Honoring Dilip B. Madan's
60th Birthday

Investments, wealth and risk tolerance

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Topics

- Utility-based measurement of performance
- Utility volume – traditional (backward) and alternative (forward) formulation
- Forward dynamic utilities and construction of such a class
- Motivational examples
- Variational and stochastic components of forward optimal utility volume
- Construction and analysis of variational utility component

References

Joint work with M. Musiela (BNP Paribas, London)

- "Investments and forward utilities"
  Preprint 2006

- "Backward and forward dynamic utilities and their associated pricing systems: Case study of the binomial model"
  Indifference Pricing, PUP (2003, 2005)

- "Investment and valuation under backward and forward dynamic utilities in a stochastic factor model"
  to appear in Dilip Madan's Festschrift (2006)

  http://www.ma.utexas.edu/users/zariphop/

Utility-based measurement of performance

Deterministic environment

Utility traits

$$u(x,t) \quad : \quad x \ \textbf{“wealth”} \quad \text{and} \quad t \ \textbf{“time”}$$

- Monotonicity $\quad u_x(x,t) > 0$
- Risk aversion $\quad u_{xx}(x,t) < 0$
- Impatience $\quad u_t(x,t) < 0$

Fisher (1913, 1918), Koopmans (1951),

Koopmans-Diamond-Williamson (1964) ...

Stochastic environment

Important ingredients

- Time evolution concurrent with the one of the investment universe
- Consistency with up to date information
- Incorporation of available opportunities and constraints
- Meaningful optimal utility volume

Dynamic utility

$U(x,t)$ is an $\mathcal{F}_t$-adapted process

- As a function of x, U is increasing and concave

- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t satisfies

$$U(X_s^\pi, s) \geq E_{\mathbb{P}}(U(X_t^\pi, t) \mid \mathcal{F}_s) \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$U(X_s^{\pi^*}, s) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) \qquad 0 \leq s \leq t$$

Traditional framework

A deterministic utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$\boldsymbol{U(x,T) = u_T(x)}$$

Backwards in time generation of optimal utility volume

$$V(x,s) = \sup_{\pi} E_{\mathbb{P}}(u_T(X_T^{\pi})|\mathcal{F}_s; X_s^{\pi} = x)$$

$$V(x,s) = \sup_{\pi} E_{\mathbb{P}}(V(X_t^{\pi},t)|\mathcal{F}_s; X_s^{\pi} = x) \qquad \text{(DPP)}$$

$$V(x,s) = E_{\mathbb{P}}(V(X_t^{\pi^*},t)|\mathcal{F}_s; X_s^{\pi^*} = x)$$

$$\Downarrow$$

$$U(x,t) \equiv V(x,t) \qquad 0 \leq t < T$$

The dynamic utility coincides with the traditional value function

Alternative framework

A deterministic utility datum $u_0(x)$ is assigned at the beginning of the trading horizon, $t = 0$

$$\boldsymbol{U(x, 0) = u_0(x)}$$

Forward in time generation of optimal utility volume

$$U(X_s^{\pi^*}, s) = E_{\mathbb{P}}(U(X_t^{\pi^*}, t)|\mathcal{F}_s) \qquad 0 \leq s \leq t$$

- Dynamic utility can be defined for all trading horizons
- Utility and allocations take a very intuitive form
- Difficulties due to the "inverse in time" nature of the problem

Motivational examples

An incomplete multiperiod binomial example

Exponential utility datum

- Traded security: S_t, $t = 0, 1, ...$

  $\xi_{t+1} = \dfrac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi_{t+1}^d, \xi_{t+1}^u$ with $0 < \xi_{t+1}^d < 1 < \xi_{t+1}^u$

  Second traded asset is riskless yielding zero interest rate

- Stochastic factor: Y_t, $t = 0, 1, ...$

  $\eta_{t+1} = \dfrac{Y_{t+1}}{Y_t}$, $\eta_{t+1} = \eta_{t+1}^d, \eta_{t+1}^u$ with $\eta_t^d < \eta_t^u$

- Probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$

  $\{S_t, Y_t : t = 0, 1, ...\}$: a two-dimensional stochastic process

- State wealth process: X_t, $t = s+1, s+2, \ldots, \ldots$

  α_i : the number of shares of the traded security held in this portfolio over the time period $[i-1, i]$

$$X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \triangle S_i$$

- Forward dynamic exponential utility $\qquad (0 \leq s \leq t)$

$$\begin{cases} U(X_s^{\alpha^*}, s) = E_{\mathbb{P}}(U(X_t^{\alpha^*}, t) | \mathcal{F}_s) \\ U(x, 0) = u_0(x) = -e^{-\gamma x} , \qquad \gamma > 0 \end{cases}$$

- A forward dynamic utility

$$U(x,t)=\begin{cases} -e^{-\gamma x} & \text{if} \quad t=0 \\ -e^{-\gamma x+\sum_{i=1}^{t} h_i} & \text{if} \quad t\geq 1 \end{cases}$$

- Auxiliary quantities : local entropies h_i

$$h_i = q_i \log \frac{q_i}{\mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)} + (1-q_i)\log\frac{1-q_i}{1-\mathbb{P}\left(A_i \,|\mathcal{F}_{i-1}\right)}$$

with

$$A_i = \{\xi_i = \xi_i^u\} \quad \text{and} \quad q_i = \mathbb{Q}\left(A_i \,|\mathcal{F}_{i-1}\right)$$

for $i=0,1,..$ and $\mathbb{Q}$ being the minimal relative entropy measure

Important insights

The forward utility process

$$U(x,t) = -e^{-\gamma x + \sum_{i=1}^{t} h_i}$$

is of the form

$$\boldsymbol{U(x,t) = u(x, A_t)}$$

where $u(x,t)$ is the deterministic utility function

$$u(x,t) = -e^{-\gamma x + \frac{1}{2}t}$$

and A_t corresponds to a time change depending on the "market input"

$$A_t = 2\sum_{i=1}^{t} h_i$$

Important insights (continued)

- The variational utility input

$$u(x,t) = -e^{-\gamma x + \frac{1}{2}t}$$

solves the partial differential equation

$$\begin{cases} u_t \; u_{xx} = \frac{1}{2} u_x^2 \\ u(x,0) = -e^{-\gamma x} \end{cases}$$

- The stochastic market input

$$A_t = 2 \sum_{i=1}^{t} h_i$$

plays now the role of "time". It depends exclusively on the market parameters.

A continuous-time example

- Investment opportunities

  Riskless bond : $r = 0$

  Risky security : $dS_t = \sigma_t S_t(\lambda_t dt + dW_t)$

- Utility datum at $t = 0$: $u_0(x)$

- Wealth process

$$\begin{cases} dX_t = \sigma_t \pi_t (\lambda_t dt + dW_t) \\ X_0 = x \end{cases}$$

- Market input : λ_t, A_t

$$\begin{cases} dA_t = \lambda_t^2 dt \\ A_0 = 0 \end{cases}$$

- Building the martingale $U(X_t^{\pi^*}, t)$

Assume that we can construct $U(x,t)$ via

$$\begin{cases} U(X_t^{\pi^*}, t) = u(X_t^{\pi^*}, A_t) \\ U(x,0) = u(x,0) = u_0(x) \end{cases}$$

where $u(x,t)$ is the variational utility input and A_t the stochastic market input

$$dU(X_t^{\pi}, t) = u_x(X_t, A_t)\sigma_t \pi_t \, dW_t$$

$$+ \underbrace{(u_t(X_t^{\pi}, A_t)\lambda_t^2 + u_x(X_t^{\pi}, A_t)\sigma_t\pi_t\lambda_t + \tfrac{1}{2}u_{xx}(X_t^{\pi}, A_t)\sigma_t^2\pi_t^2)}_{\leq 0} dt$$

- Variational utility input condition

$$\begin{cases} u_t \; u_{xx} = \frac{1}{2} u_x^2 \\ \\ u(x,0) = u_0(x) \end{cases}$$

- The optimal allocations in stock, π_t^*, and in bond, $\pi_t^{0,*}$,

$$\begin{cases} \pi_t^* = -\sigma_t^{-1} \lambda_t \dfrac{u_x(X_t^{\pi^*}, A_t)}{u_{xx}(X_t^{\pi^*}, A_t)} = \sigma_t^{-1} \lambda_t R_t \\ \\ \pi_t^{0,*} = X_t^{\pi^*} - \sigma_t^{-1} \lambda_t R_t \end{cases}$$

$$\boldsymbol{R_t = r(X_t^{\pi^*}, A_t) \quad ; \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}}$$

The local risk tolerance $r(x,t)$ and the subordinated

risk tolerance process R_t emerge as important quantities

Dynamic utility measurement

time t_1, information $\mathcal{F}_{t_1}$

asset returns

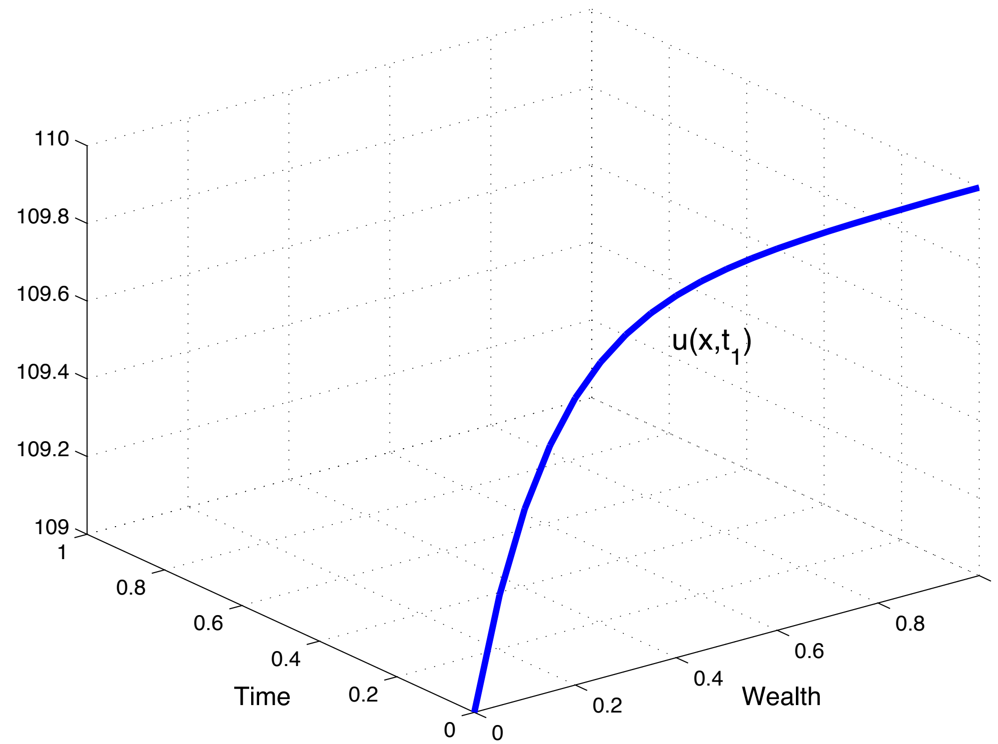
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_1) \dashrightarrow + \dashleftarrow u(x, t_1)$$

$$\Downarrow$$

$$\boxed{U(x, t_1; MI) \in \mathcal{F}_{t_1} \qquad \pi(x, t_1; MI) \in \mathcal{F}_{t_1}}$$

