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DISTRIBUTIONAL STUDY OF DE FINETTI'S DIVIDEND PROBLEM FOR A GENERAL LÉVY INSURANCE RISK PROCESS.

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Abstract

We provide a distributional study of the solution to the classical control problem due to De Finetti (1957), Azcue and Muller (2005) and Avram et al. (2006) which concerns the optimal payment of dividends from an insurance risk process prior to ruin. Specifically we build on recent work in the actuarial literature concerning calculations for the n-th moment of the net present value of dividends paid out in the optimal strategy as well as the moments of the deficit at ruin and the Laplace transform of the red period. The calculations we present go much further than existing literature in that our calculations are valid for a general spectrally negative Lévy process as opposed to the classical Cramér-Lundberg process with exponentially distributed jumps. Moreover, the technique we use appeals principally to excursion theory rather than integro-differential equations and for the case of the n-th moment of the net present value of dividends, makes a new link with the distribution of integrated exponential subordinators.

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1. Lévy insurance risk processes

Recall that the Cramér-Lundberg model corresponds to a Lévy process $X^{CL} = \{X_t^{CL} : t \geq 0\}$ with characteristic exponent given by

$$\Psi^{CL}(\theta) = -\log \int_{\mathbb{R}} e^{i\theta x}\mathbb{P}(X_1^{CL} \in dx) = -ic^{CL}\theta + \lambda^{CL} \int_{(0,\infty)} (1 - e^{-i\theta x})F(dx),$$

for $\theta \in \mathbb{R}$ such that $\lim_{t\uparrow\infty} X_t^{CL} = \infty$. In other words, X^{CL} is a compound Poisson process with arrival rate $\lambda^{CL} > 0$ and negative jumps, corresponding to claims, having common distribution function F with finite mean $1/\mu^{CL}$ as well as a drift $c^{CL} > 0$, corresponding to a steady income due to premiums, which necessarily satisfies the *net profit condition* $c^{CL} - \lambda^{CL}/\mu^{CL} > 0$. Suppose instead we work with a general spectrally

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negative Lévy process; that is a Lévy process $X = \{X_t : t \geq 0\}$ with Lévy measure Π satisfying $\Pi(0,\infty) = 0$. At such a degree of generality, the analogue of the condition $c^{CL} - \lambda^{CL}/\mu^{CL} > 0$ may be taken as $\lim_{t\uparrow\infty} X_t = \infty$. (Analytical conditions are available for this in terms of the underlying Lévy process and we return to this point in the next section). Such processes have been considered recently by Huzak et al. (2004a,b), Klüppelberg et al. (2004), Doney and Kyprianou (2006) and Klüppelberg and Kyprianou (2006), Furrer (1998) in the context of insurance risk models. In this case, the Lévy-Itô decomposition offers an interpretation for large scale insurance companies as follows. The characteristic exponent may be written in the form

$$\Psi(\theta) = -\log \int_{\mathbb{R}} e^{i\theta x}\mathbb{P}(X_1 \in dx) \;=\; \left\{\frac{1}{2}\sigma^2\theta^2\right\} + \left\{-i\theta c + \int_{(-\infty,-1)} (1 - e^{i\theta x})\Pi(dx)\right\} + \left\{\int_{(-1,0)} (1 - e^{i\theta x} + i\theta x)\Pi(dx)\right\} \quad (1)$$

