# Rick Neutral Probabilities and Option Prices

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**This talk is based on joint work with Jean Jacod**

**In Honor of Dilip Madan on his $60^{th}$ birthday**

# I. Overview

No Arbitrage assumption is equivalent to existence of a Risk Neutral Measure $Q$

If $Q$ is unique the market is **complete**

If $Q$ is not unique, the market is **incomplete:** there is a choice of an infinite number of such $Q$, and not all contingent claims can be replicated.

People believe markets are incomplete; if they thought they were complete and models were correct, using historical volatility would be equivalent to using implied volatilities.

A standard problem: how does one choose one such $Q$?

Four methods proposed to date; possibly more:

- Minimal martingale measures (Föllmer-Schweitzer)
- Minimal entropy measures
- Indifference pricing (involves personal choice of a utility function)
- Arbitrarily choose one and stick with it, for no good reason

We propose an alternative.

## The usual approach is as follows:

1. Begin with the risky asset price process and a (possibly random) savings rate;

2. By a change of numéraire argument, assume interest rates are zero;

3. Find and choose a risk neutral measure $Q$;

4. If $g(X_T)$ is a financial (European style) derivative at time $T$, the price process is *declared* to be $E_Q\{g(X_t)|\mathcal{F}_t\}$ for $0 \leq t \leq T$.

## Our approach:

1. Begin with the risky asset price process and a (possibly random) savings rate;

2. Next assume there are market given price processes for a large number of (European style) derivatives of the form $g(X_T)$, where $T$ varies;

3. Find the collection $\mathcal{Q}$ of risk neutral measures that make both the price process and all of the derivative price processes local martingales;

4. **If there are enough derivative prices, the cardinality of $\mathcal{Q}$ might be one;**

5. **Alternatively the cardinality of $\mathcal{Q}$ might be zero or $\infty$** (compatibility issues are serious here).

# Other Approaches; Some key ones follow:

- **H. Dengler and R. Jarrow** (1997): A simple jump model where the price process is pure jump, with two distinct jump sizes; two call options are used to complete the market.

- **B. Dupire** (1997): A simple stochastic volatility model, where the stock price varies but the expiration time $T$ is fixed. A PDE is obtained, and when (and if) solved, it gives a unique martingale measure;

- **E. Derman and I. Kani** (1997): the stochastic volatility case, where the strike price $K$ varies and $T$ is fixed. Smoothness in $K$ is assumed, and by twice differentiating obtain a density for the call option, leading to a PDE, which when (and if) solved leads to a martingale measure choice.

- **M. Schweizer and J. Wissel** (2006), to appear in *Math Finance*: improved on previous ideas by using different maturities, creating a term structure of volatilities, within a Brownian based stochastic volatility framework. They find conditions for equivalent martingale measures to exist, and also a condition for it to be unique.

The seminal papers of **Eberlein, Jacod and Raible** inspired the paper on which this talk is based.

Indeed, the talk is only for the continuous case, but an analogous (and more interesting) theory is developed in our paper for the general case (with jumps) as well.

## I. For simplicity of presentation, we consider here the continuous paths case.

$(\Omega, \mathcal{F}, P, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Risky asset price:

$$X_t = X_0 + \int_0^t a_s ds + \sum_{i \in I} \int_0^t \sigma_s^i dW_s^i,$$

with $\int_0^t (|a_s| + \sum_{i \in I} |\sigma_s^i|^2) ds < \infty$ a.s., all $t$.

We want $X_t > 0$ so we assume $a$ and $\sigma$ factor: $a_t = X_{t-} \bar{a}_t$ and $\sigma_t^i = X_{t-} \bar{\sigma}_t^i$.

Next we add the financial derivatives (options), always European style.

Let $g : (0, \infty) \to \mathbb{R}_+$, be nonnegative and convex.

**The price of the option $g(X_T)$ at time $t$ with expiration time $T$ is $P(T)_t$.**

We suppose there are options with different expiration times, with the same $g$, and **let $\mathcal{T}$ denote the *set of expiration times.***
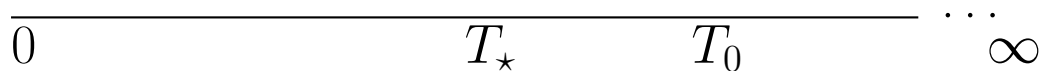
**$[0, T_\star]$ is *your* period of trading.**

$\mathcal{T}$ is a finite set in practice and can be quite small. But in the spirit of HJM models, we take $\mathcal{T}$ to be an interval in $\mathbb{R}_+$, or a countable dense subset of an interval.