Dynamic utility measurement

time t_2, information $\mathcal{F}_{t_2}$

asset returns

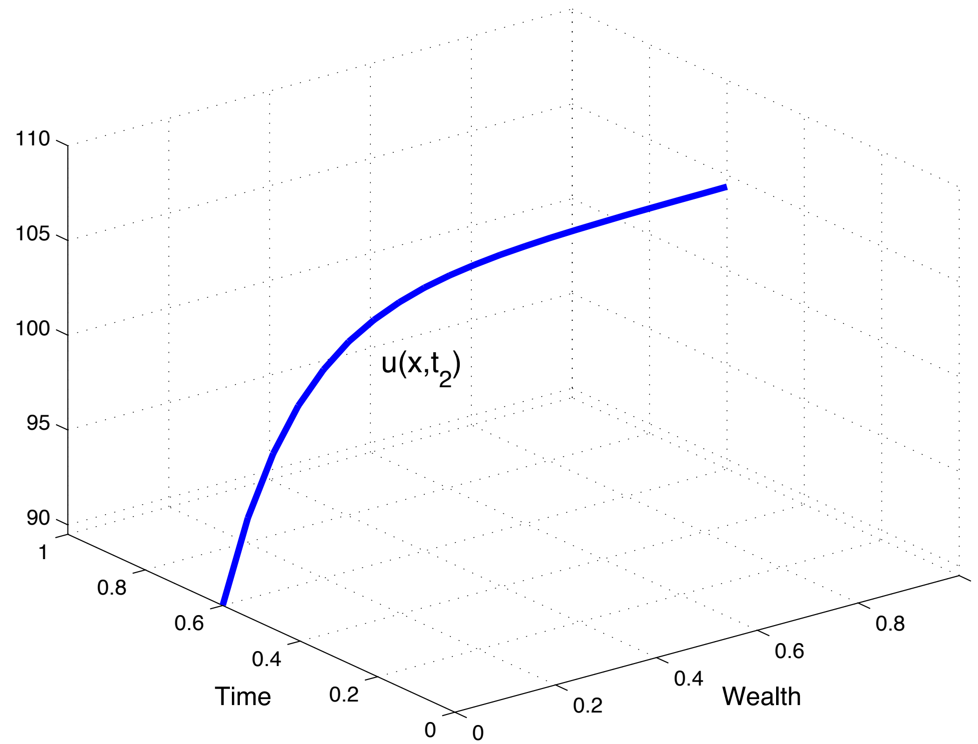
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_2) \dashrightarrow \; + \; \dashleftarrow u(x, t_2)$$

$$\Downarrow$$

$$\boxed{U(x, t_2; MI) \in \mathcal{F}_{t_2} \qquad \pi(x, t_2; MI) \in \mathcal{F}_{t_2}}$$

Dynamic utility measurement

time t_3, information $\mathcal{F}_{t_3}$

asset returns

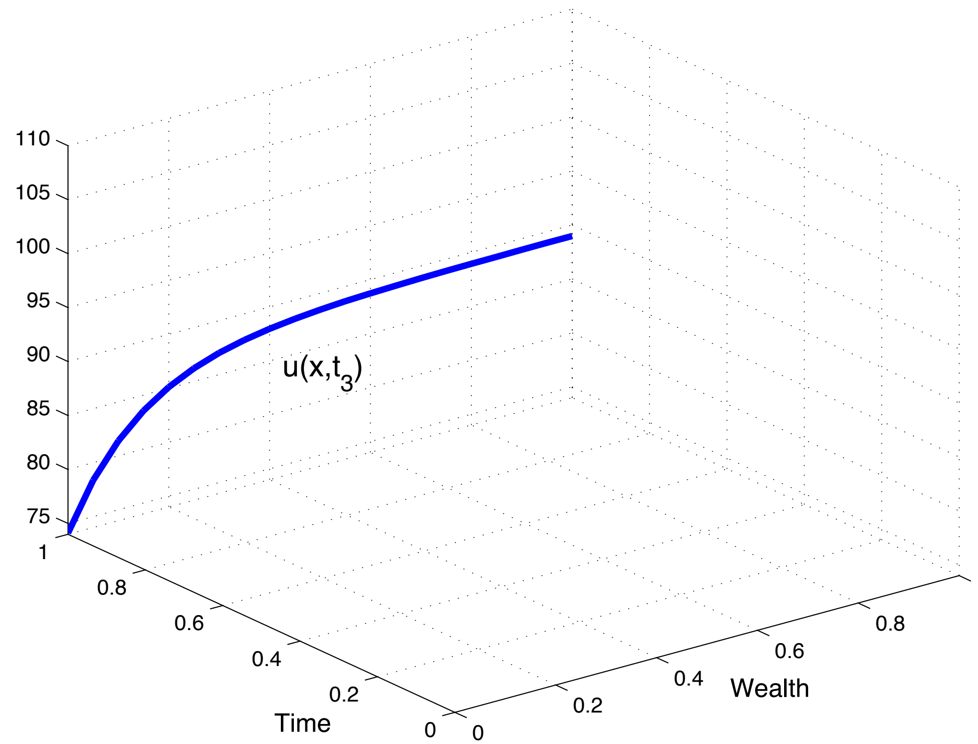
constraints

market view
away from equilibrium

benchmark numeraire

calendar time subordination

$\longrightarrow$



$$MI(t_3) \dashrightarrow + \dashleftarrow u(x, t_3)$$

$$\Downarrow$$

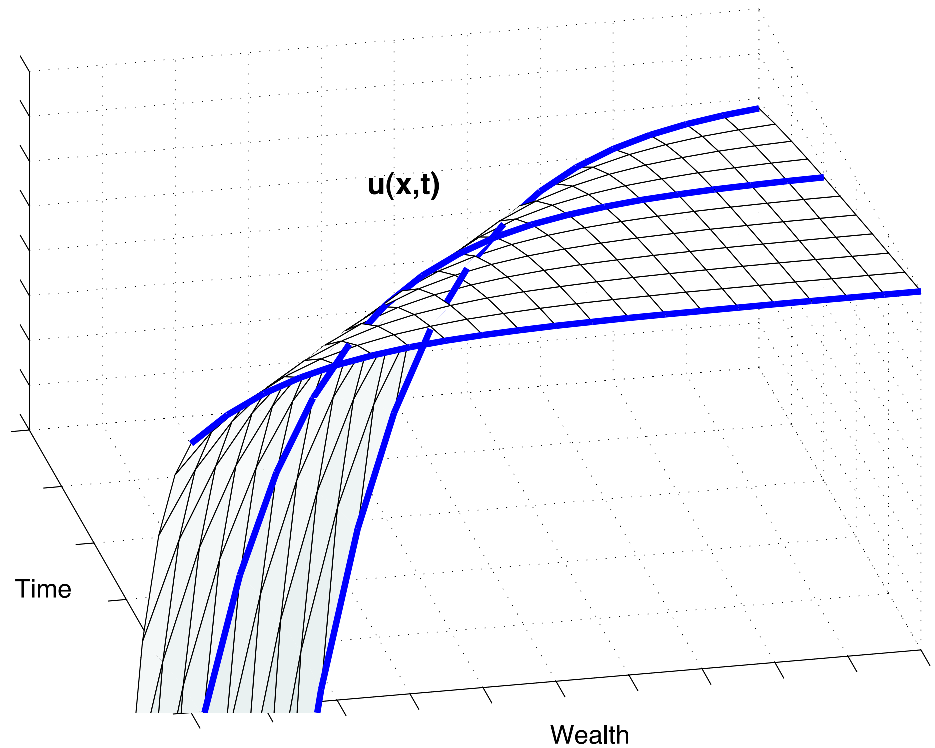
$$\boxed{U(x, t_3; MI) \in \mathcal{F}_{t_3} \qquad \pi(x, t_3; MI) \in \mathcal{F}_{t_3}}$$

Dynamic utility measurement

time t, information $\mathcal{F}_t$

asset returns additional market input

$\longrightarrow$



$$MI(t) \dashrightarrow + \dashleftarrow u(x,t)$$

$$\Downarrow$$

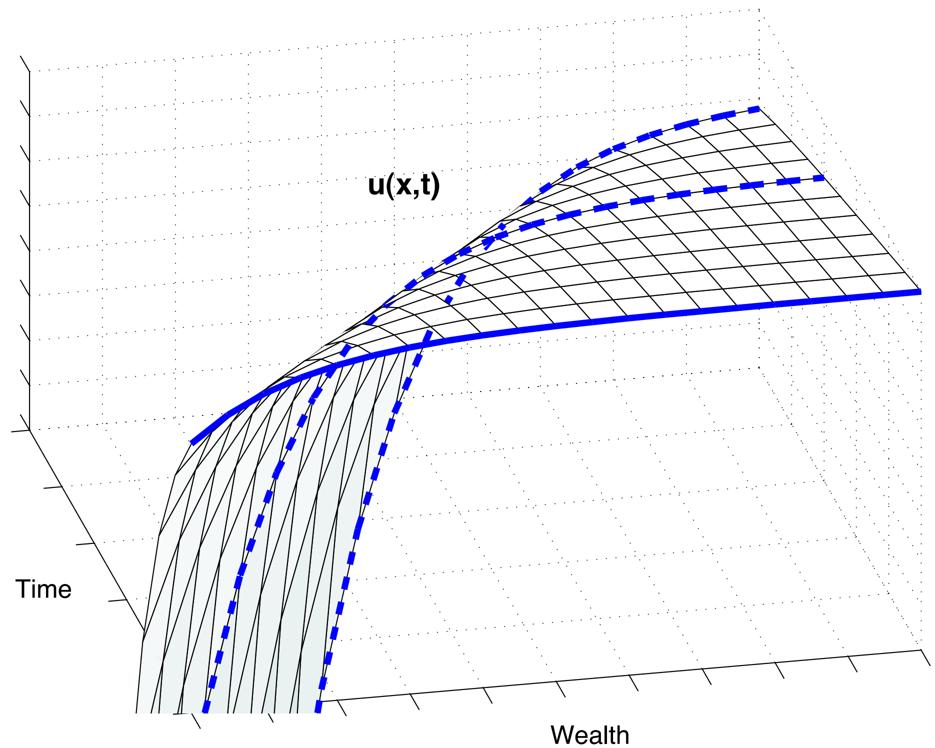
$$\boxed{U(X_t^*, t) \in \mathcal{F}_t \qquad \pi^*(X_t^*, t) \in \mathcal{F}_t}$$

