Mathematical Work of Norbert Wiener

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The Norbert Wiener Centenary Congress was held at Michigan State University, November 27–December 3, 1994. The Congress was cosponsored by the American Mathematical Society, the International Association of Cybernetics, and the World Organization of Systems and Cybernetics.

The aim of the Congress was to reveal the depth and strong coherence of thought that runs through Wiener's legacy and to exhibit its influence on current research. The Congress drew fifty-eight participants, including thirteen from Europe, two from Asia, one from Mexico, and two from Canada. There were twenty-nine addresses, sixteen contributed papers, and one round table discussion. The Congress ended with the award of the Norbert Wiener Centenary Medal to Claude E. Shannon (Shannon's wife accepted the medal on his behalf. See photo above.).

The following article, written by one of the co-organizers of the Congress, describes some of Wiener's work explored at the Congress and discusses its impact on contemporary research.

—Allyn Jackson

Norbert Wiener began his mathematical life in logic and foundations. He went to England to work with B. Russell. His contact with G. H. Hardy and others expanded the spectrum of his work to include analysis, engineering, statistics, and physics. He is known to the general public as a founder of cybernetics, but to mathematicians he is known for his fundamental contributions in analysis and for his use of randomness to expand the vision and applications of mathematics.

We shall explore this aspect of his work. His interest in randomness begins with his work on realizing Brownian motion in a function space. The insight gained raised further questions and issues which naturally led him to generalized harmonic analysis, Tauberian theorems, and later Paley-Wiener theory, which was then used to study problems involving randomness in signal analysis. In addition, Wiener introduced fundamental ideas and techniques in potential theory and homogeneous chaos, which to this day are finding applications to emerging fields in mathematics at the hands of other researchers. In order to refer systematically to Wiener's works, we shall refer to his *Collected works* [NW1] with double brackets. For example, [[34a]] shall mean reference 34a in [NW1]. We also bring at-

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tention to the biographical article by P. Masani [NW2], written at the time of Wiener's death.

We shall begin our look at Wiener's work first by examining his construction of the measure W on the space $C_0[0,1]$ of real-valued continuous functions x on [0,1] with $x(0)=0$ as a model for paths traversed by a particle following Brownian motion. This is contained in five papers from 1920–1924. First, we describe the measure space for W (Wiener measure). Let $\{t_{n,i}\}$ be a doubly indexed set of points in [0,1] with these properties:

a. for each n, $0 < t_{n,1} < t_{n,2} < \cdots < t_{n,k_n} < 1$;
b. for each n the set $\{t_{n,i}\}$ is a subset of $\{t_{n+1,i}\}$;
c. the set of all $\{t_{n,i}\}$ is dense in [0,1].

For each n, introduce the set π_n of subsets of $C_0[0,1]$: an element of π_n is of the form

$$I = \{x \in C_0[0,1] : a_i < x(t_{n,i}) < b_i, i = 1,2,\ldots,k_n\}.$$

Notice that, because of b) $\pi_n \subset \pi_{n+1}$. The Wiener measure of I, $W(I)$ is

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} p(t_1,y_1)p(t_2-t_1,y_2-y_1)\cdots p(t_n-t_{n-1},y_n-y_{n-1})dy_1\cdots dy_n$$

where

$$p(t,y) = (2\pi t)^{-1/2}\exp(-\frac{y^2}{2t}), y \in R.$$

$\cup\pi_n$ is the basis for a topology on $C_0[0,1]$, and W defines a Borel measure on this space. To show this, Wiener follows the Daniell approach to measure theory: starting with a positive functional on a (sufficiently large) linear space of Baire functions, satisfying the monotone convergence theorem of Lebesgue, Daniell extends this function as an integral on all Baire functions. Here we take the functions space to be $\mathcal{L}$, the set of simple functions based on $\cup\pi_n$; the functional is $\int f dW$ on $\mathcal{L}$: for $f = \sum v_i \chi_{I_i}$, $\int f dW = \sum v_i W(I_i)$. The crucial point is to prove continuity of W from above at 0: For $\{f_n\}$ a non-increasing sequence in $\mathcal{L}$ converging to 0, Wiener succeeds in showing that $\int f_n dW \to 0$. For this he constructs (for each integer m) a compact set $K_m \subset C_0[0,1]$ of Hölder continuous functions of order $< 1/2$, with Wiener measure greater than $1 - 1/m$. From this it follows that W is a Borel measure on $C_0[0,1]$ and that Hölder continuous paths with finite quadratic variation exist almost everywhere in Wiener space. These paths in particular are nowhere differentiable. Thus Wiener uses the topological structure of $C[0,1]$ to construct Brownian motion. In the work of A. N. Kolmogorov (see [10]) in 1933 for probability measures on $\mathbf{R}^T$ (T arbitrary), regularity in the finite-dimensional case is used to produce the compact sets, and the topological structure of $\mathbf{R}^T$ is exploited to produce a function of Baire Class two.

