University of Pittsburgh

FACULTY OF ARTS AND SCIENCES
Department of Mathematics and Statistics

Office phone: 412/624-5849
Home phone: 412/361-0465

September 20, 1982

Professor John J. Benedetto
Department of Mathematics
University of Maryland
College Park, Maryland 20742

Dear John:

The passage on antagonistic errors occurs in Cybernetics (2nd Ed., p.9; 1st Ed., p.16).

In none of his documents does Wiener quite tie this in with his discussion. The enclosed Document XI is one such. The others duplicate this in essence, or deal with other aspects of the problem, not relevant to the passage.

Best wishes,

Sincerely yours,

Pesi

P. Masani

XI

Memorandum on Circuit Theoretical Lead Mechanisms
by

Norbert Wiener

No date

ca. 1940 or ca. 1941

The problem to which this memorandum is a contribution is that of anticipating the position of a moving object in such a way that it may be accurately reached by gunfire. The method is that of linear circuit theory.

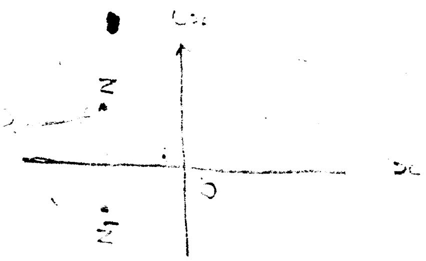
It is desired, given a function of the time, to subject this to a transformation by a linear apparatus which will yield the same function at a time later by a fixed interval. In its completeness, and without auxiliary assumptions, this problem cannot be solved. Every apparatus is influenced by the past of the impulses which it receives, and any future change in behaviour of these impulses cannot be recorded by it. However, if we restrict ourselves to the case where the impulses received by an instrument are sufficiently slow to change the character, or, in other words, where the high frequency components are sufficiently small, we may make our prediction to a useful degree of accuracy. In particular, we wish to design a device which, when used against an airplane moving on a straight course or turning with a constant angular velocity on a circle on which it is physically possible for it to turn, will aim a gun at a point a fixed time ahead on the straight or circular trajectory.

The fact that the apparatus is strictly unable to foretell the future and that it is linear means that if it receives an input of the form $f(t)$, it will yield an output of the form $\int_{-\infty}^{t} f(r)\, dg\,(t - r)$. Now if we apply the apparatus to a simple harmonic $e^{i\omega t}$, we shall obtain as an output $\int_{-\infty}^{t} e^{i\omega r}\, dg\,(t - r) = -e^{i\omega t} \int_{0}^{\infty} e^{-i\omega r}\, dg\,(r)$.

[illegible] r, $-\int_0^\infty e^{-i\omega r}\, dg(r) = F(i\omega)$, will be known as the conversion ratio of the apparatus for frequency ω. We shall then find that the function of a complex variable, $F(Z)$, has no singularity when the real part of Z is positive. It may, however, have zeros in this region.

The functions of $F(Z)$ which are obtainable as conversion ratios have electrical systems with a finite number of inductances, resistances and capacitors, or have their mechanical equivalents, such as integraph devices, with a finite number of integrators, and will be of the form $\frac{P(Z)}{Q(Z)}$, where P and Q are polynomials of finite order. When reduced to their lowest terms, $Q(Z)$ may have no zeros to the right of the imaginary axis. If the device consists entirely of resistances and capacitors, but no inductances, the zeros of $Q(Z)$ will all lie on the negative real axis. Otherwise, they may lie anywhere in the negative half plane, except, of course, the values off the axis must occur in pairs so that Q is real for real arguments.

$P(Z)$ may have zeros anywhere, and when the apparatus is designed for filter purposes, zeros in the right half plane are often very useful. This is due to the fact that such zeros may be used to cancel zeros of the denominator in the left half plane as far as their effect on the absolute value of F is concerned, while they produce an effect in the same direction as far as the phase of F is concerned. This effect is that of a lag. As we are trying to obtain a lead, and as any amplitude effect produced by a zero in the right half plane can be produced by its mirror image in the left half plane, it does not seem reasonable to expect that such zeros will be of any help to us in the design of apparatus to produce a lead.



We may, therefore, assume that both P(Z) and Q(Z) have exclusively zeros in the left half plane, or possibly, as far as P is concerned, on the imaginary axis. Even with a device consisting entirely of condensers and resistances, the zeros of P need lie on the real axis.

The logarithm of F will accordingly be free from singularities in the right half plane. This means that the logarithm of the absolute value of F will be what is known as the conjugate function to the phase of F(iω). In other words, the conversion ratio of such an apparatus is completely determined except for an additive constant in phase when it is determined in amplitude and vice versa. It is, thus, impossible to adjust these two quantities independently, and if we wish to obtain a certain phase characteristic, we must look to it that we don't so distort the amplitude characteristic of the apparatus as to produce undesired results. In particular, if either of the two quantities, $\int_{-\infty}^{\infty} \left|\log\left|F(i\omega)\right|\right|^2 d\omega$, and $\int_{-\infty}^{\infty} \left|\arg\,(F(i\omega))\right|^2 d\omega$, exists, the other must exist and have the same value. Thus, if our apparatus is to produce a considerable phase shift, its attenuation or amplitude shift will differ widely from one frequency to another. If we hold it down within a given frequency range, it will always be at the expense of a large, parasitic amplification outside the range over which we are working.

