A Generic One-Factor Lévy Model for Pricing Synthetic CDOs

Wim Schoutens

-

joint work with Hansjörg Albrecher and Sophie Ladoucette

Maryland – 30th of September 2006

www.schoutens.be

Abstract

The one-factor Gaussian model is well-known not to fit simultaneously the prices of the different tranches of a collateralized debt obligation (CDO), leading to the implied correlation smile. Recently, other one-factor models based on different distributions have been proposed. Moosbrucker used a one-factor Variance Gamma model, Kalemanova et al. and Guegan and Houdain worked with a NIG factor model and Baxter introduced the BVG model. These models bring more flexibility into the dependence structure and allow tail dependence. We unify these approaches, describe a generic one-factor Levy model and work out the large homogeneous portfolio (LHP) approximation. Then, we discuss several examples and calibrate a battery of models to market data.

CDOS

- Collateralized Credit Obligations (CDOs) are complex multivariate credit risk derivatives.

- A CDO transfers the credit risk on a reference portfolio of assets in a tranched way.

- The risk of loss on the reference portfolio is divided into tranches of increasing seniority:

  - The *equity* tranche is the first to be affected by losses in the event of one or more defaults in the portfolio.
  - If losses exceed the value of this tranche, they are absorbed by the *mezzanine* tranche(s).
  - Losses that have not been absorbed by the other tranches are sustained by the *senior* tranche and finally by the *super-senior* tranche.

CDOS

- When tranches are issued, they usually receive a rating by rating agencies.

- The CDO *issuer* typically determines the size of the senior tranche so that it is AAA-rated.

- Likewise, the CDO issuer generally designs the other tranches so that they achieve successively lower ratings.

- The CDO *investors* take on exposure to a particular tranche, effectively selling credit protection to the CDO issuer, and in turn collecting premiums (spreads).

- We are interested in pricing tranches of *synthetic* CDOs.

- A synthetic CDO is a CDO backed by credit default swaps (CDSs) rather than bonds or loans, i.e. the reference portfolio is composed of CDSs.

- Recall that a CDS offers protection against default of an underlying entity over some time horizon.

CDOS

- Take the example of the DJ iTraxx Europe index.

- It consists of a portfolio composed of 125 actively traded names in terms of CDS volume, with an equal weighting given to each

- Below, we give the standard synthetic CDO structure on the DJ iTraxx Europe index.

Reference portfolio	Tranche name	K_1	K_2
	Equity	0%	3%
125	Junior mezzanine	3%	6%
CDS	Senior mezzanine	6%	9%
names	Senior	9%	12%
	Super-senior	12%	22%

Table 1: Standard synthetic CDO structure on the DJ iTraxx Europe index.

PROBLEMS IN CDO MODELING

- The problem is high dimensional : 125 dependent underlyers.

- The behavior of the underlying firm's values typically show all the stylized features like, jumps, stochastic volatility, ...

- The tranching complicates mathematics.

- You can write down a fancy model, but prices should be generated within a sec. to be practically useful.

A GENERIC LEVY MODEL FOR PRICING CDOS

- We are going to model a homogeneous portfolio of n obligors: each obligor

  - has the same weight in the portfolio
  - has the same recovery value R
  - has the same individual default probability term structure $p(t),\ t \geq 0$, which is the probability an obligor will default before time t.

- The inhomogeneous case is also possible but a bit more involved in notation and calculations.

- Basic idea is to come up with a vector of n dependent random variables who indicate the value of the firms.

A GENERIC LEVY MODEL FOR PRICING CDOS

- The Gaussain one-factor model (Vasicek, Li) assumes the following dynamics:
  - $A_i(T) = \sqrt{\rho}\, Y + \sqrt{1-\rho}\, \epsilon_i$, $i = 1, \ldots, n$;
  - Y and ϵ_i, $i = 1, \ldots, n$ are i.i.d. standard normal with cdf Φ.