for $\theta \in \mathbb{R}$ and $\sigma \geq 0$ and necessarily the Lévy measure satisfies $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(dx) < \infty$ and the requirement that X drifts to infinity implies that $c - \int_{(-\infty,-1)} |x|\Pi(dx) > 0$. Note that when $\Pi(-\infty, 0) = \infty$ the process X enjoys a countably infinite number of jumps over each finite time horizon. The third bracket in (1) we may understand as a Lévy process representing a countably infinite number of arbitrarily small claims compensated by a deterministic positive drift (which may be infinite in the case that $\int_{(-1,0)} |x|\Pi(dx) = \infty$) corresponding to the accumulation of premiums over an infinite number of contracts. Roughly speaking, the way in which claims occur is such that in any arbitrarily small period of time dt, a claim of size $|x|$ is made independently with probability $\Pi(dx)dt + o(dt)$. The insurance company thus counterbalances such claims by ensuring that it collects premiums in such a way that in any dt, $|x|\Pi(dx)dt$ of its income is devoted to the compensation of claims of size $|x|$. The second bracket in (1) we may understand as coming from large claims which occur occasionally and are compensated against by a steady income at rate $c > 0$ as in the Cramér-Lundberg model. Here 'large' is taken to mean claims of size one or more. Finally the first bracket in (1) may be seen as a stochastic pertubation of the system of claims and premium income.

Since the first and third brackets correspond to martingales, the company may guarentee that its revenues drift to infinity over an infinite time horizon by assuming the latter behaviour applies to the compensated process of large claims corresponding to the second bracket in (1).

2. De Finetti's dividend problem

An offshoot of the classical ruin problem for the Cramér-Lundberg process was introduced by De Finetti (1957). His intention was to make the study of ruin under the Cramér-Lundberg dynamics more realistic by introducing the possibility that dividends are paid out to share holders up to the moment of ruin. Further, the payment of dividends should be made in such a way as to optimise the expected net present value of the total income of the shareholders from time zero until ruin. Mathematically speaking, De Finetti's dividend problem amounts to solving a control problem which we state in the next paragraph but within the framework of the general Lévy insurance risk process described in the previous section.

Suppose that X is a general spectrally negative Lévy process (no assumption is made on its long term behaviour) with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ such that under $\mathbb{P}_x$ we have $X_0 = x$ with probability one. (For convenience we shall write $\mathbb{P}_0 = \mathbb{P}$). Let $\pi = \{L_t^\pi : t \geq 0\}$ be a dividend strategy consisting of a left-continuous non-negative non-decreasing process adapted to the (completed and right continuous) filtration, $\{\mathcal{F}_t : t \geq 0\}$, of X. The quantity L_t^π thus represents the cumulative dividends paid out up to time t by the insurance company whose risk-process is modelled by X. The controlled risk process when taking account of dividend strategy π is thus $U^\pi = \{U_t^\pi : t \geq 0\}$ where $U_t^\pi = X_t - L_t^\pi$. Write $\sigma^\pi = \inf\{t > 0 : U_t^\pi < 0\}$ for the time at which ruin occurs when taking account of dividend payments. A dividend strategy is called admissible if at any time before ruin a lump sum dividend payment is smaller than the size of the available reserves; in other words $L_{t+}^\pi - L_t^\pi < U_t^\pi$ for $t < \sigma^\pi$. Denoting the set of all admissible strategies by Π, the expected value discounted at rate $q > 0$ associated with the dividend policy $\pi \in \Pi$ with initial capital $x \geq 0$ is given by

$$v_\pi(x) = \mathbb{E}_x \left(\int_0^{\sigma^\pi} e^{-qt} dL_t^\pi \right),$$

where $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$ and $q > 0$. *De Finetti's dividend problem* consists of solving the following stochastic control problem: characterise

$$v^*(x) := \sup_{\pi \in \Pi} v_\pi(x)$$

and, further, if it exists, establish a strategy, π^*, such that $v^*(x) = v_{\pi^*}(x)$.