Dynamic utility measurement

time t_1, information $\mathcal{F}_{t_1}$

asset returns
additional
market input

$\longrightarrow$



$$MI(t_1) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x, t_1)$$

$$\Downarrow$$

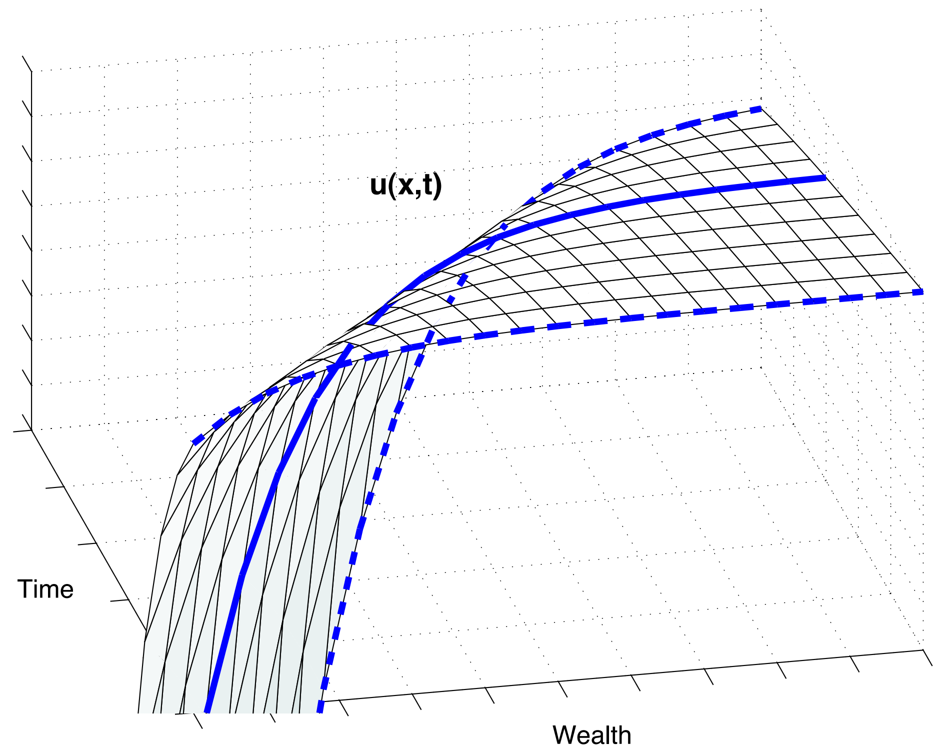
$$\boxed{U(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1} \qquad \pi^*(X^*_{t_1}, t_1) \in \mathcal{F}_{t_1}}$$

Dynamic utility measurement

time t_2, information $\mathcal{F}_{t_2}$

asset returns additional market input

$\longrightarrow$



$MI(t_2)$ $\dashrightarrow$ $+$ $\dashleftarrow$ $u(x, t_2)$

$\Downarrow$

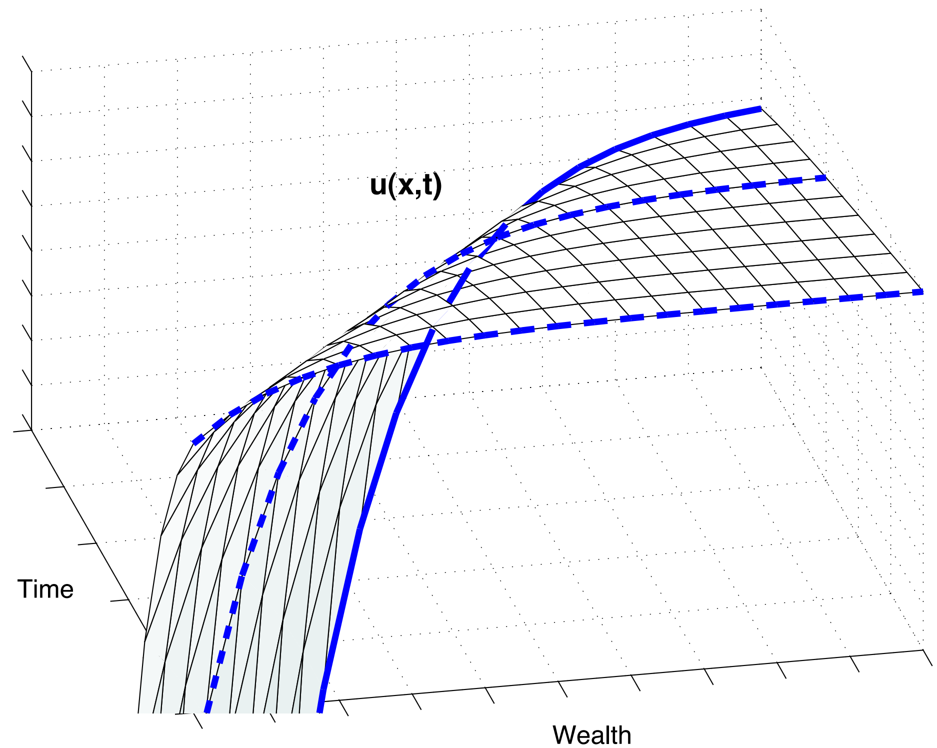
$$\boxed{U(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2} \qquad \pi^*(X^*_{t_2}, t_2) \in \mathcal{F}_{t_2}}$$

Dynamic utility measurement

time t_3, information $\mathcal{F}_{t_3}$

asset returns
additional
market input

$\longrightarrow$



$MI(t_3)$ $\dashrightarrow$ $+$ $\dashleftarrow$ $u(x, t_3)$

$\Downarrow$

$$\boxed{U(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3} \qquad \pi^*(X^*_{t_3}, t_3) \in \mathcal{F}_{t_3}}$$

Construction of a class of forward dynamic utilities

Creating the martingale that yields the optimal utility volume

Minimal model assumptions

Stochastic optimization problem “inverse” in time

Key idea

Stochastic input

Market

Variational input

Individual

Maximal utility — Optimal allocation

Variational input – utility surfaces

Utility surface

A model independent variational constraint on

impatience, risk aversion and monotonicity

- Initial utility datum

$$u_0(x) = u(x, 0)$$

- Fully non-linear pde

$$\begin{cases} u_t \; u_{xx} = \dfrac{1}{2} u_x^2 \\ u(x, 0) = u_0(x) \end{cases}$$

Utility transport equation

The utility equation can be alternatively viewed as a transport equation with slope of its characteristics equal to (half of) the risk tolerance

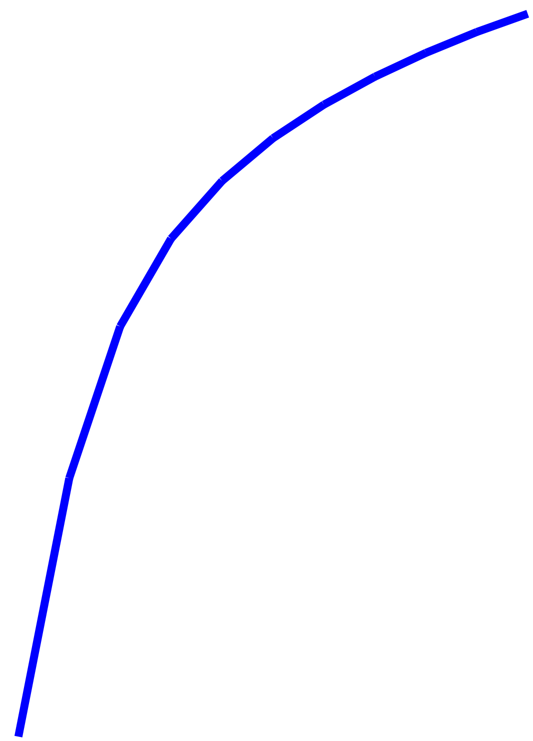
$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

$$\begin{cases} u_t + \dfrac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

Characteristic curves: $$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t),t)$$

Construction of utility surface $u(x, t)$ using characteristics

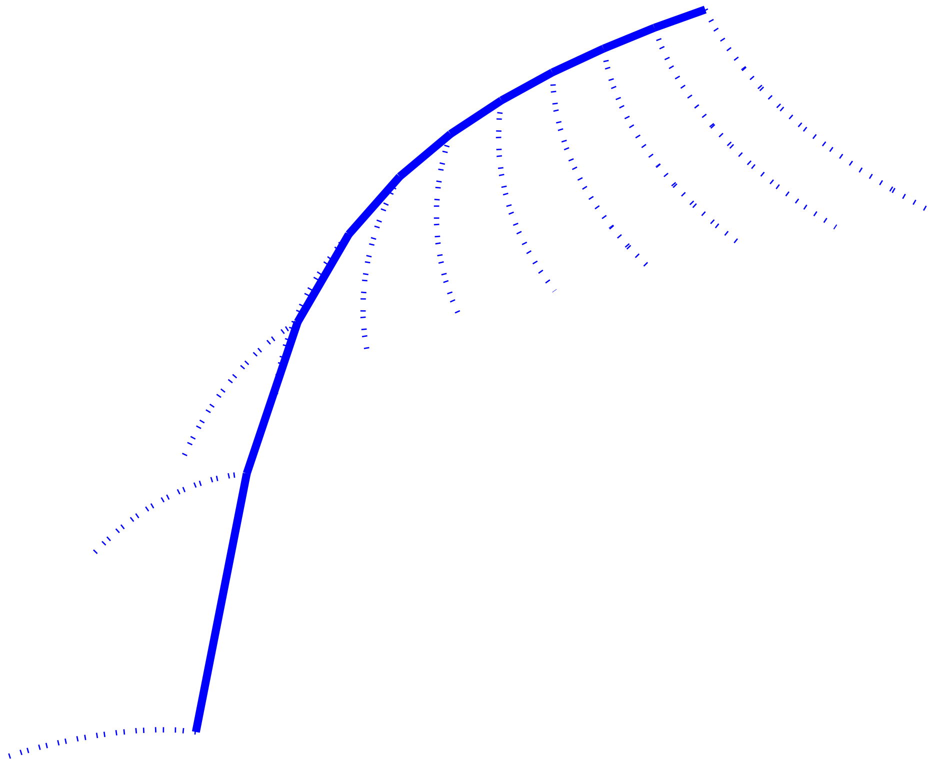
$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$