- The ith obligor *defaults at time T* if the firm value $A_i(T)$ falls below some preset barrier $K_i(T)$ (extracted from CDS quotes - see later): $A_i(T) \leq K_i(T)$

- This model is actual based on the Gaussian Copula with its known problems (cfr. correlation smile).

- The underlying reason is the too-light tail behavior of the standard normal rv's. (Note that a large number of joint defaults will be caused by a very negative common factor Y).

- Therefore we look for models where the distribution of the factors has more heavy tails.

HOW TO GENERATE MULTIVARIATE FIRM VALUES

- We want to generate standardized (zero mean, variance one) multivariate random vectors with a prescribed correlation.

- Basic idea: correlate by letting Lévy processes run some time together and then let them free (independence)
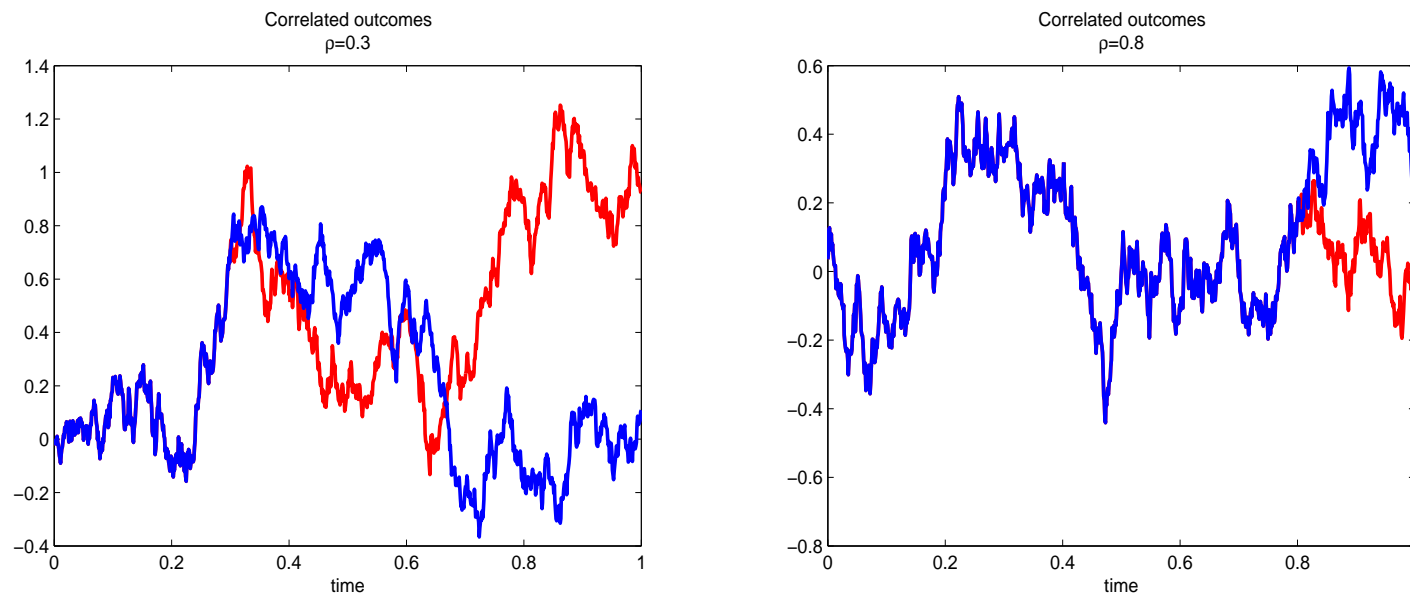

Figure 1: Correlated (Normal) random variables

A GENERIC LEVY MODEL FOR PRICING CDOS

- Let us start with a mother infinitely divisible distribution L.

- Let $X = \{X_t, t \in [0,1]\}$ be a Lévy process based on that infinitely divisible distribution: $X_1 \sim L$.

- Note that we will only work with Lévy processes with time running over the unit interval.