This problem has been re-considered very recently by Azcue and Muler (2005) when X is the classical Cramér-Lundberg risk process and Avram et al. (2006) for the general spectrally negative case. Both have shown that in very general circumstances that the optimal strategy consists of a barrier strategy. Specifically, there exists a constant $a^* \in (0, \infty)$ such that the optimal dividend strategy π^* corresponds to the reducing the risk process $U_t^{\pi^*}$ to the level a^* if $x > a^*$ at time $t = 0$ by paying the amount $(x - a^*)^+$ and then to the payment of the surplus of the process X above a^* so long as it remains above this level. From now on we assume that $x \leq a^*$, otherwise one takes $v^*(x) = (x - a^*) + v^*(a)$. It is well known that for $0 < x \leq a^*$ the corresponding controlled risk process, say U^{π^*} under $\mathbb{P}_x$ equal in law to the process $a^* - Y = \{a^* - Y_t : t \geq 0\}$ under $\mathbb{P}_0$ where

$$Y_t = ((a^* - x) \vee \overline{X}_t) - X_t$$

and $\overline{X}_t = \sup_{s \leq t} X_s$ is the running supremum of the risk process. Moreover, for $x \in (0, a^*)$ we may write

$$v^*(x) = \mathbb{E} \left(\int_0^{\sigma_{a^*}} e^{-qt} dL_t^* \right),$$

where now $\sigma_{a^*} = \inf\{t > 0 : Y_t > a^*\}$ and $L_t^* = (a^* - x) \vee \overline{X}_t$.

The optimal strategy associated with De Finetti's dividend problem then yields a net present value of the paid dividends which is given by the random variable $\int_0^{\sigma_{a^*}} e^{-qt} dL_t^*$. The paper of Dickson and Waters (2004) studies distributional aspects of this random variable, in particular its moments, for the special case that X is a Cramér-Lundberg process and the associated jumps are exponentially distributed. In this paper we build on their results and give a more detailed account of distributional properties of the net present value of the paid dividends in the the general case, as well as associated quantities such as the time of ruin, the time at which dividends were last paid and the time from the latter to ruin. We also analyse the deficit at the ruin time and so-called 'red period'. Dickson and Waters (2004) take an iterative approach to their calculations with the help of integro-differential equations, making heavy use of the fact that jump times form a discrete set and that the jumps themselves are exponentially distributed. Our approach appeals to Itô's excursion theory and exposes an interesting link with recent work on integrated exponential subordinators.

We conclude this section with a brief outline of the remainder of the paper. In the next section we state our two main theorems. In Section 4 we provide a distributional analysis of an integrated bivariate subordinator which forms a substantial part of the proof of the first of main results. The proof of the latter is given in Section 5 and the proof of the second main theorem is given in Section 6. We conclude with some further discussion in Section 7.

3. Main results

For the sake of clarity we abstract ourselves from De Finetti's dividend problem and suppose that $X = \{X_t : t \geq 0\}$ is any spectrally negative Lévy process, and for a given fixed $a > 0$ we write $Y_t = ((a - X_0) \vee \overline{X}_t) - X_t$. Note that the latter process when considered under $\mathbb{P}_x$ means that we are taking $X_0 = x$.

3.1. Distributional identities

Fundamentally, for $n = 0, 1, 2, ...,$ $x \in (0, a]$ and $\lambda, \mu, \kappa \geq 0$ we are interested in the quantity

$$\Theta(x; n, \lambda, \mu, \kappa) := \mathbb{E}_x \left[e^{-\lambda L^{-1}_{L_{\sigma_a}} - \mu(\sigma_a - L^{-1}_{L_{\sigma_a}}) - \kappa Y_{\sigma_a}} \left(\int_0^{\sigma_a} e^{-qt} dL_t \right)^n \right],$$

where for $t \geq 0$, $L_t = (a - X_0) \vee \overline{X}_t$ is a version of the Markov local time of Y spent at zero,

$$\sigma_a = \inf\{t > 0 : Y_t > a\}.$$

Note that in relation to De Finetti's dividend problem, L plays the role of the cumulant dividend payments, σ_a is the time of ruin of the controlled ruin process, $L^{-1}_{L_{\sigma_a}}$ plays the role of the time that dividends were last paid before ruin and $\sigma_a - L^{-1}_{L_{\sigma_a}}$ plays the role of time from the last payment of dividends to the time of ruin of the controlled risk process. Finally, $Y_{\sigma_a} - a$ is the deficit at the ruin time.