Before we go to Wiener's contribution to potential theory in the twenties, we want to examine a trail of ideas starting from the above construction to his major work of the thirties on generalized harmonic analysis.

Immediately after his paper [[24d]], Wiener gave a Fourier series representation for Brownian motion ([[24e]], p. 570), in terms of independent identically distributed (i. i. d.) standard normal random variables $\{a_n\}$ (see also [[34a]], p. 21). In [[27a]] Wiener gives the definition of the spectrum of a sequence of complex numbers $\{f_n\}$ satisfying

$$\lim_{N\to\infty} \sum_{-N}^{N} f_{n+k}\bar{f}_n.$$

The above-mentioned sequence a_n satisfies Wiener's condition a. e., and from the representation in [[34a]], one gets

$$a_n = \int_0^{2\pi} e^{inu} dB(u)$$

where B is the Brownian motion and the integral is in the sense of Wiener as given in [[23a]] using integration by parts. Thus Wiener seems to have been aware of the harmonic analysis of a sequence where the "spectral measure" B may not be a measure. In recent years one defines the spectrum in the generalized sense [14] using the theory of generalized functions. The latter theory was unavailable to Wiener in a systematic the techniques of the Plancherel Theorem. This had two advantages: first, it led to Tauberian theorems, and second, it found applications to electrical engineering, as Fourier theory was an established method for engineers.

In the mind of an analyst of the 1920s, a Tauberian theorem is one which deduces the convergence of an infinite series on the basis of the properties of the function it defines (via power series, Fourier series, or as coefficients in any standard expansion of a function) and any kind of auxiliary hypothesis which prevents the general term of the series from converging to zero too slowly. Here is Wiener's discussion in his 1932 paper, "Tauberian Theorems", in the *Annals of Mathematics*:

> Tauberian theorems gain their name from a theorem published by A. Tauber in 1897, to the effect that if
>
> (0.07) $$\lim_{x\to 1-} \Sigma_0^\infty a_n x^n = A$$
>
> and
>
> (0.08) $$a_n = o(\frac{1}{n}),$$
>
> then
>
> (0.09) $$\Sigma_0^\infty a_n = A.$$
>
> This is a conditioned converse of Abel's theorem, which stated that (0.07) follows from (0.09) without the mediation of any hypothesis such as (0.08). Such conditioned inverses of Abel's theorem, and of other analogous theorems which assert that the convergence of a series implies its summability by a certain method to the same sum, have been especially studied by G. H. Hardy and J. E. Littlewood and have been termed by them *Tauberian.*
>
> It is the service of Hardy and Littlewood to have replaced hypothesis (0.08) by hypotheses of the form
>
> (0.10) $$a_n = O(\frac{1}{n}),$$
>
> or even of the form $na_n > -K$. The importance of these generalizations is scarcely to be exaggerated. They far exceed in significance Tauber's original technique and is capable, among other things, of supplying the gaps in Poisson's imperfect discussion of the convergence of the Fourier series. For these reasons, I feel that it would be far more appropriate to term these theorems Hardy-Littlewood theorems, were it not that usage has sanctioned the other appellation.

Wiener's central Tauberian theorem is an analogous theorem about indefinite integrals. It goes like this. Let f be a bounded measurable function on the real line ($f \in L^\infty$), and let K be an L^1 function (that is, $\int_{-\infty}^{\infty} |K(\xi)|d\xi < \infty$). What is at issue is the limit of $f * K(x)$ as $x \to \infty$:

(1) $$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(\xi)K(\xi - x)d\xi.$$

The hope is that this limit is $A\int_{-\infty}^{\infty} K(\xi)d\xi$ for some number A independent of K. The theorem of Wiener is that if this is true for some $K \in L^1$ whose Fourier transform never vanishes, then it is true for all $K \in L^1$.