- Denote the cdf of X_t by $H_t(x)$, $t \in [0,1]$, and assume it is continuous.

- Assume the distribution is standardized: $E[X_1] = 0$ and $\mathrm{Var}[X_1] = 1$.

- Then, it is not that hard to prove that $\mathrm{Var}[X_t] = t$.

- Let $X = \{X_t, t \in [0,1]\}$ and $X^{(i)} = \{X_t^{(i)}, t \in [0,1]\}$, $i = 1, 2, \ldots, n$ be independent and identically distributed Lévy processes (so all processes are independent of each other and are based on the same mother infinitely divisible distribution L).

- Let $0 < \rho < 1$, be the correlation that we assume between the defaults of the obligors.

A GENERIC LEVY MODEL FOR PRICING CDOS

- We propose the generic one-factor Lévy model.

- We assume that the asset value of obligor $i = 1, \ldots, n$ is of the form:
$$A_i(T) = X_\rho + X_{1-\rho}^{(i)}, \quad i = 1, \ldots, n.$$

- Each $A_i = A_i(T)$ has by the stationary and independent increments property the same distribution as the mother distribution L with distribution function $H_1(x)$

- Indeed the sum of an increment of the process over a time interval of length ρ and an independent increment over a time interval of length $1 - \rho$ follows the distribution of an increment over an interval of unit length, i.e. is following the law L.

- As a consequence, $E[A_i] = 0$ and $\mathrm{Var}[A_i] = 1$.

- Further we have
$$\mathrm{Corr}[A_i, A_j] = \frac{E[A_i A_j] - E[A_i]E[A_j]}{\sqrt{\mathrm{Var}[A_i]}\sqrt{\mathrm{Var}[A_j]}} = E[A_i A_j] = E[X_\rho^2] = \rho.$$

A GENERIC LEVY MODEL FOR PRICING CDOS

- So, starting from any mother standardized infinitely divisible law, we can set up a one-factor model with the required correlation

- Recall, we say that the ith obligor *defaults at time T* if the firm's value $A_i(T)$ falls below some preset barrier $K_i(T)$: $A_i(T) \leq K_i(T)$

- In order to match default probabilities under this model with default probabilities $p(T)$ observed in the market, set $K = K_i = K_i(T) := H_1^{[-1]}(p(T))$.

A GENERIC LEVY MODEL FOR PRICING CDOS

- Let us denote with M the number of defaults in the portfolio. We have that the probability of having k defaults until time T equals:

$$P(M = k) = \int_{-\infty}^{+\infty} P(M = k | X_\rho = y)\, \mathrm{d}H_\rho(y), \quad k = 0, \ldots, n.$$

- Conditional on $\{X_\rho = y\}$, the probability of having k defaults is (because of independence):

$$P(M = k | X_\rho = y) = \binom{n}{k} p(y; T)^k \left(1 - p(y; T)\right)^{n-k}$$

  where $p(y; T)$ is the probability that the firm defaults before time T given that the systematic factor X_ρ takes the value y.

- As in the classical Gaussian case one could prove:

$$p(y; T) = P(A_i \leq K | X_\rho = y) = H_{1-\rho}(K - y).$$

- Substituting yields:

$$P(M = k) = \int_{-\infty}^{+\infty} \binom{n}{k} \left(H_{1-\rho}(K - y)\right)^k \left(1 - H_{1-\rho}(K - y)\right)^{n-k} \mathrm{d}H_\rho(y), \quad k = 0, \ldots, n.$$

A GENERIC LEVY MODEL FOR PRICING CDOS

- Let Z_n denote the fraction of the defaulted securities at time T in the portfolio.