**We have two cases:**

**Full Models:** $T_\star = \infty$ **and** $\mathcal{T} = (0, \infty)$;

**Partial Models:** $T_\star < \infty$ **and** $\mathcal{T} = [T_0, \infty)$ **with** $T_0 > T_\star$.

$$\overline{\phantom{0 \hspace{8cm} T_\star \hspace{2cm} T_0 \hspace{2cm}}} \quad \cdots$$

$0 \hspace{5cm} T_\star \hspace{2cm} T_0 \hspace{3cm} \infty$

Let $P(T)_t =$ the price of an option $g(X_T)$ at time $t$.

We must have $P(T)_T = g(X_T)$.

We also take $P(T)_t = g(X_T) = P(T)_T$ for $t \geq T$.

## The Standard Approach:

Choose $Q$ equivalent to $P$ such that the price process $X$ is a $Q$ martingale (or a $Q$ local martingale), with $E_Q(g(X_T)) < \infty$ and

$$P(T)_t = E_Q(g(X_T)|\mathcal{F}_t) \quad \text{for} \quad t \le T.$$

$(P(T))_{t \ge 0}$ is then a $Q$ martingale on $[0, T]$.

## Change from the standard approach:

Now fix $t$ and consider $T \to P(T)_t$, on the interval $[t, \infty)$. Since $X$ is continuous, and $g$ is convex (and hence continuous),

$$T \to g(X_T) \quad \text{is continuous.}$$

Since $X$ is a $Q$ martingale and $g$ is convex, $T \to g(X_T)$ is a continuous $Q$ submartingale. Hence

$$T \to P(T)_t \textbf{ is non decreasing and}$$
$$\textbf{continuous for } T \ge t, Q \textbf{ a.s.}$$

and hence also $P$ a.s.

So if $P(T)_t$ are option prices, $T \to P(T)_t$ is a.s. continuous and nondecreasing.

**Assume $T \to P(T)_t$ is absolutely continuous on $(0, \infty)$. Thus:**

$$P(T)_t = g(X_t) + \int_t^T f(t, s)ds$$

for $t \leq T$, where $f(t, s) \in \mathcal{F}_t$, with $f \geq 0$.

What do the processes $f(t, s)$ look like?

For $t < u$ assume $f$ can be expressed with a decomposition of the form:

$$f(t, u) = f(0, u) + \int_0^t \alpha(r, u)dr + \sum_{i \in I} \int_0^t \gamma^i(r, u)dW_r^i.$$

For the **Black-Merton-Scholes model**, let
$C(x,t) = E\{g(xe^{\sigma U\sqrt{t}-\frac{\sigma^2 t}{2}})\}$ where $U$ is $N(0,1)$.

One has $C'_t(x,0) = \frac{\sigma^2}{2}x^2 g''(x)$ when $g$ is $\mathcal{C}^2$;

When $g$ is convex but not $\mathcal{C}^2$, then
$$C(x,t) - g(x) = O(\sqrt{t}) \quad \text{as} \quad t \to 0.$$

More generally, $C'_t(x,0)$ exists if $g$ is $\mathcal{C}^2$ at $x$, and does not exist if $g$ is not.

And if $g(x) = (x-K)_+$, then
$$\sqrt{t}C'_t(x,t) \to \frac{\sigma}{2\sqrt{2\pi}}1_{\{x=K\}}x \quad \text{as} \quad t \downarrow 0.$$

One has:
$$
\begin{aligned}
f(t,T) &= C'_t(X_t, T-t) \\
f(0,T) &= C'_t(X_0, T) \\
\alpha(r,T) &= -C''_{tt}(X_r, T-r) + \frac{1}{2}C'''_{txx}(X_r, T-r)\sigma^2 X_r^2 \\
\gamma(r,T) &= C''_{tx}(X_r, T-r)\sigma X_r
\end{aligned}
$$

**Definition 1.** *A **full option model** for $(X, g)$ is a family of processes $P(T), T > 0$, given by $P(T)_t = g(X_t) + \int_t^T f(t, s) ds$, where, for $t < s$,*

$$f(t, s) = f(0, s) + \int_0^t \alpha(u, s) du + \sum \int_0^t \gamma^i(u, s) dW_u,$$