Utility datum $u_0(x)$

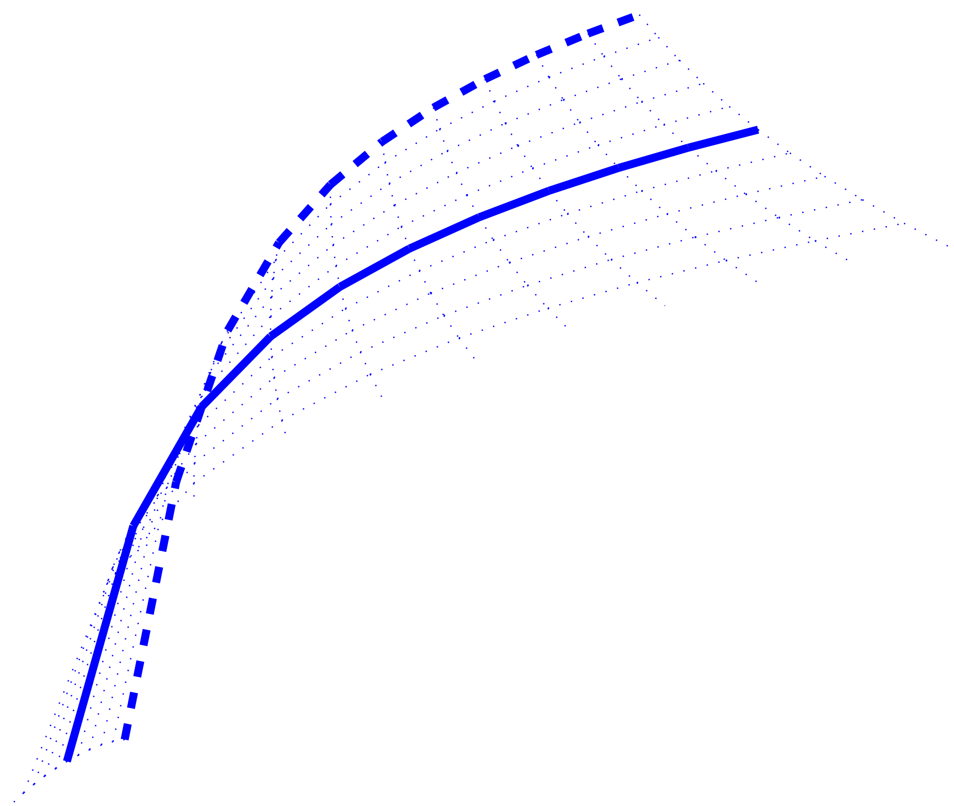
Construction of characteristics

$$\frac{dx(t)}{dt} = \frac{1}{2} r(x(t), t)$$

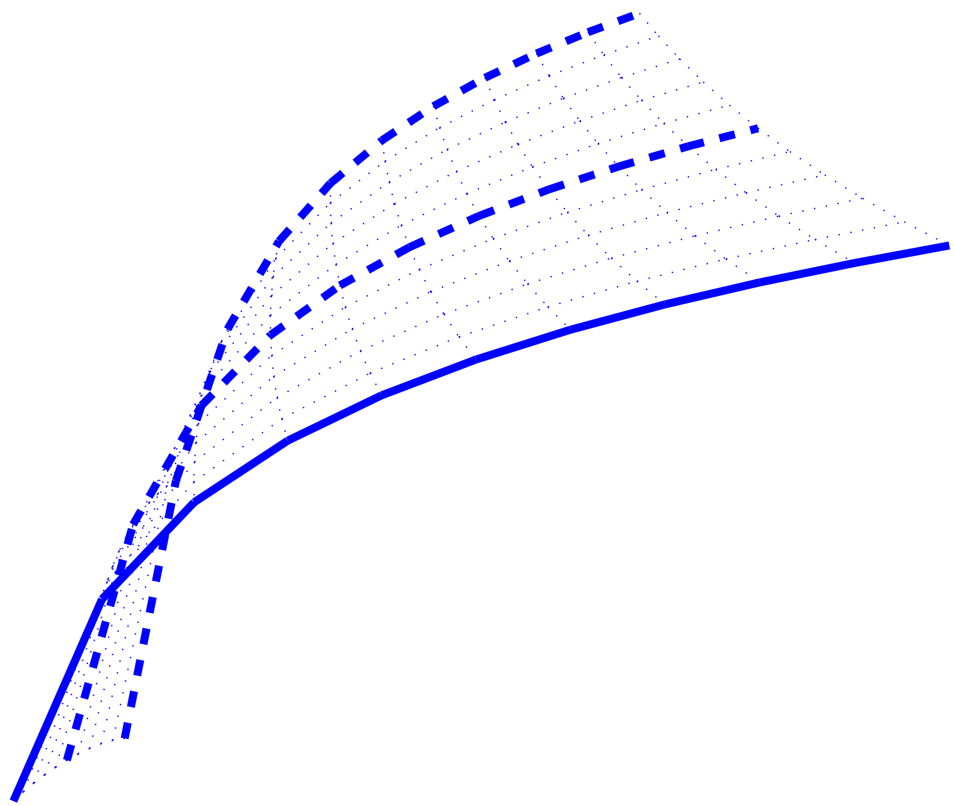


Utility datum $u(x, 0)$
Characteristic curves

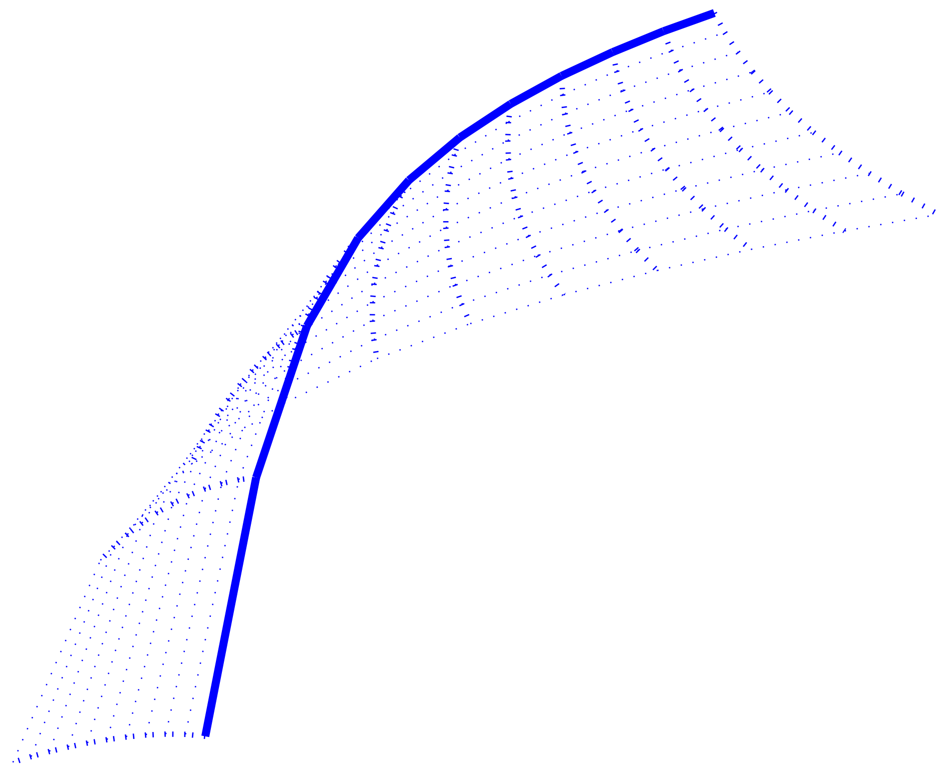
Propagation of utility datum along characteristics



Propagation of utility datum along characteristics



Utility surface $u(x, t)$



The risk tolerance pde

- Recall the utility equation

$$\begin{cases} u_t \ u_{xx} = \frac{1}{2} u_x^2 \\ u(x,t) = u_0(x) \end{cases}$$

- The local risk tolerance $\boldsymbol{r(x,t) = -u_x(x,t)/u_{xx}(x,t)}$ solves the autonomous equation of "fast diffusion type"

$$\begin{cases} \boldsymbol{r_t + \frac{1}{2} r^2 r_{xx} = 0} \\ \boldsymbol{r(x,0) = r_0(x)} \end{cases} \qquad \text{(FDE)}$$

The risk aversion pde

$$\gamma(x,t) = \frac{1}{r(x,t)}$$

Porous medium equation

$$\begin{cases} \gamma_t = \left(\dfrac{1}{\gamma}\right)_{xx} \\ \\ \gamma(x,0) = \dfrac{1}{r_0(x)} \end{cases} \qquad \text{(PME)}$$

Fast diffusion/Porous medium pde

$$v_t = \nabla \cdot (v^{-n} \nabla v)$$

- $n = 0$: heat conduction equation (infinite propagation speed)
- $n < 0$: gas propagation equation in a porous medium

  (finite propagation speed)
- $n > 0$: fast diffusion

  Cases $0 < n \leq 1$ widely studied

  (thermalised electron cloud, gas kinetics, limit

  of Carleman's model of the Boltzman eqn)

Risk tolerance/risk aversion pdes

$n = 2$

$$\gamma_t + \nabla(\gamma^{-2}\nabla\gamma) = 0$$

$$\Updownarrow \; r = \gamma^{-1}$$

$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

Difficulties

- Very limited results for $n > 1$
- Solutions blow up in finite time
- Inverse in time problem

Solutions to the risk tolerance equation

Classes of solutions

- “Additively” separable
  (special cases: log, power and exponential)

- Multiplicatively separable
  (special cases: log, power and exponential)

- Travelling waves

- Self-similar

“Additively separable” risk tolerance

$$r^2(x,t;\alpha,\beta) = m(x;\alpha,\beta) + n(t;\alpha,\beta)$$

$$m(x;\alpha,\beta) = \alpha x^2 \qquad n(x;\alpha,\beta) = \beta e^{-\alpha t}$$

$$r(x,t;\alpha,\beta) = \sqrt{\alpha x^2 + \beta e^{-\alpha t}} \qquad \alpha,\beta > 0$$