- Letting $n \to \infty$ (as in the classical Gaussian case) one could show the following cdf F_T for the fraction of the defaulted securities Z in the limiting portfolio:

$$F_T(z) := P(Z \leq z) = 1 - H_\rho \left(H_1^{[-1]}(p(T)) - H_{1-\rho}^{[-1]}(z) \right), \quad z \in [0, 1].$$

- The cdf of the percentage losses of the portfolio at time T, taking into account the recovery R, is then simply:

$$F_T^{LHP}(z) = F_T \left(\frac{z}{1 - R} \right), \quad z \in [0, 1 - R].$$

A GENERIC LEVY MODEL FOR PRICING CDOS

- The Gaussain one-factor model (Vasicek, Li) assumes the following dynamics:
  - $A_i(T) = \sqrt{\rho}\, Y + \sqrt{1 - \rho}\, \epsilon_i,\ i = 1, \ldots, n$;
  - Y and $\epsilon_i,\ i = 1, \ldots, n$ are i.i.d. standard normal with cdf Φ.

- This model can be casted in the above general Lévy framework. The mother infinitely divisible distribution is here the standard normal distribution and the associated Lévy process is the standard Brownian motion $W = \{W_t, t \in [0,1]\}$:
  - W_ρ follows a Normal$(0, \rho)$ distribution as does $\sqrt{\rho}\, Y$;
  - $W_{1-\rho}^{(i)}$ follows a Normal$(0, 1 - \rho)$ distribution as does $\sqrt{1 - \rho}\, \epsilon_i$.
  - Adding these independent rv's lead to a standard normal rv.

- Using the classical properties of normal random variables, the cumulative distribution function F_T for the fraction of the defaulted securities Z transforms into:

$$F_T(z) = 1 - \Phi\left(\frac{\Phi^{[-1]}(p(T)) - \sqrt{1 - \rho}\,\Phi^{[-1]}(z)}{\sqrt{\rho}}\right) = \Phi\left(\frac{\sqrt{1 - \rho}\,\Phi^{[-1]}(z) - \Phi^{[-1]}(p(T))}{\sqrt{\rho}}\right).$$

A GENERIC LEVY MODEL FOR PRICING CDOS

- The density function of the Gamma distribution Gamma(a, b) with parameters $a > 0$ and $b > 0$ is given by:

$$f_{Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \quad x > 0.$$

- Let us denote the corresponding cumulative distribution function by $\mathcal{H}_G(x; a, b)$.

- The Gamma-process $G = \{G_t, t \geq 0\}$ with parameters $a, b > 0$ is defined as the stochastic process which starts at zero and has stationary, independent Gamma-distributed increments. More precisely, the time enters in the first parameter: G_t follows a Gamma(at, b) distribution.

- Some properties of the Gamma(a, b) distribution:

	Gamma(a, b)
mean	a/b
variance	a/b^2

A GENERIC LEVY MODEL FOR PRICING CDOS

- Let us start with a unit variance Gamma-process $G = \{G_t, t \in [0,1]\}$ with parameters $a > 0$ and $b = \sqrt{a}$, such that $\mathrm{Var}[G_1] = 1$.

- The mean of the process is then $\sqrt{a}$. As driving Lévy process, we then take the following shifted Gamma process :

$$X_t = \sqrt{a}t - G_t, \quad t \in [0,1].$$

- The interpretation in terms of firm's value is that there is a deterministic up trend $(\sqrt{a}t)$ with random downward shocks (G_t).

- The one-factor shifted Gamma-Lévy model is:

$$A_i = X_\rho + X_{1-\rho}^{(i)},$$

where $X_\rho, X_{1-\rho}^{(i)}, i = 1, \ldots, n$ are independent shifted Gamma-processes.