Our main first main result is expressed in terms of the scale functions for spectrally negative processes which we shall briefly introduce. The reader is otherwise referred to Chapter 8 of Kyprianou (2006) for a fuller account.

Firstly let

$$\psi(\theta) = -\Psi(-i\theta) = \log \mathbb{E}(e^{\theta X_1})$$

be the Laplace exponent of X which is known to be finite for at least $\theta \in [0, \infty)$. The asymptotic behaviour of X is characterised by $\psi'(0+)$, so that X drifts to $\pm\infty$ (oscillates) accordingly as $\pm\psi'(0+) > 0$ ($\psi'(0+) = 0$).

For every $q \geq 0$ there exists a function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and otherwise is almost everywhere differentiable on $[0, \infty)$ satisfying,

$$\int_0^\infty e^{-\lambda x} W^{(q)}(x)dx = \frac{1}{\psi(\lambda) - q}, \quad \text{for} \quad \lambda > \Phi(q), \tag{2}$$

where $\Phi(q)$ is the largest solution to the equation $\psi(\theta) = q$ (there are at most two). We shall write for short $W^{(0)} = W$ and further assume that the jump measure Π of X has no atoms when X has paths of bounded variation which is a necessary and sufficient condition for $W^{(q)}$ to be continuously differentiable. Associated with $W^{(q)}$ is the function

$$Z^{(q)}(x) := 1 + q \int_{-\infty}^{x} W^{(q)}(y)dy, \; x \in \mathbb{R}, \; q \geq 0.$$

By a procedure of analytical extension, it is also possible define them for all $q \in \mathbb{C}$ via the following representation

$$W^{(q)}(x) = \sum_{k \geq 0} q^k W^{*(k+1)}(x)$$

where $W^{*n}(x)$ is the n-th convolution of W; see Chapter 8 of Kyprianou (1996) for further details. The function $Z^{(q)}$ keeps the same definition as above when $q \in \mathbb{C}$. The functions $W^{(q)}$ and $Z^{(q)}$ are known as scale functions and appear in a number of fundamental identities concerning exit problems which earns them their title by analogy with the role played by scale functions for one dimensional diffusions.

There exists a well known exponential change of measure that one may perform for spectrally negative Lévy processes,

$$\left.\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{\vartheta X_t - \psi(\vartheta)t}$$

for $\vartheta \geq 0$ under which X remains within the class of spectrally negative Lévy processes. It turns out that it is important to introduce an additional parameter to the scale functions described above in the light of this change of measure. Henceforth we shall refer to the functions $W_\vartheta^{(q)}$ and $Z_\vartheta^{(q)}$ where $q \geq 0$ and $\vartheta \geq 0$ as the functions that play the role of the scale functions defined in the previous paragraph but when considered under the measure $\mathbb{P}^\vartheta$.

We are now ready to state the first of our two main results.

Theorem 1. *For $n = 1, 2, 3, ...,$ $\lambda, \mu, \kappa \geq 0$ we have*

$$\begin{aligned} \Theta(x; n, \lambda, \mu, \kappa) &= n! \frac{W^{(\lambda+qn)}(x)}{W^{(\lambda+qn)}(a)} \prod_{k=1}^{n} \frac{W^{(\lambda+qk)}(a)}{W^{(\lambda+qk)\prime}(a)} \frac{e^{\Phi(\lambda)a} W'_{\Phi(\lambda)}(a) W(a)}{W^{(\lambda)\prime}(a) W'(a)} \\ &\quad \times \left\{ Z_\kappa^{(\mu-\psi(\kappa))}(a) \frac{W_\kappa^{(\mu-\psi(\kappa))\prime}(a)}{W_\kappa^{(\mu-\psi(\kappa))}(a)} - (\mu - \psi(\kappa)) W_\kappa^{(\mu-\psi(\kappa))}(a) \right\}. \end{aligned}$$

Note in particular, we have the following corollary giving a somewhat simpler expression from which one easily derives the moments of the net present value of paid dividends associated with a barrier strategy.