*and*

*1. $f(0, s) \geq 0$, non-random, locally integrable in $s$*

*2. appropriate measurability of $\alpha$ and $\gamma^i$*

*3. $\alpha$ and $\gamma^i$ are such that the integrals make sense*

*4. $f(t, s) \geq 0$ for all $t < s$*

*5. $\int_t^T f(t, s) ds < \infty$ a.s. for all $t \leq T$.*

Also, we want to define

$$\chi(s)_t = \int_0^t (|\alpha(u, s)| + \sum_{i \in I} |\gamma^i(u, s)|^2) du.$$

Note that (3) above is equivalent to $\chi(s)_t < \infty$ a.s. for all $t < s$.

$$\chi(s)_t = \int_0^t (|\alpha(u,s)| + \sum_{i \in I} |\gamma^i(u,s)|^2)du.$$

Also note that $t \to \chi(s)_t$ is increasing, but $s \to \chi(s)_t$ is not increasing in general.

**Definition 2.** *A full option model for $(X,g)$ is*

1. **regular** *if $\chi(s)_s < \infty$ a.s., for almost all $s$. (This implies $f(s,s)$ is well defined.)*

2. **fair** *if for all $T$, $\int_t^T \chi(s)_t ds < \infty$ a.s.*

3. **strongly regular** *if for all $T$, $\int_t^T \chi(s)_s ds < \infty$ a.s.*

$$\text{Strongly Regular} \quad \Rightarrow \quad \begin{cases} \text{Fair} \\ \text{Regular} \end{cases}$$

## Relation to Black-Scholes

- $P(T)$ is a full option model, which is **always fair**
- If $g \in \mathcal{C}^2$, it is **strongly regular**
- If $g \notin \mathcal{C}^2$, then it is **not even regular**

# Easier situation: Partial Models

**Trading takes place up to time $T_\star$**

**Expiration dates all have $T \geq T_0$, with $T_0 > T_\star$.**

We only need to model $P(T)$ for $T \geq T_0$, with

$$P(T)_t = P(T_0)_t + \int_{T_0}^{T} f(t,s)ds.$$

So if we know the dynamics of $P(T_0)$, the model will be specified by the dynamics of $t \to f(t,s)$ for $s > T_0$.

**Definition 3.** *A $(T_\star, T_0)$ **partial option model associated with** $(X,g)$ is a family of processes $(P(T) : T \geq T_0)$, where $f$ is as in our definition of full option models, and for $t \leq T_\star$:*

$$P(T)_t = P(T_0)_t + \int_0^t \overline{\alpha}_s ds + \sum_{i \in I} \int_0^t \overline{\gamma}_s^i dW_s^i,$$

*with the integrability condition:*

$$\int_0^{T_\star} (|\overline{\alpha_t}| + \sum_{i \in I} |\overline{\gamma}_t^i|^2)dt < \infty,$$

*and finally*

$$t \in [0, T_\star] \quad \Rightarrow \quad P(T_0)_t \geq g(X_t).$$

The **model is fair** if we have $\int_0^T \chi(s)_{T_\star} ds < \infty$ a.s. for all $T > T_0$.

The notions of regular or strongly regular do not apply.

# II. Equivalent Local Martingale Measures

The Delbaen-Schachermayer theory says that no arbitrage is equivalent to the existence of at least one measure $Q$, equivalent to $P$, making the price process a local martingale.

We diverge a bit: $Q$ **is locally equivalent to** $P$ if $Q$ and $P$ are equivalent on each $\mathcal{F}_t, t < \infty$.

**Example:** $W$ and $Z$, where $Z_t = W_t + \lambda t$ on $[0, \infty)$.

$\mathcal{M}_{loc}$ is the set of all probability measures $Q$ locally equivalent to $P$, under which $X$ **and** $P(T)$ for all $T \in \mathcal{T}$ are $Q$ **local martingales**.

$\mathcal{M}_{loc}(T_\star, T_0)$ is the set of all $Q$ on $(\Omega, \mathcal{F}_{T_\star})$ equivalent to $P$ on $\mathcal{F}_{T_\star}$ and under which $X$ **and** $P(T)$, for all $T \geq T_\star$, are $Q$ **local martingales on the time interval** $[0, T_\star]$.