Utility surface

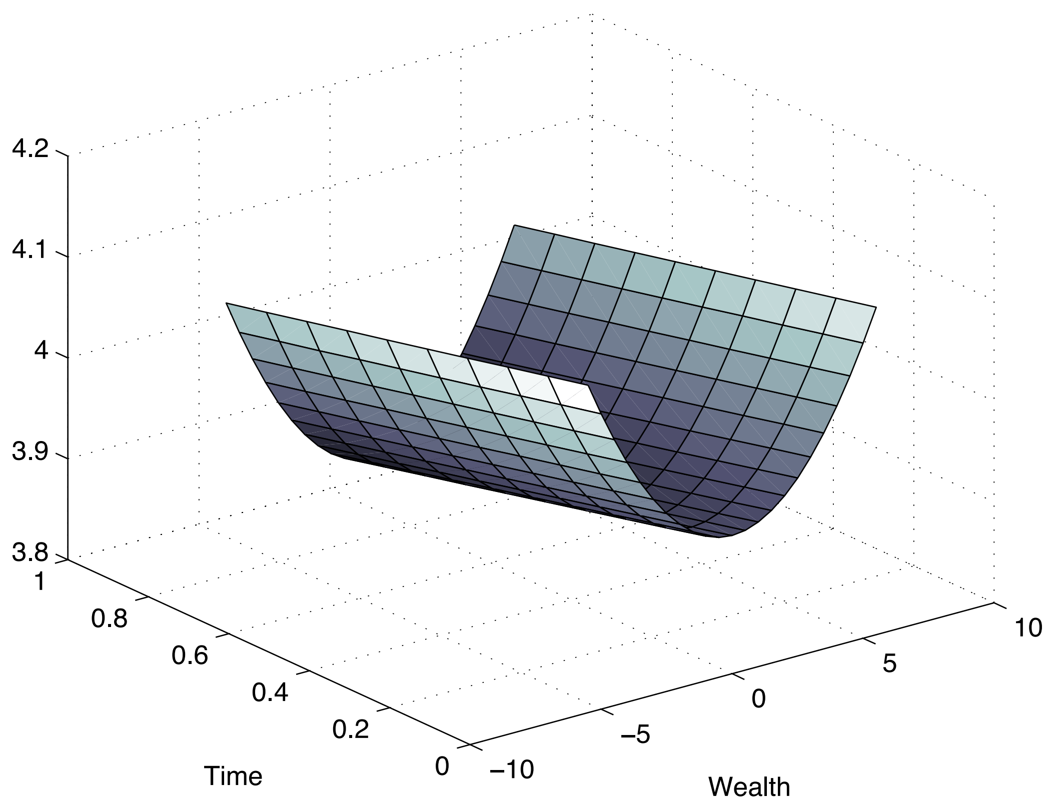
$$-\frac{u_x(x,t)}{u_{xx}(x,t)} = \sqrt{\alpha x^2 + \beta e^{-\alpha t}}$$

$$\Downarrow$$

$$u(x,t) = \int^x (\sqrt{\alpha}\; z + \sqrt{\alpha z^2 + \beta e^{-\alpha t}}\,)^{-1/\sqrt{\alpha}} + K_1(t)\, dz + K_2(t)$$

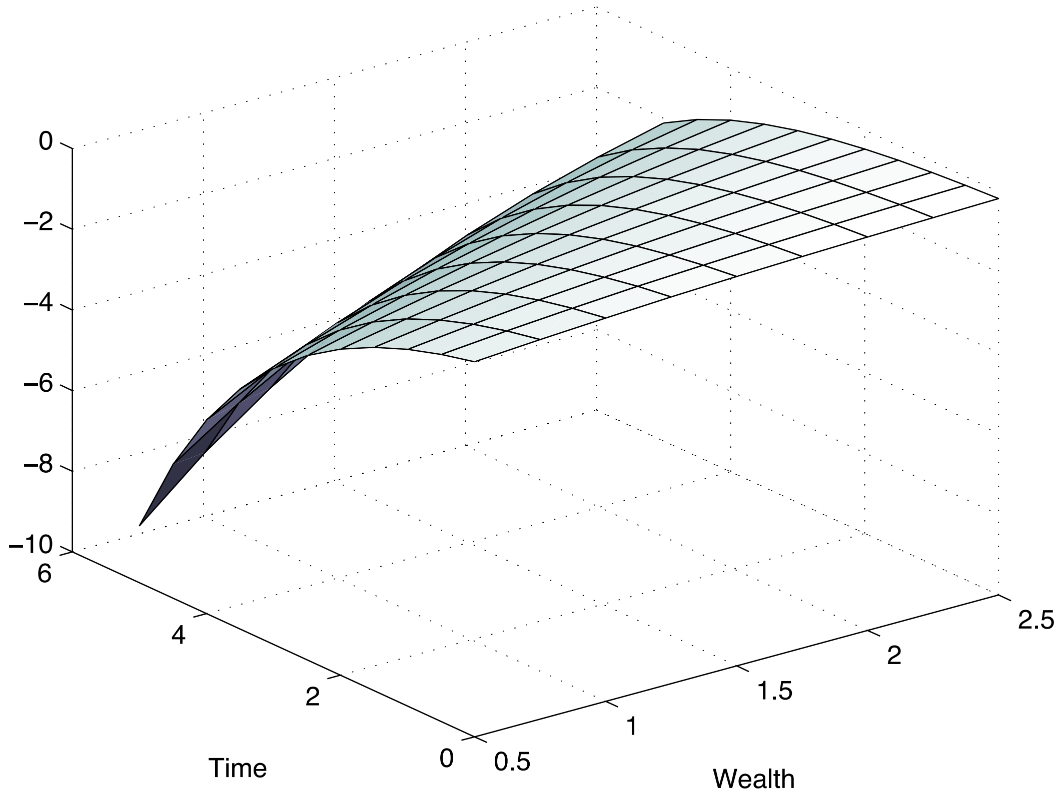
Need to analyze limiting cases for (α,β)

Risk tolerance $r(x,t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}}$



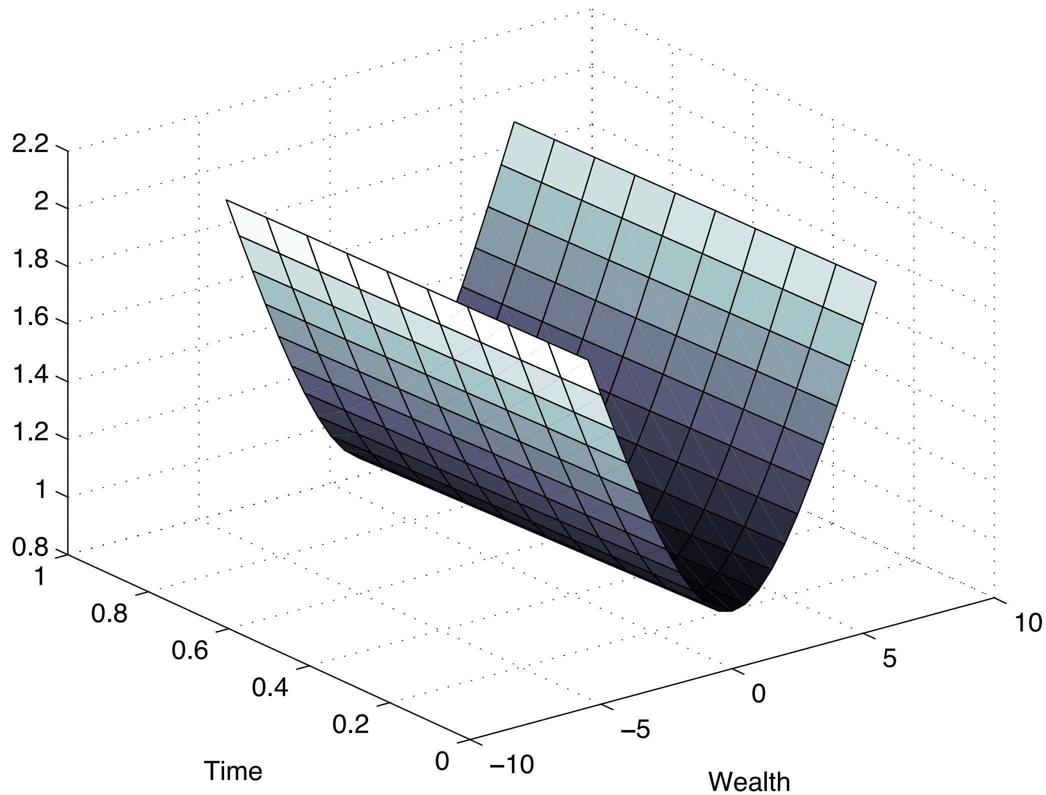
Utility surface $u(x,t)$ generated by

risk tolerance $\quad r(x,t) = \sqrt{0.05x^2 + 15.5e^{-0.05t}}$



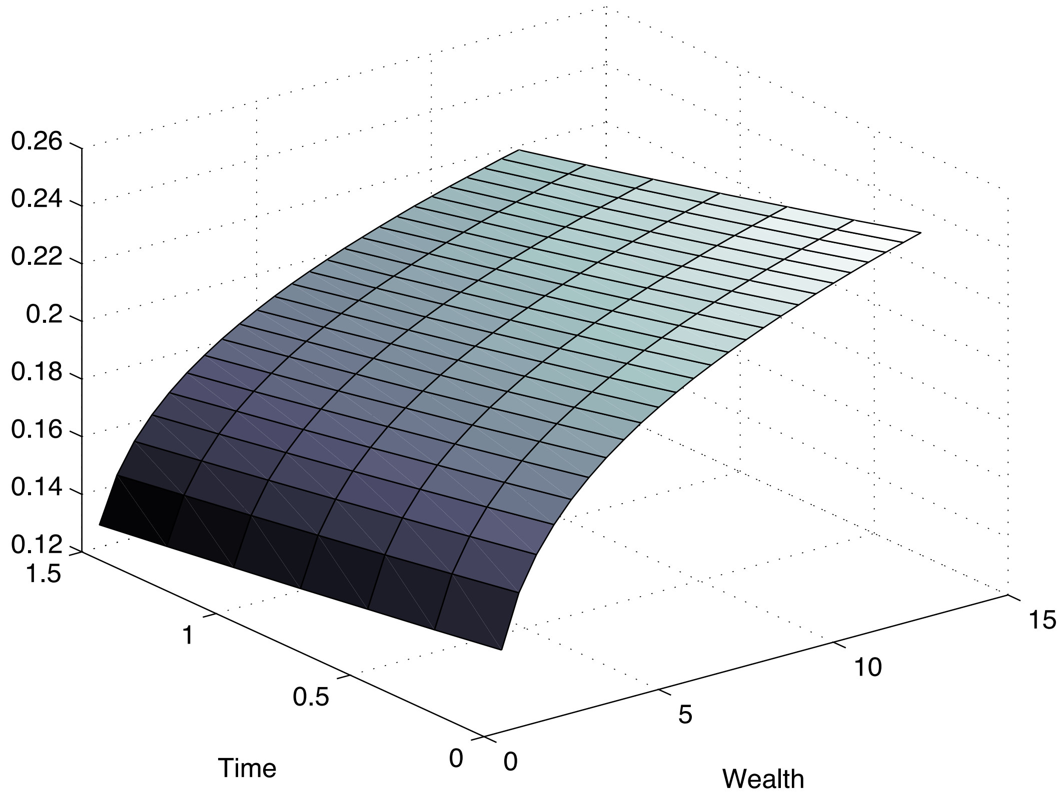
Characteristics: $\quad \dfrac{dx(t)}{dt} = \dfrac{1}{2}\sqrt{0.05x(t)^2 + 15.5e^{-0.05t}}$

