Some remarkable properties
of
GAMMA PROCESSES

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On the occasion of D. MADAN's

60^{th} birthday

– University of Maryland –

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Motivation, interest in Gamma processes :

Searching for simple, yet pertinent examples of subordinators, which posses certain desirable properties, e.g:

(i) explicit expressions for their 1- dimensional marginals;

(ii) simple Lévy measure :

Recall

- $$E\left[\exp(-\lambda T_t)\right]$$

$$= \exp\left(-t\psi(\lambda)\right) \equiv \exp\left(-t\int \mathcal{L}(dx)\,(1-e^{-\lambda x}$$

- $$\langle \mathcal{L}, F\rangle = \lim_{(t\downarrow 0)} \frac{1}{t} E\left[F(T_t\right.$$

1) The Gamma process as a limit in law of tilted stable (α) subordinat

- If $(T_t, t \geq 0)$ is a subordinator, with Lévy exponent $(\Phi(\lambda), \lambda \geq 0)$, i.e.:

$$E[\exp(-\lambda T_t)] = \exp(-t\,\Phi(\lambda))$$

then its ν-Esscher transform satisfies

$$E[\exp(-\lambda T_t^{(\nu)})] = \exp(-t(\Phi(\lambda+\nu) - \Phi(\nu))$$

- If $(T_\alpha(u), u \geq 0)$ denotes the stable (α)-subordinator ($: 0 < \alpha < 1$, and: $\Phi_\alpha(\lambda) = \lambda^\alpha$)

then:
$$\left\{ T_\alpha^{(\nu)}\left(\frac{u}{\alpha}\right)\right\} \xrightarrow[(\alpha \to 0)]{f.d.} \left\{\frac{\gamma_u}{\nu}\right\}$$

- At the level of Lévy measures:

T_α	$T_\alpha^{(\nu)}$	$\frac{\gamma_u}{\nu}$
$C_\alpha \frac{dx}{x^{\alpha+1}}$	$C_\alpha \frac{dx}{x^{\alpha+1}} e^{-\nu x}$	$\frac{dx}{x} e^{-\nu x}$

2) Symmetrisation:

The variance-gamma process

It is not difficult to show – and might also be seen as a consequence of the previous convergence result – that:

$$\boxed{\left(\gamma_t - \gamma'_t,\ t \geq 0\right) \overset{(d)}{=} \left(\sqrt{2}\ \beta_{\gamma_t},\ t \geq 0\right)}$$

where on the LHS, γ and γ' are two independent gamma processes,

on the RHS, (β_u) is a BM indpdt of γ.

More about this latter – In particular, the bounded variation property of $(\beta_{\gamma_t}, t \geq 0)$

3) Brownian and Gamma bridges: a parallel.

- $\boxed{\text{Brownian bridges}}$

If $(B_u, u \geq 0)$ denotes standard Brownian motion, then:

$$\boxed{b_y^{(t)}(u) = \left(B_u - \frac{u}{t} B_t\right) + \frac{u}{t} y \quad , \quad u \leq t}$$

is an incarnation of $BB^{(t)}_{0 \to y}$, the Brownian bridge of duration t, starting at 0, ending at y.

Pf: $\left(B_u - \frac{u}{t} B_t, u \leq t\right)$ and B_t are independent.

- $\boxed{\text{Gamma bridges}}$

If $(\gamma_u, u \geq 0)$ denotes standard Gamma process, then an incarnation of $\Gamma B^{(t)}_{0 \to y}$ is:

$$\boxed{\left(y \frac{\gamma_u}{\gamma_t} , u \leq t \right)}$$

Pf: $D_u^{(t)} \equiv \gamma_u / \gamma_t$, $u \leq t$, the Dirichlet process of duration t, is independent from γ_t.

• $\boxed{\text{An important difference}}$

→ Integrals of BM and BB with deterministic integrands are Gaussian Variables

→ On the other hand,

$$\int_0^t h(u)\, d\gamma_u$$ is not so easily characterized!

Note:

$$\int_0^t h(u)\, d\gamma_u \overset{(d)}{=} \left(\int_0^t h(u)\, dD_u^{(t)} \right) \gamma_t$$

(Independence)

This brings us to the notion of

$\boxed{\text{Generalized Gamma Convolutions}}$

$\boxed{\mathcal{GGC}}$

Among Subordinators,

there are two important subclasses

$$\boxed{\mathcal{GGC} \subset \mathcal{SD} \subset \mathcal{S}}$$

$\mathcal{S}$ ≡ the whole family /set/ of subordinators

$\mathcal{SD}$ ≡ the self-decomposable ones, ie:

$$T_1 \overset{(d)}{=} c\,T_1 + \underbrace{R_c}_{\text{"Residual"}} \qquad (0 < c < 1)$$

iff $\boxed{\mathcal{L}(dx) = \frac{dx}{x}\, h(x) \ , \quad h \downarrow}$

$\mathcal{GGC}$: $\underline{h(x) = \int \nu(dy)\, e^{-xy}}$ $\quad \underline{\nu(dy) \geq 0, \ \sigma\text{-finite}}$

In particular:

$$\underline{h(x) = \delta\, E\left[\exp(-x\, G)\right]}$$

for some $\delta > 0$, $G \geq 0$.