An example of the characteristic of a possible piece of apparatus of this sort is:

$$\frac{1 + 2.3562\, i\omega - 2.6619\, \omega^2 - 1.5861\, i\omega^3 + .9067\, \omega^4}{1 + .7854\, i\omega - .2314\, \omega^2 - .0302\, i\omega^3 + .0015\, \omega^4}$$

This has been designed to approximate to the function $e^{\frac{\pi i\omega}{2}}$ of the

XI p.4

range $-1 \leq \omega \leq 1$. If this operator is applied to $e^{i\omega t}$, it will yield

$$e^{i\omega(t + \frac{\pi}{2})}$$

and that is, if $\omega = 1$, it will yield a lead of 1/4 of a period. The accuracy of the approximating function is computed in the following way. The absolute error of approximation to $e^{i\omega t}$ is divided by ω and estimated as a percent. Let it be noted that if this method is applied to one coordinate of a body moving in a circle with a fixed speed, it will estimate on the same scale the error in the position of the body for circles or all radii. That is, if a body is moving nearly in a straight line, the amplitude of the circle which it is describing will be large, and the same percentage error as referred to the radius of the circle will be a much larger actual distance error than if the body is moving in a circle or small radius. On this basis, the absolute error divided by the frequency will yield the following table:

ω	$F(i\omega)$	$\frac{\lvert Error \rvert}{\omega}$ (per cent)
Near 0	Near 1 + 1.5708 i	0
1/3	.8717 + .5128 i	4.21
1/2	.7144 + .7251 i	3.88
2/3	.5167 + .9346 i	10.59
3/4	.4131 + .9893 i	10.92
.9	.1559 + .9877 i	5.99
1	i	0

The characteristic here given was obtained by a crude process of interpolation between values at given points. There is a more precise process of approximation which will give a result which is in some sense the best possible, and, by this way, we ought to be able to reduce the percentage error divided by the frequency to 6 or 7 per cent.

In this example, it is worthwhile considering what the amplification is for frequencies outside of the range over which we desire a close imitation of a phase shift. The order of magnitude is given roughly by the following table:

ω	$\lvert F(i\omega)\rvert$
1	nearly 1
2	nearly 10
3	nearly 45
4	nearly 100
5	nearly 175
6	nearly 180
∞	nearly 600

While the parasitic amplification appears to be enormous, it must be reflected that this is an amplification of the amplitude and that, since the acceleration of a plane is fixed by mechanical considerations, the amplitude must go down at least as fast as the square of the frequency and, indeed, practically as fast as the cube of the frequency, unless the plane is engaged in wild and probably impossible maneuvers. Under these conditions, the reduced magnitude of the parasitic amplification is no greater than a small integer, and is not serious, since no device whatever will make it possible to hit a plane where the maneuvering time of the pilot after the gun is fired is long and the pilot takes advantage of this maneuvering time to stunt.

In the characteristic $F(i\omega)$, which has been given, the numerator and denominator are taken to be of the same degree. The denominator is, in fact, $(1 + \frac{\pi i\omega}{16})^4$. It would be desirable, from some points of view, to use a denominator of higher degree than the numerator, say a denominator $(1 + \frac{\pi i\omega}{20})^5$, with a 4th degree numerator, in order to introduce a heavy damping for high frequencies such as may be put in by the observer in cranking his instrument.

It will be noted that both of these denominators give F zeros only on the real axis. They are, thus, obtainable by networks in which only capacitors and resistances occur, together with such amplifiers as are necessary to prevent the excessive attenuation of the input signal, characteristic of such networks. Such a network may be given a sufficiently large time constant to be operative without an excessive enlargement of its parts, whereas this would be impossible about zero frequency if we introduced inductances.

In this network, we have imitated $e^{\frac{\pi i \omega}{2}}$. It is easy, with the same denominator, to obtain a numerator so that we may imitate $e^{ai\omega}$, where a is any number less than $\frac{\pi}{2}$. This means that, with the same structure of capacitors and resistances, but with a different takeoff from these members we are able to obtain any lead less than $\frac{\pi}{2}$, and so do with the aid of gangs of rheostats on a single shaft so that the lead is continuously adjustable. In an apparatus of this type, the information received and stored by the machine is not changed by changing the lead, and the apparatus will come instantaneously to the same position which it would have had if it had been initially adjusted for the new lead. This is extremely important where there is a feed-back connection between the lead mechanism and an auxiliary mechanism which determines the proper lead for a given expected position of the target.

All the integrations in this apparatus may be performed with disc integrators of the Bush type, instead of capacitors. The disadvantage of the use of this type integrator is the mechanical one of obtaining a linear combination of several different output functions where the coefficients are not simple ratios between small transfers. However, a device using Bush integrators has been set up for the characteristic function

$$\frac{(1 + \frac{i\omega}{2})^4}{(1 + \frac{i\omega}{4})^4}$$

and the results are available to accompany this. Such an apparatus, while far from having an adequately precise characteristic, at least shows the method of operation of the apparatus.

The fact that such a prediction apparatus is primarily used on subjects like airplanes moving at very nearly constant speed may be employed to further refine the performance of the apparatus. It is possible, by means of impedances already contained in the apparatus designed, to measure the components of acceleration of the plane and to put these together into a single quantity giving the absolute value of the acceleration. As this acceleration is almost entirely radial, it gives a measure of the radius of curvature of the path of the plane and this again may be used to set a correction in the output of the device so as to reduce the spacial error by a factor that may easily be as large as 10. This is, of course, a departure from a strictly linear apparatus, but it is certainly desirable to use any characteristic of the motion of the plane which gives us more information as to what to expect.