Risk tolerance $r(x,t) = \sqrt{10x^2 + e^{-10t}}$



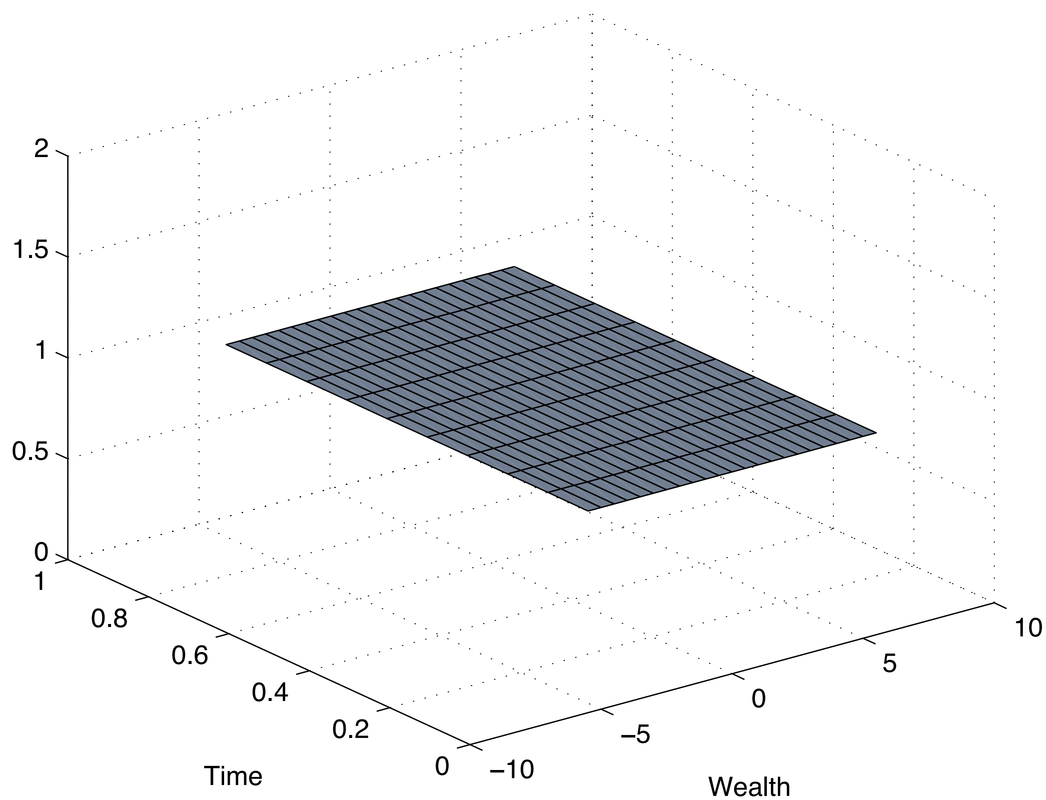
Utility surface $u(x,t)$ generated by

risk tolerance $r(x,t) = \sqrt{10x^2 + e^{-10t}}$



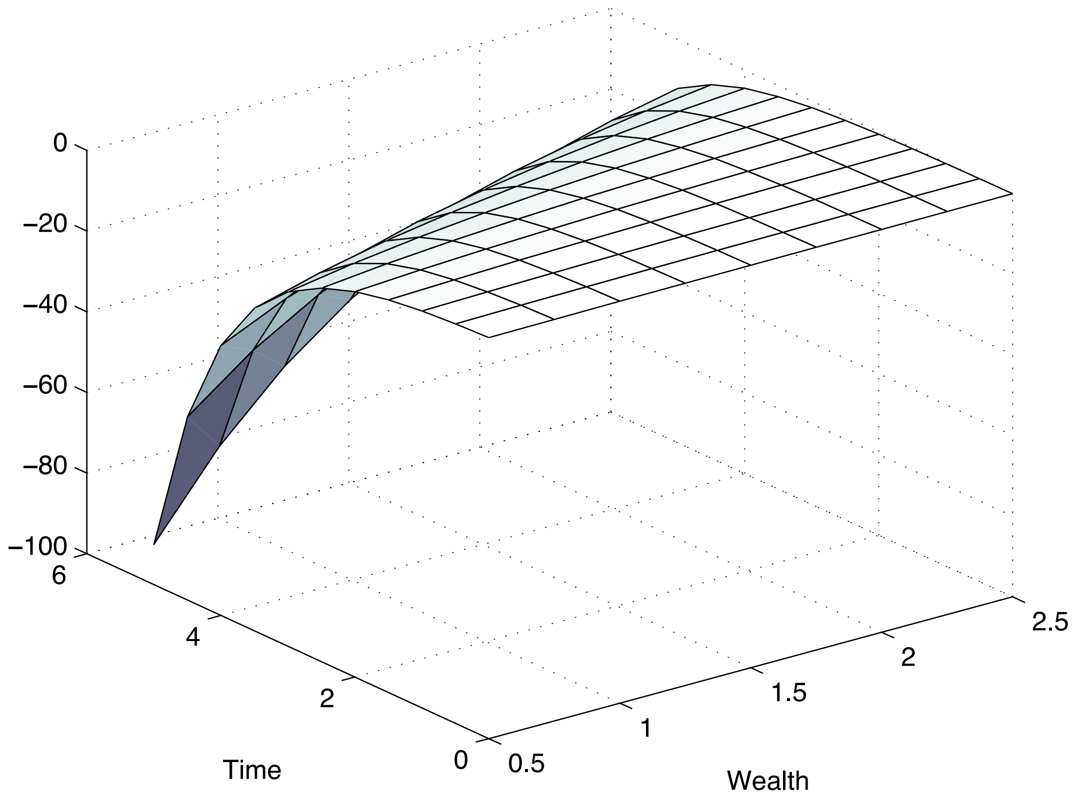
Characteristics: $$\frac{dx(t)}{dt} = \frac{1}{2}\sqrt{10x(t)^2 + e^{-10t}}$$

Risk tolerance $r(x,t;0,1)=\sqrt{0x^2+1}=1$



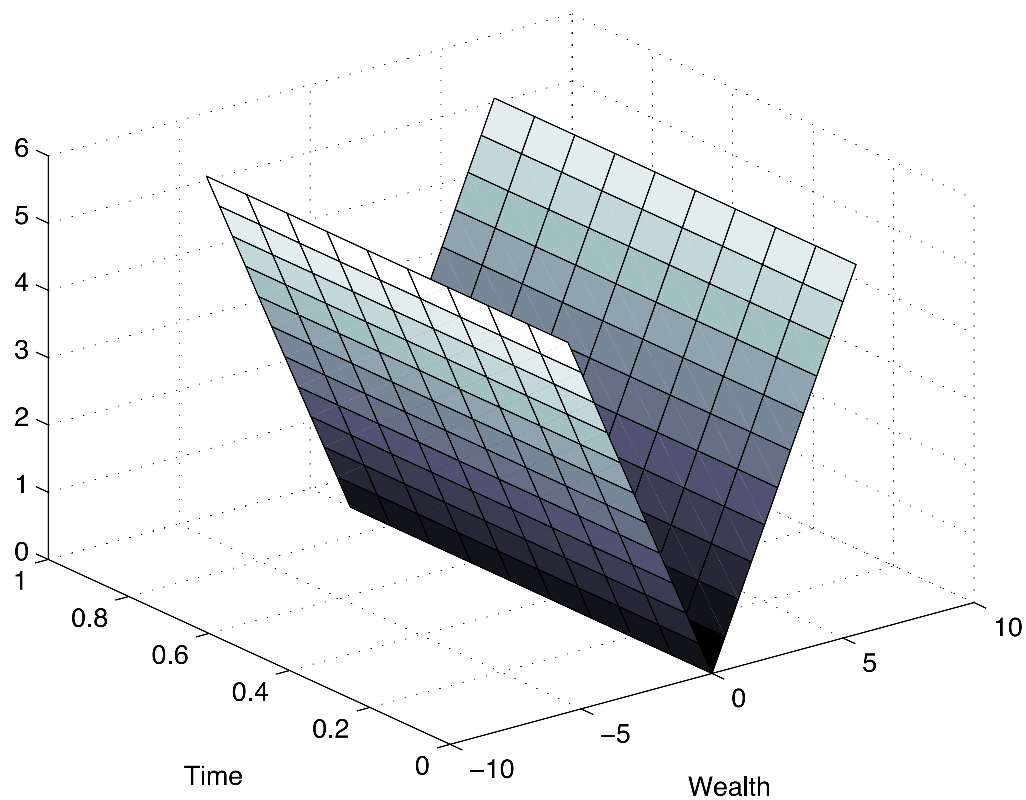
Utility surface $u(x,t) = -e^{-x+\frac{t}{2}}$ generated by