Corollary 1. *For $n = 1, 2, \ldots$ we have*

$$\mathbb{E}_x\left[\left(\int_0^{\sigma_a} e^{-qt} dL_t\right)^n\right] = n! \frac{W^{(qn)}(x)}{W^{(qn)}(a)} \prod_{k=1}^{n} \frac{W^{(qk)}(a)}{W^{(qk)\prime}(a)}.$$

For the sake of completeness and for later reference, we present the result of Avram et al. (2004).

Lemma 1. *For $\mu, \kappa \geq 0$,*

$$\mathbb{E}_x\left[e^{-\mu\sigma_a - \kappa(Y_{\sigma_a} - a)}\right] = e^{\kappa x}\left(Z_\kappa^{(\upsilon)}(x) - C_{\kappa,\mu}(a) W_\kappa^{(\upsilon)}(x)\right),$$

where $\upsilon = \mu - \psi(\kappa)$ and $C_{\kappa,\mu}(a) = (\upsilon W_\kappa^{(\upsilon)}(a) + \kappa Z_\kappa^{(\upsilon)}(a))/(W_\kappa^{(\upsilon)\prime}(a) + \kappa W_\kappa^{(\upsilon)}(a))$.

In particular this lemma gives the joint Laplace transform of the time to ruin and the deficit at ruin.

3.2. Tail asymptotics

The second of our main results concerns the case that the process X satisfies the so-called Spitzer-Doney condition *and hence is oscillating* (cf. Chapter III of Bertoin (1996)). This is not compatible with the net profit condition in the case that X the classical Cramér-Lundberg process. The Spitzer-Doney condition stipulates that there exists a $\rho \in (0, 1)$ such that

$$\lim_{t\uparrow\infty} \frac{1}{t} \int_0^t \mathbb{P}(X_s \geq 0) ds = \rho$$

or equivalently (cf. Doney (1995)) that

$$\lim_{t\uparrow\infty} \mathbb{P}(X_s \geq 0) = \rho.$$

A classical example of a spectrally negative Lévy process fulfilling this condition is that of a spectrally negative stable process of index $\alpha \in (1, 2)$.

Our second main result now follows.

Theorem 2. *Suppose that X fulfills the Doney-Spitzer condition with index $\rho \in (0, 1)$. Let $I_q = \int_0^{\sigma_a} e^{-qt} dL_t$. Then for $x \in (0, a]$, as $y \uparrow \infty$,*

$$-\log \mathbb{P}_x(I_q > y) \sim (1 - \rho)\varphi(y),$$

where $\varphi(x)$ is the unique solution in $(0, \infty)$ to the equation $\psi(\theta) = \theta x$.

Note in particular that the asymptotic above is independent of q.

4. Some calculations for an integrated bivariate subordinator

In order to prove Theorem 1 we first need to move to yet one more degree of abstraction. To motivate why, we must briefly turn to Itô's excursion theory. Denote by $\{(t, \epsilon_t) : t \geq 0\}$ the Poisson point process of excursions from zero of Y indexed by local time at zero whose intensity measure necessarily takes the form $dt \times n(d\epsilon)$ on the product space $[0,\infty) \times \mathcal{E}$ where $\mathcal{E}$ is the space of excursions. The reader is referred to Chapters 5 and 6 of Bertoin (1996) for the necessary background. Assuming that $X_0 = a$ so that $L_t = \overline{X}_t$ we may make a change of variables $t \mapsto L_t^{-1}$ to deduce that

$$I_q = \int_0^\infty e^{-qL_t^{-1}} \mathbf{1}_{(\sup_{s\leq t} \overline{\epsilon}_s \leq a)} dt,$$

where $\overline{\epsilon}_t$ is the height of the excursion at local time t. As a consequence of the fact that excursions form a Poisson point process one may identify the above integral as belonging to the class of integrals of the form

$$\int_0^\infty e^{-q\xi_t} \mathbf{1}_{(\sup_{s\leq t} \Delta\eta_s \leq a)} dt, \tag{3}$$

where $\Delta\eta_t = \eta_t - \eta_{t-}$ and $(\xi,\eta) = \{(\xi_t,\eta_t) : t \geq 0\}$ is a bivariate subordinator with respect to some probability measure which we shall denote by $\mathbf{P}$ in the sequel.