We can characterize the local martingale measures simply, as follows:

**Definition 4.** *Let $(b_s^i)_{i \in I}$ be predictable processes with $\sum_{i \in I} \int_0^t (b_s^i)^2 ds < \infty$. Two families $\mathcal{B}$ and $\mathcal{B}'$ of such processes are* **equivalent** *if*

$$i \in I \quad \Rightarrow \quad b^i = b^{i\prime}$$

*The set of equivalent classes of these families of processes is $\Upsilon$.*

**Theorem 5.** *There is a one-to-one correspondence between $\Upsilon$ and the $Q$ which are locally equivalent to $P$.*

*Moreover if $\mathcal{B} = (b_s^i)_{i \in I}$ is in $\Upsilon$ then under $Q$, the processes $W_t^{i\prime} = W_t^i - \int_0^t b_s^i ds$ are independent Brownian motions.*

$\Upsilon_{loc}$ is the set of all $\mathcal{B} \in \Upsilon$ which correspond to a probability measure in $\mathcal{M}_{loc}$. Analogously $\Upsilon'_{loc}(T_\star, T_0)$ is the set of all $\mathcal{B}' \in \Upsilon'(T_\star)$ which correspond to a probability measure in $\mathcal{M}_{loc}(T_\star, T_0)$.

**Theorem 6.** *For a strongly regular model and if g is $\mathcal{C}^2$, the set $\Upsilon_{loc}$ is the set of all $(b^i) \in \Upsilon$ which satisfy*

$$a_s + \sum_{i \in I} \sigma_s^i b_s^i = 0 \tag{1}$$

$$f(s,s) = \frac{1}{2} g''(X_s) \sum_{i \in I} (\sigma_s^i)^2 \tag{2}$$

*and for all $T \geq s$:*

$$\alpha(s,T) + \sum_{i \in I} \gamma^i(s,T) b_s^i = 0. \tag{3}$$

**Theorem 7.** *For a $(T_\star, T_0)$ partial fair model, the set $\Upsilon'_{loc}(\mathcal{T}, T_\star)$ is the set of all $(b^i) \in \Upsilon'$ which satisfy (1) above for $s \leq T_\star$, (3) for $s \leq T_\star$ and $T \geq T_0$, and finally, for $s \leq T_\star$:*

$$\overline{\alpha}_s + \sum_{i \in I} \overline{\gamma}_s^i b_s^i = 0$$

# III. Completeness for $(T_\star, T_0)$ partial models.

**Theorem 8.** *Let us be given a $(T_\star, T_0)$ partial fair model such that the set $\mathcal{M}_{loc}(T_\star, T_0) \neq \emptyset$. Then there are two alternatives:*

*(a) Either for all $s \leq T_\star$ and $\omega$ the closed linear subspace of $L(I)$ spanned by the vectors $(\sigma^i(\omega)_s)_{i\in I}, \overline{\gamma}^i(\omega)_s)_{i\in I}$ and $(\overline{\gamma}^i(\omega, s, T)_{i\in I}$ for $T \geq T_0$, is equal to $L(I)$ itself; in this case $\mathcal{M}_{loc}(T_\star, T_0)$ is a singleton;*

*(b) Or, this property fails, and the set $\mathcal{M}_{loc}(T_\star, T_0)$ is infinite.*

If, for example, the coefficients $\sigma^i_s$ and $\overline{\gamma}^i_s$ and $\gamma^i(s, T)$ are taken continuous in $s$ and $T$, then there is a nice version of everything and **condition $(a)$ can be checked**.

The hard part will be checking that the set $\mathcal{M}_{loc}$ is not empty! When all the coefficients can be taken to be continuous in $s$ and $T$, and if there is a countable dense subset $D(s, \omega)$ of $(T_0, \infty)$ such that the vectors $\sigma_s(\omega), \overline{\gamma}_s(\omega)$, and $\gamma(\omega, s, T)$ for $T \in D(\omega, s)$ are linearly independent, **then $\mathcal{M}_{loc}$ is not empty**.

# IV. Full Regular Models

A full model describes what happens when time increases and reaches the expiration times $T \in \mathcal{T}$, and thus **some compatibility between the stock price and the option model is required**, in addition to $P(T)_t \geq g(X_t)$. This restricts the possible option models associated with $(X, g)$ if we want them to be free of arbitrage. We know from the Black-Merton-Scholes example that there are some models, but existence in general is delicate.