Using the Tauberian theorems, Wiener gave a proof in this paper of the Prime Number Theorem. His proof reduces it to the convergence of a certain definite integral (derived from the Riemann zeta function) based on function-theoretic information about the function the indefinite integral defines.

Wiener's Tauberian theorems followed from two results which themselves have had great impact on the development of modern analysis.

I. Suppose that f is a continuous function on the circle which has an absolutely convergent Fourier series. If f never vanishes on the unit circle, then $1/f$ also has an absolutely convergent Fourier series.

Wiener's argument proceeds from a local statement to the global one, using some very technical convergence arguments. This result is a lemma on the way to proving:

II. Let f be an L^1 function on the real line. The linear span of the set of translates of f (functions of x of the form $f(x+y)$) is dense in L^1 if and only if the Fourier transform of f vanishes on a set of measure zero.

It is quite easy to establish the necessary condition, for if F represents the Fourier transform of f, then the Fourier transform S of the span of all translates of f is the linear span of the set of all functions of the form $e^{iy\xi}F(\xi)$. Since the closure of the linear span of the exponentials $e^{iy\xi}$ in the uniform norm on any compact set coincides with all continuous functions on that compact set, all functions in S will vanish on the intersection of that compact set with the zero set of F. If that set has positive measure, then there certainly are functions in L^1 whose Fourier transform is identically 1 there; such a function cannot be in the closure of the linear span of the translates of f. The real issue is the sufficiency.

The property of the Fourier transform, that it changes translation operators into multiplication operators, is fundamental to all the uses of the Fourier transform. What was apparently first understood by Wiener is that it changes geometric questions about function spaces on the circle or in R^n into algebraic questions about the ring of multiplication operators on the transformed space. This ultimately led to considerations of abstract normed rings and great generalizations and simplifications of these works of Wiener (first by the Uppsala school of analysts and completed by Gelfand).

Thus, starting from the construction of Brownian motion, Wiener was led to several significant contributions to analysis, harmonic analysis, almost periodic functions, and number theory. His generalized harmonic analysis has had major impact in signal analysis and time series ([NW2] and [[49g]]), since in practical situations one can only observe precisely the correlation functions associated with a signal. Here one needs to analyze the spectrum. In all of Wiener's work, the concept of ergodicity, to which we shall come in a while, was implicit.

Wiener's work with Paley started with [[33a]], where they considered problems of random Fourier series with i. i. d. Gaussian random variables replacing Rademacher functions. In his work on differential space [[23d]], Wiener had given a definition of a stochastic integral. He considered the process

$$Y(t,\cdot) = \int_{-\infty}^{+\infty} \varphi(u-t)B(du,\cdot)$$

for $\varphi \in L^2(-\infty,+\infty)$. Using the Paley-Wiener result that there is a measure-preserving ergodic flow on $[0,1]$ given by translations of Brownian motion, one can see that this is a stationary process. In fact, considering this representation of the signal, one can show that a correlation function exists a. e. and is constant. It has absolutely continuous spectrum with density $|\hat{\varphi}(\lambda)|^2$. To study causality, one needs to assume that $\varphi(u)=0, u \geq 0$. In this context the following two basic results of Paley and Wiener are indispensable [[34d]].

1. $\varphi \in L_2(-\infty,\infty)$ is the boundary value of a function φ_+ analytic in the upper half plane which is uniformly L_2 on every horizontal line (the Hardy class H_2) if the Fourier transform $\hat{\varphi}$ vanishes on $(-\infty,0)$.

2. The necessary and sufficient condition that a nonnegative function f on $(-\infty,\infty)$ and in $L_1(-\infty,\infty)$ is of the form $f=|\varphi|^2$ a. e. where φ is as above is $\log f(\lambda)/(1+\lambda^2) \in L_1(-\infty,\infty)$.

These also turned out to be important results in the study of one variable prediction theory and filtering problems. In his work on multivariate prediction theory with P. Masani and E. J. Aukowitz, Wiener took the approach of Kolmogorov [11]. Let us illustrate this in the one-variable case.