- The cumulative distribution function $H_t(x; a)$ of X_t, $t \in [0,1]$, can easily be obtained from the Gamma cumulative distribution function:

$$H_t(x; a) = P(\sqrt{a}t - G_t \leq x) = 1 - \mathcal{H}_G(\sqrt{a}t - x; at, \sqrt{a}), \quad x \in (-\infty, \sqrt{a}t).$$
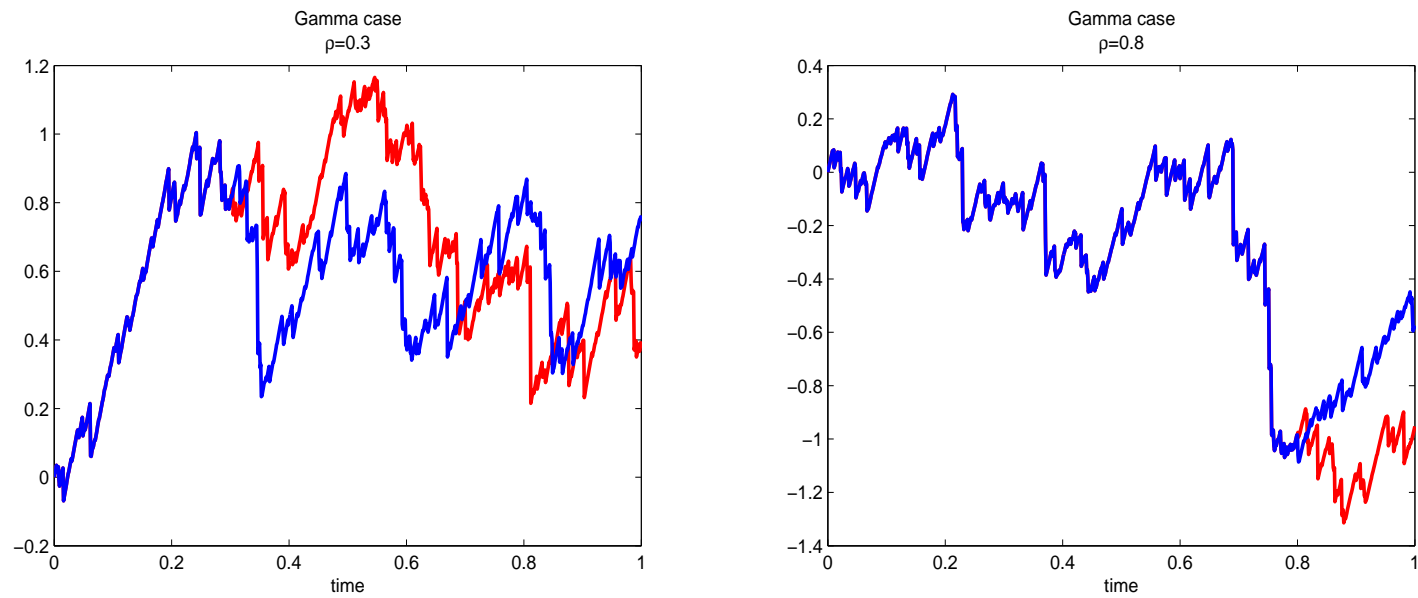
A GENERIC LEVY MODEL FOR PRICING CDOS

Figure 2: Correlated Gamma random variables

A GENERIC LEVY MODEL FOR PRICING CDOS

- Moosbrucker (2006) assumes a factor model where the asset value of obligor $i = 1, \ldots, n$ is of the form:
$$A_i = c\,Y + \sqrt{1 - c^2}\,X_i$$

  where

  $- X_i \sim \mathrm{VG}(\sqrt{1 - \nu\theta^2}, \nu/(1 - c^2), \theta\sqrt{1 - c^2}, -\theta\sqrt{1 - c^2})$
  $- Y \sim \mathrm{VG}\left(\sqrt{1 - \nu\theta^2}, \nu/c^2, \theta c, -\theta c\right).$
  $-$ The X_i's and Y are independent

- In this setting, the random variable A_i is $\mathrm{VG}(\sqrt{1 - \nu\theta^2}, \nu, \theta, -\theta)$-distributed.

- Note that these random variables have indeed zero mean and unit variance, but that there is a constraint on the parameters, namely $\nu\theta^2 < 1$.