Recall the Meyer-Tanaka formula:

$$g(X_t) = g(X_0) + \int_0^t \alpha_s ds + M_t + L_t,$$

where $M$ is a local martingale, $\alpha$ is of course integrable, and $L$ is an adapted, increasing, singular process (eg, a local time). If $g$ is $\mathcal{C}^2$ then $L = 0$, but otherwise $L$ does not vanish.

**Theorem 9.** *a) If the process $L$ does not vanish a.s. for any full strongly regular option model associated with $(X, g)$, then the set $\mathcal{M}_{loc}$ is empty;*

*b) If $g$ is $\mathcal{C}^2$ and there exists at least one locally equivalent probability such that **the price process $X$ is a local martingale**, then there is a sequence of stopping times $(S_n)_{n \geq 1} \nearrow \infty$ a.s. and for each $n$ a **full regular option model associated with** $(X^{S_n}, g)$ such that $\mathcal{M}_{loc}$ is not empty;*

*c) If $g$ is $\mathcal{C}^2$ with at most linear growth and there exists at least one locally equivalent probability such that $X$ **is a martingale**, then there exists a **full regular option model associated with** $(X, g)$ such that $\mathcal{M}_{loc}$ is not empty.*

**Theorem 10.** *Assume that $g$ is of class $\mathcal{C}^2$. A locally equivalent probability $Q$ cannot be in $\mathcal{M}_{loc}$ for two different full strongly regular models associated with $(X, g)$.*

**Caution:** Theorem 10 does not mean that there is a single arbitrage free strongly regular option model associated with $(X, g)$. There could be several, even infinitely many, with each one corresponding to a different equivalent local martingale measure.

## Example: Markov type stochastic volatility models

The price process is

$$X_t = X_0 + \int_0^t a(X_s, Y_s)ds + \int_0^t \sigma(X_s, Y_s)dW_s^1 \quad (4)$$

$Y$ can be multidimensional.

$$Y_t = Y_0 + \int_0^t h(X_s, Y_s)ds + \sum_{i \in I} \int_0^t k_i(X_s, Y_s)dW_s^i$$

$$+ \int_0^t \int_{\mathbb{R}} H(X_s, Y_{s-}, x)(\mu - \nu)(ds, dx)$$

with the coefficients nice enough for unique strong solutions of both equations, and also that $Y$ is strong Markov.

**In the classic paradigm**, choose locally equivalent $Q$ with $X$ a local martingale, and **we are not concerned with whether or not $Y$ is a local martingale under $Q$**. Then evaluate the prices $P(T)_t$ of options under $Q$. **These prices may be discontinuous**.

Among the many such $Q$, there are some that preserve the Markov property, namely the "extremal" ones in the convex set of all locally equivalent probability measures. Let $(\mathcal{L}, \mathcal{D}_\mathcal{L})$ be the infinitesimal generator of $(X, Y)$ under $Q$. Assume too that $\mathcal{D}_\mathcal{L} \supset \mathcal{C}^2$.

Then the price $P(T)_t$ takes the form (under $Q$):

$$P(T)_t = E\{g(X_T)|\mathcal{F}_t\} = Q_{T-t}G(X_t, Y_t),$$

where $G(x, y) = g(x)$. If $g \in \mathcal{C}^2$, hence $G$ as well, then $G \in \mathcal{D}_\mathcal{L}$, and therefore:

$$P(T)_t = g(X_t) + \int_t^T Q_{s-t}\mathcal{L}G(X_t, Y_t)ds.$$

So in this case, $f(t, s) = Q_{s-t}\mathcal{L}G(X_t, Y_t)$.

$(f(t, s) :\leq t \leq s)$ is a $Q$ martingale which is discontinuous as soon as $Y$ is.

We conclude, after some work and using results from the discontinuous case, that we have as many full regular option models associated with $(X, g)$ with no arbitrage as there are locally equivalent probabilities for which $(X, Y)$ is a Markov process and $X$ is a local martingale.

When $g$ is convex but not $\mathcal{C}^2$, then in general $G \notin \mathcal{D}_{\mathcal{L}}$. But under some ellipticity, $Q_\varepsilon G$ is in it for all $\varepsilon > 0$. We can get a full option model, but it will not be regular.