Assume, for simplicity: $\delta = 1$.

and $\mathcal{G}(x) = P(G \leq x)$

assumed to be strictly increasing, and continuous

Then:

$$T_t \overset{(d)}{=} \left(\int_0^t \frac{d\gamma_u}{\mathcal{G}^{-1}\left(\frac{u}{t}\right)} \right)$$

$$\equiv \left(\int_0^t \frac{dD_u^{(t)}}{\mathcal{G}^{-1}\left(\frac{u}{t}\right)} \right) \cdot \gamma_t$$

independence ---

$\boxed{\text{An interesting example}}$

$$E\left(\exp(-\lambda T_t)\right) \equiv \left(\sqrt{1+\lambda} - \sqrt{\lambda}\right)^{2t}$$

corresponds to: • $G = \beta_{(1/2,\, 1/2)}$

• $T_t \overset{(d)}{=} \int_0^t \frac{d\gamma_u}{\sin^2\left(\frac{\pi u}{2t}\right)}$

4) The BB filtration and the ΓB filtration

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• $\boxed{\text{The BB filtration}}$

With the previous notation, we introduce:

$$\mathcal{B}_t = \sigma\{B_s, s \le t\},$$

$$\hat{\mathcal{B}}_t = \sigma\{b_y^{(t)}(u), u \le t\} \equiv \sigma\{b_0^{(t)}(u), u \le t\}$$

Note that $\{\hat{\mathcal{B}}_t\}$ is a strict subfiltration of $\{\mathcal{B}_t\}$

and that it is the natural filtration of:

$$\boxed{\left\{\hat{B}_t = B_t - \int_0^t \frac{ds}{s} B_s,\ t \ge 0\right\}} \text{ a new BM.}$$

• $\boxed{\text{The ΓB filtration}}$

Likewise, introduce: $\mathcal{G}_t = \sigma\{\gamma_u, u \le t\}$

the natural filtration of γ,

and: $\hat{\mathcal{G}}_t = \sigma\left\{D_u^{(t)} \equiv \frac{\gamma_u}{\gamma_t}, u \le t\right\}$

$\{\hat{\mathcal{G}}_t\}$ is a strict subfiltration of $\{\mathcal{G}_t\}$

Theorem (Emery-Y): For each $s>0$, the formula:

$$\int_{u_s(x)}^{\infty} \frac{e^{-z}dz}{z} = \int_x^{\infty} \frac{dy}{y(1+y)^s}$$

defines a bijection $u_s: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, and:

$$\hat{\gamma} \quad \sum \quad u\left(\frac{\Delta\gamma_s}{\ldots}\right)$$

is a new Gamma process, which generates $\{\hat{\mathcal{G}}_t\}$.

Question: What happens when one iterates these transforms??

ie: on the BM side:

$$
\begin{array}{ccc}
(B_t) & \longrightarrow & \mathcal{T}(B)_t = \hat{B}_t \\
 & & \downarrow \\
 & & \mathcal{T}^2(B) \\
 & & \downarrow \\
 & & \mathcal{T}^n(B)
\end{array}
$$

For each t, $B_t, B_t^{(1)}, \ldots, B_t^{(n)}, \ldots$ are independent,

$$B_t^{(n)} = \int_0^t L_n\left(\log \frac{t}{s}\right) dB_s$$

(Laguerre polynomials...)

Thm: $\mathcal{T}$ is ergodic

5) Brownian Motion and the Gamma process as special Harnesses.

- If $(X_t, t \geq 0)$ is a Lévy process such that $E(|X_1|) < \infty$, then it is a Harness in the sense of Hammersley, i.e:

$$\boxed{E\left[\frac{X_c - X_b}{c-b} \,\Big|\, \mathcal{F}^X_{a,d}\right] = \frac{X_d - X_a}{d-a}}$$

where $\mathcal{F}^X_{a,d} \equiv \sigma\{X_u, u \leq a ; X_v, v \geq d\}$

$$a < b < c < d.$$

- (B_t) and (γ_t) are special harnesses as:

(*) if $X = B$, $\dfrac{X_c - X_b}{c-b} - \dfrac{X_d - X_a}{d-a}$

(**) if $X = \gamma$, $\dfrac{X_c - X_b}{X_d - X_a}$

are independent from $\mathcal{F}^X_{a,d}$.