risk tolerance $r(x,t) = 1$

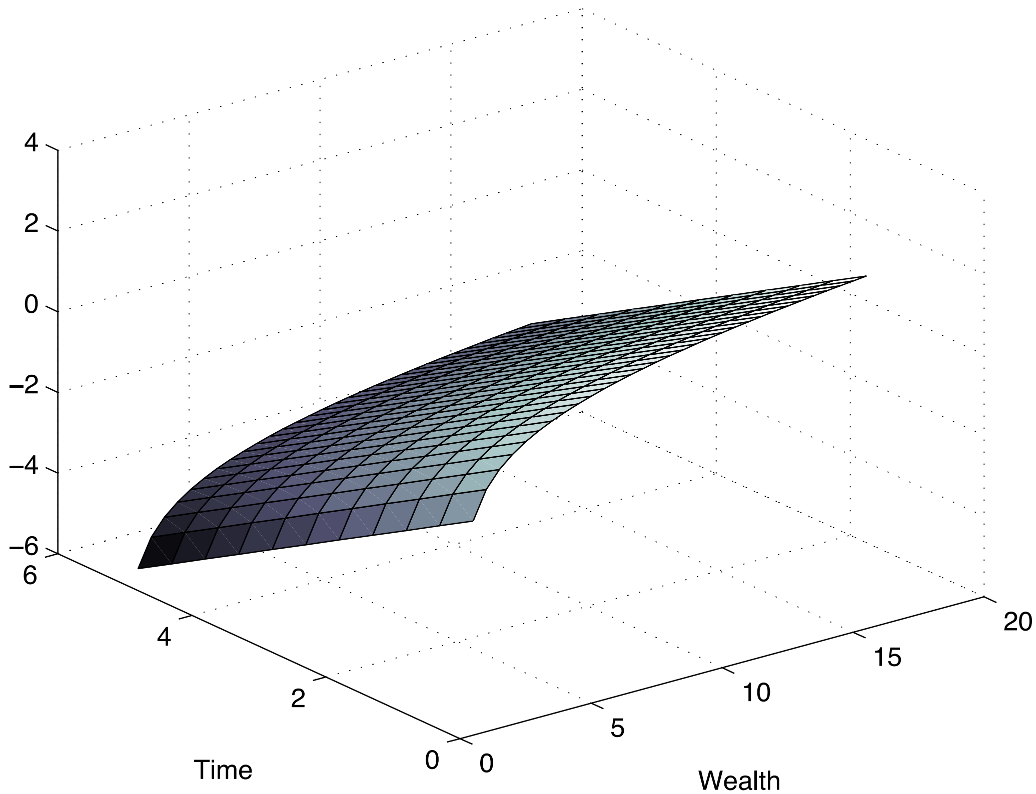


Characteristics: $\dfrac{dx(t)}{dt} = \dfrac{1}{2}$

Risk tolerance $r(x,t;1,0)=\sqrt{x^2+0e^{-t}}=|x|$

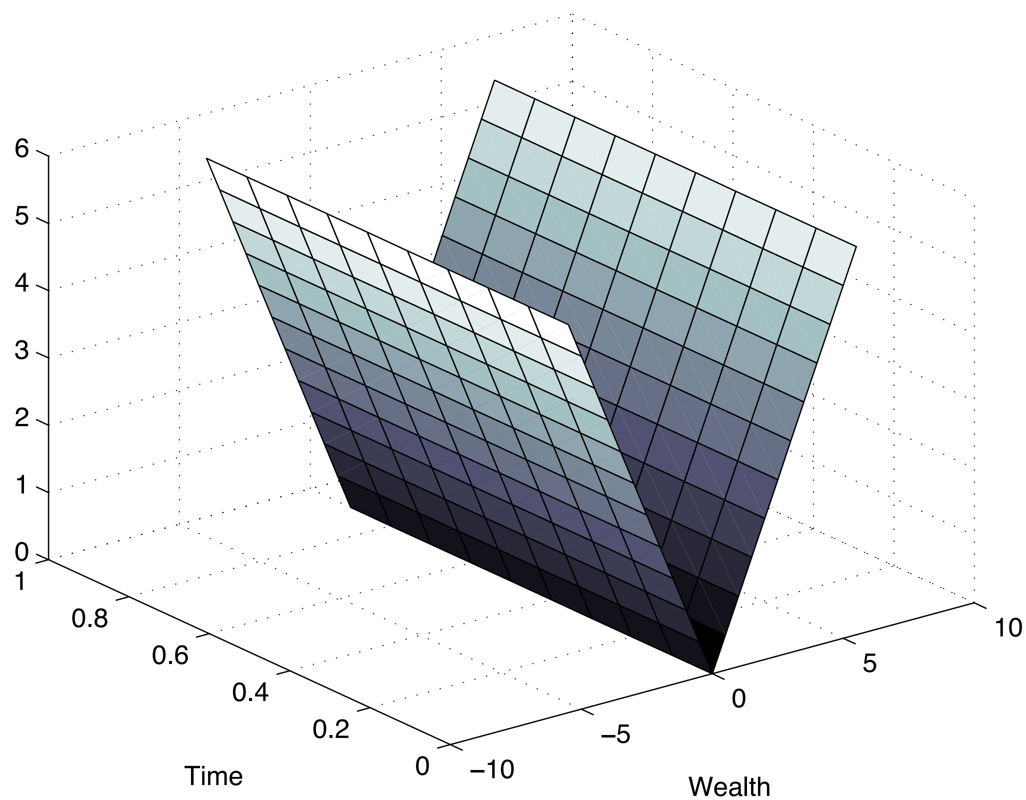


Utility surface $u(x,t) = \log x - \frac{t}{2}$, $x > 0$ generated by risk tolerance $\quad r(x) = x$



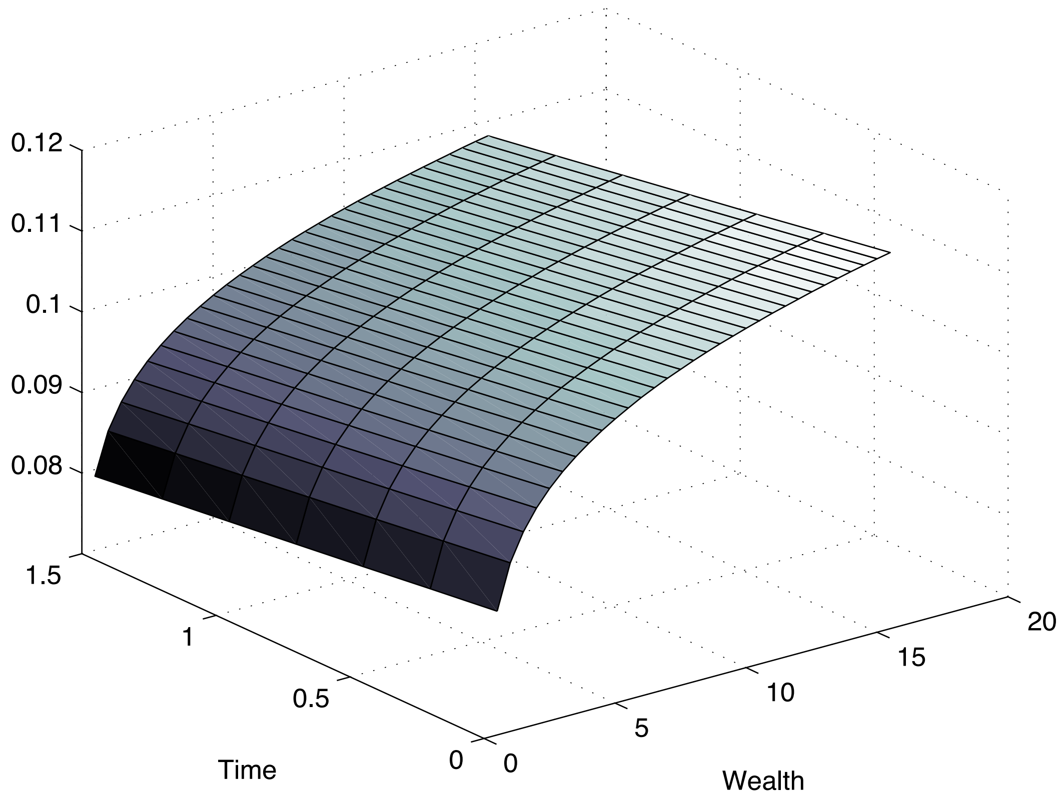
Characteristics: $\quad \dfrac{dx(t)}{dt} = \dfrac{1}{2}x(t)$

Risk tolerance $r(x,t;4,0)=\sqrt{4x^2+0e^{-4t}}=2|x|$



Utility surface $u(x,t) = 2\sqrt{x}\, e^{-\frac{t}{2}}$, $x > 0$ generated by

risk tolerance $r(x,t) = 2x$



Characteristics: $$\frac{dx(t)}{dt} = x(t)$$

Multiplicatively separable risk tolerance

$$r(x,t;\alpha,\beta) = m(x;\alpha)n(t;\beta)$$

$$m(x;\alpha) = \varphi(\Phi^{-1}(x;\alpha)) \qquad n(t;\beta) = \frac{1}{\sqrt{t+\beta}}\,, \qquad \beta > 0$$

$$\Phi(x;\alpha) = \int_{\alpha}^{x} e^{z^2/2}\,dz \qquad \varphi = \Phi'$$

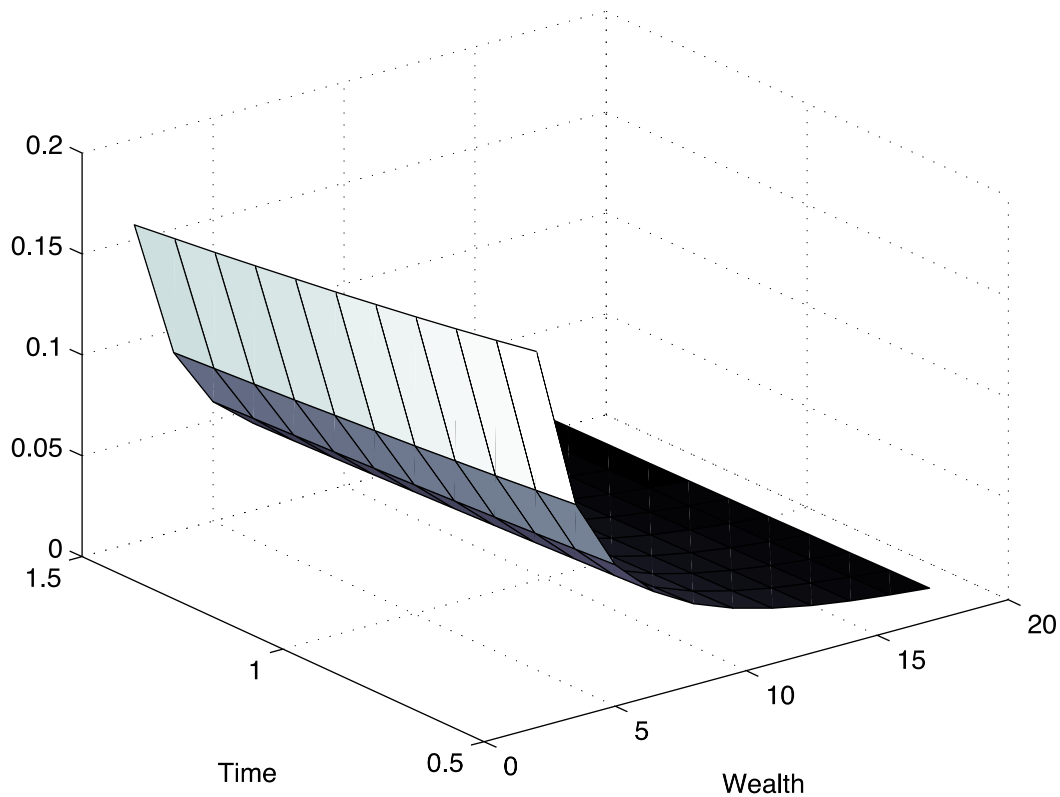
$$r(x,t;\alpha,\beta) = \frac{\varphi(\Phi^{-1}(x;\alpha))}{\sqrt{t+\beta}}$$