Let $\{X_n, n\in \mathbf{Z}\}$ be a sequence in the complex Hilbert space $H=L^2(\Omega,\mathcal{F},P)$ with $(\Omega,\mathcal{F},P)$ a probability space. Let $L(X:n)$ be the past of the process, namely, $\overline{\mathrm{sp}}\{X_k, k\leq n\}$ where $\overline{\mathrm{sp}}$ denotes the closure of the linear span. Under Wiener's condition on ergodicity

$$EX_n\bar{X}_0 = (X_n,X_0) = \int^{2\pi} e^{inx}dS(\lambda)$$

where S is the spectral measure. Thus $\overline{\mathrm{sp}}\{X_k, k\in\mathbf{Z}\}$ is isometric to $L^2([0,2\pi],S)$, sending $X_n \to e^{inx}$. Following Wold, one says the process is nondeterministic if $\cap_n L(X:n)=\{0\}$ (there is no remote past) and deterministic if $L(X:n)=L(X:n+1)$ for all n (the remote past contains all the information of the present and the future). By projecting X_n onto the remote past $L(X:-\infty)=\cap_n L(X:n)$ and its orthogonal complement in $L(X)=\bigvee_n L(X:n)$, one gets an orthogonal decomposition $X_n=Y_n+Z_n$ where $\{Y_n\}$ is deterministic and $\{Z_n\}$ is purely nondeterministic. Y_n is uninteresting for prediction theory. Let ξ_n be the new part (innovation) of Z_n, the part of Z_n which is orthogonal to $L(Z:n-1)$. Writing $\{Z_n\}$ in terms of $\{\xi_n\}$, one sees that the spectral measure of the process is absolutely continuous with respect to Lebesgue measure and its spectral density factors. Being geometric, this technique goes through in the multivariate case with minor changes. In terms of correlation, a process with factorable density is purely nondeterministic. The important result used here, (the Szegö Theorem) that $\int_0^{2\pi} \ell n f(x)dx > -\infty$ is equivalent to the factorization of f, requires ideas from Paley-Wiener theory. For a vector process, the spectral measure is matrix (or, in general, operator) valued. This method for factoring density survives in the full rank case. However, the Szegö Theorem gets delicate but can also be reduced to univariate case using a technique of Lowdenslager [13] and some geometric arguments. Because of the generality of these arguments, this has implications to the invariant subspace problem, scattering theory, and harmonic analysis [1].

Extension of the idea of expressing a (strictly) stationary process in terms of orthogonal (independent) innovations for nonlinear prediction problem was initiated in the work of Wiener with Kallianpur and was successfully carried out in special cases by M. Rosenblatt and D. Hanassumptions, the best (nonlinear) predictor of X_n with lead $h > 0$ is given by

$$E(X_n | \mathcal{F}_0^X) = L_n^2 - \lim Q_n(X_0, X_1, ..., X_{-m_n})$$

where (i) $\mathcal{F}_0^X$ is the α-algebra generated by $\{X_m, M \leq 0\}$, (ii) $m_n \geq 0$ integer based on n, (iii) Q_n is a polynomial in $m_n + 1$ variables. Here $\{X_n, n \in \mathbf{Z}\}$ is a strictly stationary process $\subseteq L^\infty(\Omega, \mathcal{F}, P)$. The most general result in this direction is due to Ito and Stein and is stated in Kallianpur [9].

However, synthesis of the black box is a delicate problem [19] and Wiener gave his thoughts on it in his monograph [[58a]] using the homogeneous chaos paper. Wiener also used this to study the continuous analogue of the nonlinear prediction problem. This is one of many of Wiener's papers in which his ideas were shown to be basic to the development of the foundations of the area. His first work of this type was in potential theory.

Almost simultaneously with his work on Brownian motion, Wiener undertook his fundamental work on potential theory. Interestingly, he again used the Daniell integral to attack the Dirichlet problem. Let D be a domain in the plane with boundary Γ. For what continuous functions φ on Γ is there a function u^φ continuous on the closure of D, equal to φ on Γ, and satisfying Laplace's equation in D? It had been known for some time that this question (Dirichlet's problem) can be solved for all continuous functions on Γ when D is a smoothly bounded domain. Now, for a general domain D, Wiener selected an increasing sequence of smoothly bounded domains D_n filling out D from the interior. Now fix a point y in D. For φ defined on Γ, let $I_y(\varphi) = lim_{n \to \infty} u_n^\varphi$, where u_n^φ solves Dirichlet's problem on D_n for some continuous extension of the given f on Γ. Wiener showed that I_y is well defined, is independent of the choices of the sequence D_n and the continuous extension of φ, and meets all the conditions of the Daniell integral on $C(\Gamma)$. This gives for each y a probability measure μ_y such that