Question: Do these properties characterize BM and Γ among Lévy processes?

Conjecture: YES

6) Absolute continuity relations.

"Cameron-Martin" theorems

- For every $t \geq 0$, measurable functional F, and every $a > 0$, one has:

$$(\dagger)\quad \boxed{E\left[F\left(a\gamma_u, u \leq t\right)\right] = E\left[F\left(\gamma_u, u \leq t\right)\frac{\exp\left(-\left(\frac{1}{a}-1\right)\gamma_t\right)}{a^t}\right]}$$

- For every $t \geq 0$, measurable functional F, for any $\nu \in \mathbb{R}$:

$$E\left[F\left(B_u + u\nu;\, u \leq t\right)\right] = E\left[F\left(B_u, u \leq t\right)\exp\left(\nu B_t - \frac{\nu^2 t}{2}\right)\right]$$

Pf of $(\dagger)$: Write $a\gamma_u = a\left(\frac{\gamma_u}{\gamma_t}\right)\gamma_t$, and use the independence property ---

7) Space-time Harmonic functions

In relation with the $\boxed{\text{exponential martingales}}$

$(\lambda \in \mathbb{R})$ $\qquad \exp\left(\lambda B_t - \frac{\lambda^2 t}{2}\right), \ t \geq 0$

$(\lambda \geq 0)$ $\qquad (1+\lambda)^u \exp(-\lambda \gamma_u), \ u \geq 0$

there are the classical series expansions:

$$\begin{cases} \exp\left(\lambda x - \frac{\lambda^2 t}{2}\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x,t) \\ (1+\lambda)^u \exp(-\lambda g) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \tilde{C}_n(u,g) \end{cases}$$

where: • $\boxed{H_n(x,t) = t^{n/2} h_n\left(\frac{x}{\sqrt{t}}\right)}$,

and $(h_n; n=0,1,2,\dots)$ are the classical Hermite polynomials;

• $\boxed{\left(\tilde{C}_n(u,g)\right)}$ are the monic Charlier polynomials

$(\longrightarrow$ Schoutens / LNStat 126, (2000))

Thus:

$\boxed{\begin{array}{l} H_n(B_t, t), \ t \geq 0 \\ \tilde{C}_n(t, \gamma_t), \ t \geq 0, \end{array} \quad \text{are martingales}}$

Note that there are Poissonian counterparts to the previous Gamma martingales, which may be obtained by inverting time and space — precisely, if $(N_t, t \geq 0)$ denotes the standard Poisson process:

$(1+\lambda)^{N_t} \exp(-\lambda t)$	$(1+\lambda)^{u} \exp(-\lambda \gamma_u)$
$\tilde{C}_n(N_t, t)$	$\tilde{C}_n(u, \gamma_u)$

are martingales

[Explanation: The sequence of jump times $\{T_n\}$ of $(N_t, t \geq 0)$ is the trace of $(\gamma_u, u \geq 0)$ on the integers ...

8) The Gamma process as the inverse local time of a diffusion

- Let $0<\alpha<1$, and $(R_{\alpha}(t), t\geq 0)$ denote the BES process with index α, or dimension $d=2(1-\alpha)$. Then, this process reflects instantaneously ~~at~~ in 0, and admits a local time $(\ell_t, t\geq 0)$ such that $((R_{\alpha}(t))^{2\alpha}-\ell_t, t\geq 0)$ is a mart....

  Let $\tau_{\ell}=\inf\{t : \ell_t>\ell\}$, $\ell\geq 0$.

  Then, $(\tau_{\ell}, \ell\geq 0)$ is a stable(α)-subordinator.

- By lifting the convergence result of Titled stable(α) subordinators towards the Gamma process, we obtain the following

Theorem (C. Donati-Martin, Y. / Publ. RIMS; 2006)