Utility surface

$$u(x,t) = \Phi(\Phi^{-1}(x;\alpha) - \sqrt{t+\beta})$$

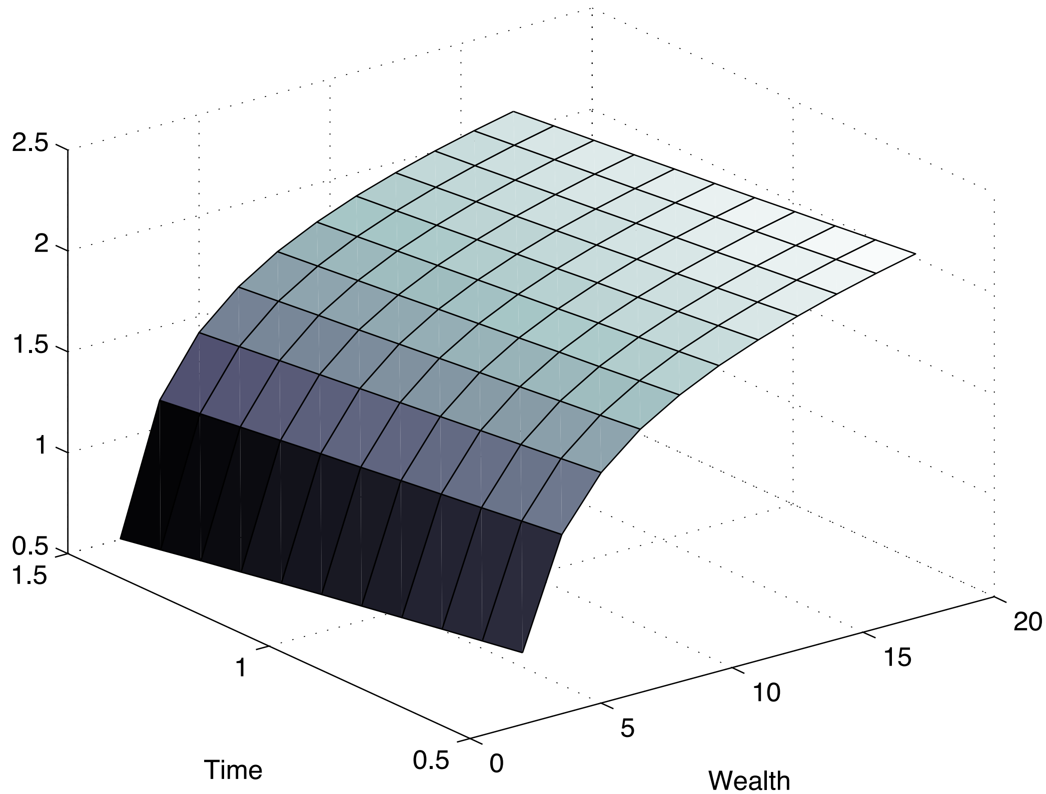
General classes by incorporating t-dependent integration constants

Risk tolerance $\quad r(x,t) = \dfrac{\varphi(\Phi^{-1}(x;0.5)}{\sqrt{t+5}}$



Utility surface $u(x,t) = \Phi(\Phi^{-1}(x;0.5) - \sqrt{t+5})$

generated by risk tolerance $r(x,t) = \dfrac{\varphi(\Phi^{-1}(x;0.5))}{\sqrt{t+5}}$

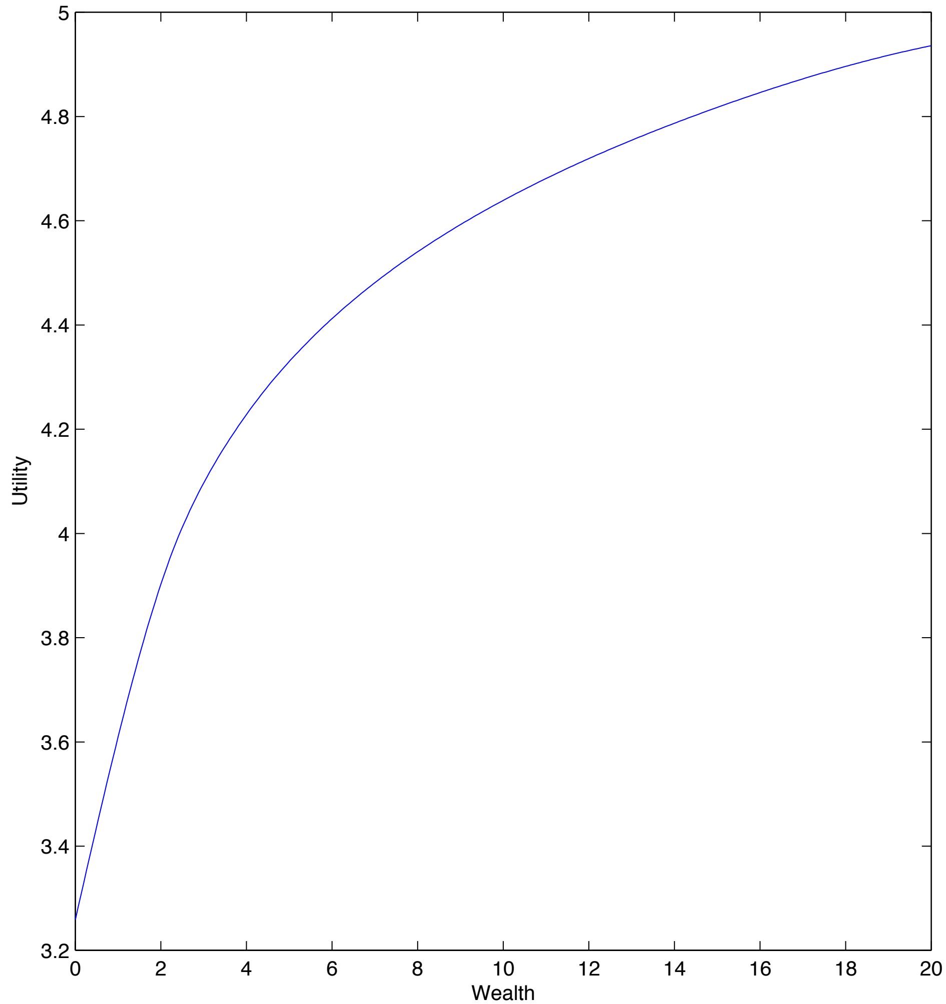


Characteristics: $$\frac{dx(t)}{dt} = \frac{\varphi(\Phi^{-1}(x(t);0.5))}{\sqrt{t+5}}$$

Utility function $u(x, t_0)$

(fixed time)

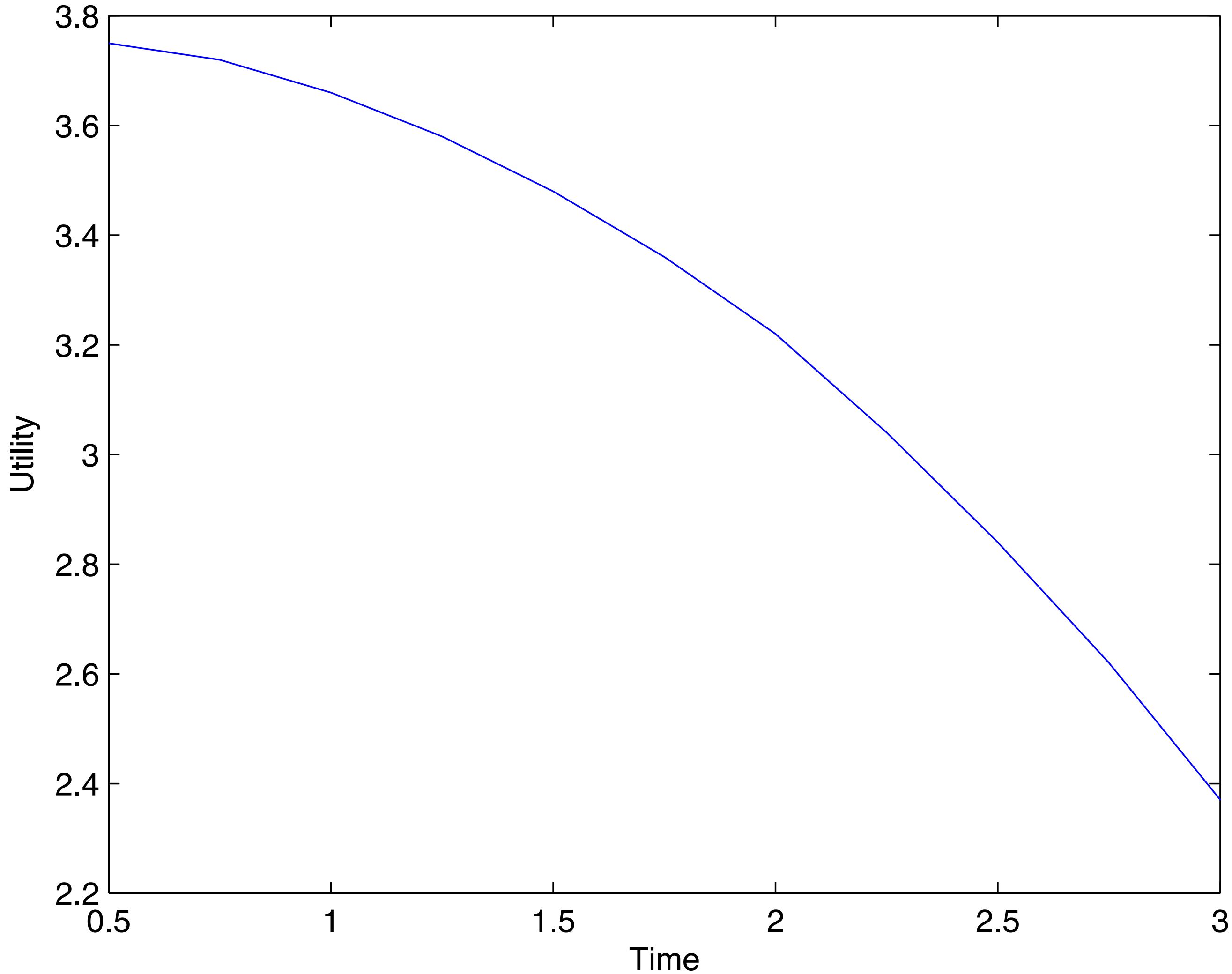
$t_0 = 2$



Utility function $u(x_0, t)$

(fixed wealth level)

$x_0 = 3.5$



- **Travelling wave solutions**

$$r(x,t) = \sqrt{\alpha|x| + \beta t + c}$$

$c > 0$, $\beta = \frac{\alpha^2}{4}$, $\alpha > 0$

- **Self-similar solutions**

$$r(x,t) = (t-c)^{\alpha} F(\frac{|x|}{(t-c)^{\beta}})$$

$2\beta = 2\alpha + 1$

F solves the second order ODE ; $\quad z = |x|(t-c)^{-\beta}$

$$\alpha F(z) = \beta z F'(z) + \frac{1}{2} z F^2(z) F''(z)$$

Summary on variational utility input

- Key state variables: wealth and risk tolerance

- Risk tolerance solves a fast diffusion equation posed inversely in time

$$\begin{cases} r_t + \frac{1}{2} r^2 r_{xx} = 0 \\ r(x,0) = -\frac{u_0'(x)}{u_0''(x)} \end{cases}$$

- Utility surface generated by a transport equation

$$\begin{cases} u_t + \frac{1}{2} r(x,t) u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

- Forward dynamic utility process constructed by compiling variational utility input and stochastic market input

Summary on optimal allocations

- Optimal portfolio $(\pi_t^*, \pi_t^{0,*})$ is directly computed and represented as a linear combination of the optimal wealth, $X_t^{\pi^*}$, and the subordinated risk tolerance process, $R_t = r(X_t^{\pi^*}, A_t)$

$$\pi_t^* = K_t^1 X_t^{\pi^*} + K_t^2 R_t$$

A_t, K_t^i : processes depending exclusively on market input

- Need to study the stochastic evolution of the solutions of the system

$$(X_t^{\pi^*}, R_t)$$

- An efficient frontier emerges.