A GENERIC LEVY MODEL FOR PRICING CDOS

- Many variations are possible: one could start with a zero mean $\mathrm{VG}(\kappa\sigma, \nu, \kappa\theta, -\kappa\theta)$ distribution for A_i with $\kappa = 1/\sqrt{\sigma^2 + \nu\theta^2}$ in order to force unit variance.

- The model always remains of the form:

$$A_i = X_\rho + X_{1-\rho}^{(i)}.$$

- Here $X_\rho, X_{1-\rho}^{(i)}, i = 1, \ldots, n$ are independent VG random variables with the following parameters

  - the common factor X_ρ follows a $\mathrm{VG}(\kappa\sqrt{\rho}\sigma, \nu/\rho, \kappa\rho\theta, -\kappa\rho\theta)$ distribution
  - the idiosyncratic factors $X_{1-\rho}^{(i)}$ all follow a $\mathrm{VG}(\kappa\sqrt{1-\rho}\sigma, \nu/(1-\rho), \kappa(1-\rho)\theta, -\kappa(1-\rho)\theta)$ distribution.

A GENERIC LEVY MODEL FOR PRICING CDOS

- NIG : Guégan and Houdain and Kalemanova et al.
- Meixner, GH, CGMY : to be explored
- B-VG: Baxter
- IG: Schoutens.

A GENERIC LEVY MODEL FOR PRICING CDOS

- Once we have set up a model for the dependency between the underlying assets, one could try to price CDOs.

- CDOs are very popular multivariate credit derivatives and are very challenging objects to model.

- They reshuffle the credit risk of a pool of companies into different tranches, all representing different risks.

- Lower (equity tranches) will be first affected by credit events but provide higher premium than higher tranches (mezzanine and senior).

- The Gaussian-model is for the moment the market standard but has many shortcomings (cfr. correlation smile).

A GENERIC LEVY MODEL FOR PRICING CDOS

- Finally, we report on a small calibration exercise of the Gaussian, the shifted Gamma, the shifted IG, the NIG and the VG cases. We calibrate the model to the iTraxx of the 4th of May 2006.

- Below one finds the market quotes together with the calibrated model quotes for the different tranches (For the 0-3% tranche the upfront is quoted with a 500 bp running).

Model/Quotes	0-3%	3-6%	6-9%	9-12%	12-22%	absolute error
Market	17%	44.0 bp	12.8 bp	6.0 bp	2.0 bp	
Gaussian	17%	105.7 bp	22.4 bp	5.7 bp	0.7 bp	73.7 bp
Shifted Gamma	17%	44.0 bp	19.7 bp	11.9 bp	6.0 bp	16.8 bp
Shifted IG	17%	44.0 bp	19.8 bp	12.2 bp	6.5 bp	17.7 bp
NIG	17%	44.0 bp	24.1 bp	17.1 bp	11.7 bp	32.1 bp
VG	17%	43.9 bp	21.8 bp	14.1 bp	7.8 bp	23.0 bp

- Some improvements are possible: no LHP but use bucket algorithm.

- There is even possibility to include some time-dynamics: $A_i(t) = X_{\rho t} + X^{(i)}_{(1-\rho)t}$.

CONCLUSION

- We have looked at several multivariate advanced models in equity and credit risk.

- Dynamic Lévy models incorporate skewness, kurtosis, jumps, ...

- Fitting on vanillas in equity and CDSs in credit can be done quite satisfactory.

- In order to price/calibrate CDOs, we need faster (and hence simpler) models/algorithms. We have built a semi-dynamic model that generalizes the well-known Gaussian setting.

- Simply replacing the Normal distribution with a shifted Gamma results in a much better fit without almost no loss in computation time.

- For more info and technical reports : `www.schoutens.be`

- THE END -

CONTACT:

Wim Schoutens

K.U.Leuven, U.C.S.,

W. De Croylaan 54,

B-3001 Leuven, Belgium.

`E-mail:  wim@schoutens.be`