Let $\nu>0$. The diffusion on $[0,\infty)$, with 0 instantaneously reflecting, and inf. gen:

$$\mathcal{L}^{\nu\downarrow}=\frac{1}{2}\frac{d^2}{dx^2}+\left(\frac{1}{2x}+\sqrt{2\nu}\,\frac{K_0'}{K_0}\left(\sqrt{2\nu}\,x\right)\right)\frac{d}{dx}$$

admits $\left(\frac{1}{\nu}\gamma_t, t\geq 0\right)$ as inverse local time at 0

9) Gamma processes time-changes

- Subordinators of the form:

$$\left(T_{\gamma_t} , t \geq 0 \right)$$

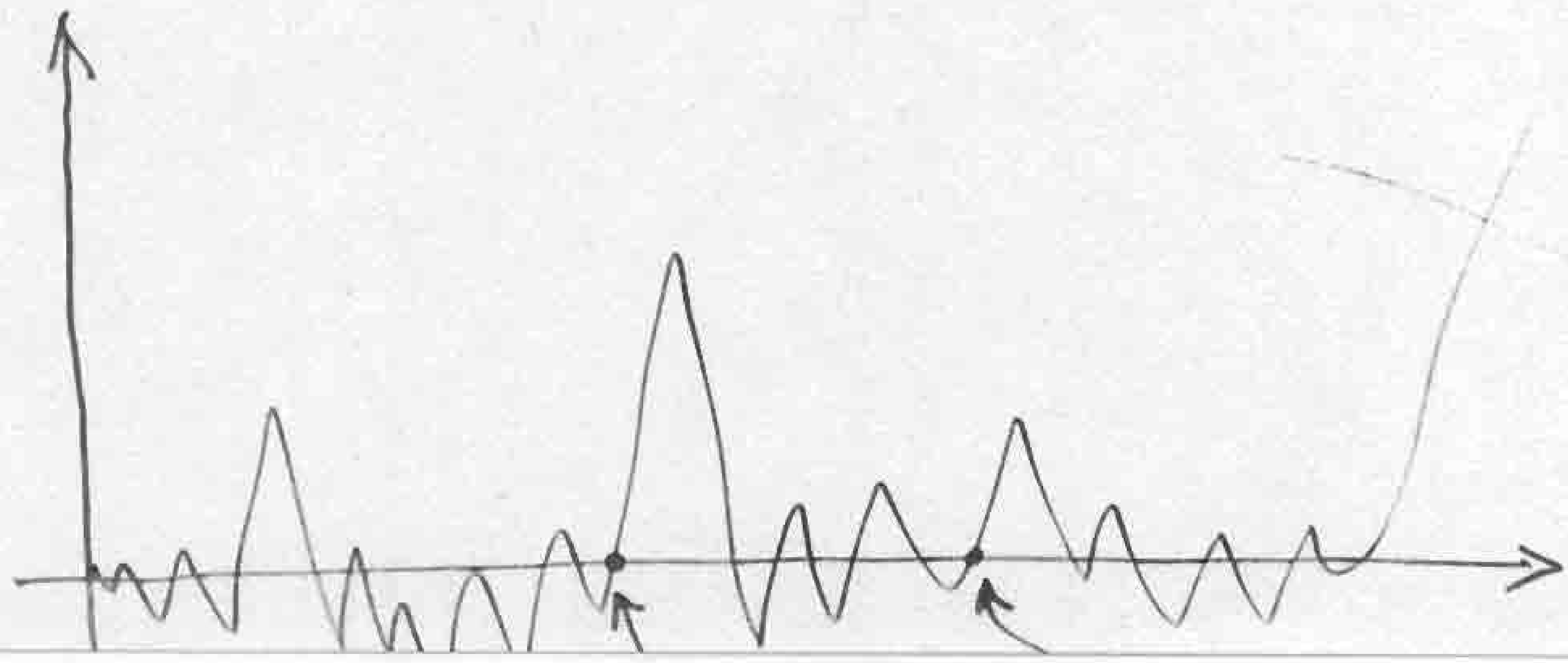
where $(T_u, u \geq 0)$ is independent from $(\gamma_t, t \geq 0)$ arise quite naturally as follows:

consider a nice real-valued diffusion (X_u) and (A_t) an add. functional, e.g:

$$A_t = \int_0^t ds\, g(X_s).$$

Then, $\left(A_{\tau_\ell}, \ell \geq 0\right)$ is a subordinator –

(the basis of Itô's excursion theory ---).



Now consider an indpdt exponential time $\mathbb{e}_\theta$

$$\mathbb{e}_\theta \sim \exp(\theta).$$ (17).

Then:

(i) $\underline{L_{\mathbb{e}_\theta}} \sim \exp(\psi(\theta))$

(exp^t of (τ_ℓ))

(ii)

$$\boxed{E\left[\exp\left(-\lambda A_{g_{\mathbb{e}_\theta}}\right) \,\middle|\, L_{\mathbb{e}_\theta} = \ell\right] = E\left[\exp\left(-A_{\tau_\ell} - \theta\tau_\ell\right)\right] \Big/ E\left[e^{-\theta\tau_\ell}\right]}$$

Thus: $A_{g_{\mathbb{e}_\theta}}$ appears as a subordinator taken at indpt exp time —

$$T_{\mathbb{e}} \longrightarrow T_{\gamma_L}$$