in *Wavelet transforms and time-frequency signal analysis*, (L. Debnath, Editor), Birkhäuser, Boston, 2000.

[12] C. de Boor, R. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2\left(\mathbb{R}^d\right)$, *J. Func. Anal.* **119** (1994), 37-78.

[13] C. de Boor, R. DeVore, and A. Ron, Approximations from shift-invariant subspaces of $L_2\left(\mathbb{R}^d\right)$, *Trans. Amer. Math. Soc.* **341** (1994), 787-806.

[14] M. Bownik, The structure of shift invariant subspaces of $L^2\left(\mathbb{R}^n\right)$, *J. Func. Anal.* **177** (2000), 282-309.

[15] M. Bownik, A characterization of affine dual frames in $L^2\left(\mathbb{R}^d\right)$, *Appl. Comp. Harm. Anal.* **8** (2000), 203-221.

[16] M. Bownik, Z. Rzeszotnik, D. Speegle, A Characterization of Dimension Functions of Wavelets, *Appl. Comp. Harm. Anal.* **10** (2001), 71-92.

[17] P. G. Casazza, O. Christensen, and N. Kalton, Frames of translates, *Collectanea Mathematica* **52** (2001), 35-54.

[18] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[19] J. Courter, Construction of Dilation-d Orthonormal Wavelets, Ph.D. Thesis, University of Colorado, Boulder, CO, 1999.

[20] X. Dai, D. R. Larson, and D. Speegle, Wavelet sets in $\mathbb{R}^n$, J. Fou. Anal. Appl., **3** (1997), 451-456.

[21] I. Daubechies, Ten lectures on Wavelets, SIAM, Philadelphia, 1992.

[22] I. Daubechies, B. Han, and R. Shen, Framelets: MRA-based constructions of wavelets frames, *Appl. Comp. Harm. Anal.* **14** (2003), 235-263.

[23] C. de Boor, R. A. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2\left(\mathbb{R}^d\right)$, *J. Funct. Anal.* **119** (1994), 37-78.

[24] C. de Boor, R. DeVore, and A. Ron, Approximations from shift-invariant subspaces of $L_2\left(\mathbb{R}^d\right)$, *Trans. Amer. Math. Soc.* **341** (1994), 787-806.

[25] R. J. Duffin and A. C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* **72** (1952), 341-366.

[26] D, Han and D. R. Larson, Frames bases and group representations, *Mem. Amer. Math. Soc.* **147**, No. 640 (2000).

[27] C. Heil, J. Ramanathan and P. Topiwala, Linear independence of time-frequency translates, *Proc. Amer. Math. Soc.* **124** (1996), 2787-2795.

[28] H. Helson, The Spectral Theorem, Springer-Verlag, New York, 1986.

[29] H. Helson, Lectures on Invariant Subspaces, Academic Press, New York, 1964.

[30] E. Hernández, D. Labate, and G. Weiss, A unified characterization of reproducing systems generated by a finite family. II. *J. Geom. Anal.* **12**(2002), 615-662.

[31] H. O. Kim and J. K. Lim, Frame multiresolution analysis, *Commun. Korean Math. Soc.* **15** (2000), 285-308.

[32] H. O. Kim and J. K. Lim, On frame wavelets associated with frame multiresolution analysis, *Appl. Comp. Harm. Anal.* **10** (2001), 61-70.

[33] H. O. Kim, R. Y. Kim and J. K. Lim, Local analysis of frame multiresolution analysis with a general dilation matrix, *Bull. Austral. Math. Soc.* **67** (2003), 285-295.

[34] D. Labate, A unified characterization of reproducing systems generated by a finite family. *J. Geom. Anal.* **12**(2002), 469-491.

[35] S. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$, *Trans. Amer. Math. Soc.* **315** (1989), 69-87.

[36] Y. Meyer, Wavelets and Operators, Advanced mathematics. Cambridge University Press, 1992.

[37] M. Papadakis, On the dimension function of orthonormal wavelets, *Proc. Amer. Math. Soc.*, **128** (1999), 2043-2049.

[38] M. Papadakis, Generalized frame multiresolution analysis of abstract Hilbert space and its applications, *SPIE Proc.* **4119** (2000), in *Wavelet Application in Signal and Image Proccessing VIII*, (A. Aldroubi, A. Laine, M. Unser, Editors).

[39] M. Papadakis, Generalized frame multiresolution analysis of abstract Hilbert spaces, in *Sampling, Wavelets, and Tomography*, Appl. Numer. Harmon. Anal., Birkhäuser, Boston, Boston, MA, 2004, 179-223.

[40] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L_2\left(\mathbb{R}^d\right)$, *Canad. J. Math.* **47** (1995), 1051-1094.

[41] M. Paluszy´nski, H. Šiki´c, G. Weiss, and S. Xiao, Tight frame wavelets, their dimension functions, MRA tight frame wavelets and connectivity properties. *Frames. Adv. Comput. Math.* **18** (2003), 297-327.

[42] X. Wang, The Study of Wavelets from the Properties of Their Fourier Transforms, Ph.D. Thesis, Washington University, St. Louis, MO, 1995.

[43] E. Weber, Applications of the wavelet multiplicity function, in *The functional and harmonic analysis of wavelets and frames* (San Antonio, TX, 1999), Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999, 297-306.