$$u^\varphi(y) = \int_{\partial D} \varphi(x) \mu_y(dx)$$

is harmonic in D. The probability measure μ_y is called harmonic measure. Kakutani related it to the exit time of Brownian motion in 1944. A point x on the boundary is called regular if $u^\varphi(y) \to \varphi(x)$, as $y \to x$ for all continuous functions φ on ∂D. Wiener gave a necessary and sufficient condition for a boundary point to be regular in terms of capacity, a notion he adapted to this purpose based on physical intuition.The

at x. Then x is regular if and only if $\Sigma \kappa_n^2$ diverges.

Thus, Wiener introduced the central concepts of the modern potential theory: harmonic measure, generalized solutions, capacity of sets, and regular points. At just about the same time (although the paper was published a year earlier than Wiener's), O. Perron found another method for solving the Dirichlet problem on an arbitrary domain. His condition for regularity at a point x was stated in terms of the local existence of a barrier function: a function subharmonic near x which, on the closure of D, has a strict maximum at x. In his paper [[25a]], Wiener showed that the two constructions and conditions for regularity are equivalent.

Let us now turn to Wiener's work on homogeneous chaos, motivated by his interest in turbulence and statistical mechanics. Wiener defines a Gaussian function for α on [0,1] whose values are set functions $\{\xi(A, \alpha)\}$ on Borel sets in $\mathbf{R}^n$ of finite Lebesgue measure with this property: $\int_0^1 \xi(A, \alpha)\xi(B, \alpha) d\alpha$ is the Lebesgue measure of $A \cap B$. This is called chaos and he then defines what can be called the multiple Wiener integral. It should be noted that if the chaos is constructed from Brownian motion, i.e., $B(A) = \int_A dB(t)$ with $n = 1$, then $\int\int\int\int 1_{A \times A \times \cdots \times A} dBdB \cdots dB = B(A)^k$. This is different from the definition of K. Ito and I. Segal, where the above integral is $H_k(B(A))$, H_k being the Hermite polynomial of order k. If one examines Wiener's later work [[58a]], one can see that he orthogonalizes his integral, relating it to the Ito multiple integral. What Wiener defines is called in current literature the Stratonowich multiple integral. The relation between the Wiener and Ito integrals is precisely the Hu-Meyer formula [6], a special case of which for $k = 2$ occurs in [[58a]]. It should be noted that coefficients in the Hu-Meyer formula are precisely those expressing polynomials through Hermite polynomials. Wiener's aim was to prove a multidimensional generalization of the Birkhoff Ergodic Theorem and apply it to larger classes of functions on chaos using approximations through his integral. Although Wiener was not able to attack the problem of statistical turbulence, the ideas of Wiener have influenced physics problems. This can be seen in the work of Segal [22] on quantum field theory and Hu-Meyer on the computation of Feynman integrals, justifying the work of Hida-Streit. An interesting consequence of this work on multiple integrals is the work of A. V. Skorokhod [23], which extends Ito's generalization of the Wiener integral. The latter integral can be used to study nonlinear filter theory. It should also be observed that Segal introduced a finitely additive (cylindrical) measure in his work on quantum field theory. Countably additive extension of this measure led L. Gross to the study of abstract Wiener spaces—an algebraic generalization of Wiener's original idea on Brownian motion! Expansion of nonlinear functions of chaos has made it possible to create a calculus on the abstract Wiener space called Malliavin calculus (see [24]), which has proven useful in the study of statistical mechanics. The just-mentioned work studies "generalized functions" on abstract Wiener space, the analogue of which for Lebesgue measure was also in Wiener's work (see [20], p. 427). Although Wiener proved a generalization of Birkhoff's Ergodic Theorem for multiparameter flows, he ended up proving a very general Ergodic Theorem [3ga].

We are unable to cover the work of Wiener on foundations, cybernetics, prosthesis, and economic philosophy; the bases of these also have deep mathematical ideas. Excellent references for Wiener's complete work, of course, are [17] and [21].

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