We thus devote the remainder of this section to a study of results concerning the object in (3) that will be of use later on. Note that such integrals are very close in nature to objects which are called integrated exponential subordinators and which have received quite some attention in recently. See for example the survey in Bertoin and Yor (2005). Not surprisingly some of our calculations are quite close in nature to those that have are been exposed in the latter article.

Let us henceforth assume $\Lambda(q)$ is the Laplace exponent of ξ. Assume further that $\nu_{\Lambda(q)}$ is the jump measure of η under the change of measure

$$\left.\frac{d\mathbf{P}^{\Lambda(q)}}{d\mathbf{P}}\right|_{\mathcal{G}_t} = e^{\Lambda(q)t - q\xi_t},$$

where $q > 0$ and $\{\mathcal{G}_t : t \geq 0\}$ is the filtration (satisfying the usual conditions) generated by the process (ξ,η).

Theorem 3. *For $n = 1,2,3,...$ and $q > 0$,*

$$\mathbf{E}\left[\left(\int_0^\infty e^{-q\xi_t} \mathbf{1}_{(\sup_{s\leq t} \Delta\eta_s \leq a)} dt\right)^n\right] = n! \prod_{k=1}^n \frac{1}{\Lambda(qk) + \nu_{\Lambda(qk)}(a,\infty)}.$$

Proof. We follow a similar argument for the calculation of the moments of an integrated exponential Lévy process as given in Bertoin and Yor (2005). To this end, define

$$J_t = \int_t^\infty e^{-q\xi_s} \mathbf{1}_{(\sup_{u\leq s} \Delta\eta_u \leq a)} ds.$$

Since,

$$\frac{d}{dt} J_t^n = -nJ_t^{n-1} e^{-q\xi_t} \mathbf{1}_{(\sup_{u\leq t} \Delta\eta_u \leq a)}$$

we obtain

$$J_0^n - J_t^n = n\int_0^t e^{-q\xi_s} \mathbf{1}_{(\sup_{u\leq s} \Delta\eta_u \leq a)} J_s^{n-1} ds.$$

Note also that by stationary and independent increments we can write

$$J_t = e^{-q\xi_t}\mathbf{1}_{(\sup_{u\le t}\Delta\eta_u\le a)}J_0^*,$$

where J_0^* is independent of $\mathcal{G}_t$ and has the same distribution as J_0. In conclusion, if we let

$$\Psi_n = \mathbf{E}\left(J_0^n\right)$$

then

$$\Psi_n\left(1-\mathbf{E}(e^{-n\xi_t}\mathbf{1}_{(\sup_{u\le t}\Delta\eta_u\le a)})\right) = n\Psi_{n-1}\int_0^t \mathbf{E}(e^{-n\xi_s}\mathbf{1}_{(\sup_{u\le s}\Delta\eta_u\le a)})ds. \tag{4}$$

Now using the above change of measure, and the fact that the first arrival of a jump of size a for the process η occurs after an exponentially distributed period of time, we note that

$$\begin{aligned}\mathbf{E}(e^{-qn\xi_t}\mathbf{1}_{(\sup_{u\le t}\Delta\eta_u\le a)}) &= e^{-\Lambda(qn)t}\mathbf{P}^{\Lambda(qn)}(\sup_{u\le t}\Delta\eta_u\le a)\\ &= e^{-[\Lambda(qn)+\nu_{\Lambda(qn)}(a,\infty)]t}.\end{aligned}$$

Plugging the above back into (4) we now see the iteration

$$\Psi_n = n\Psi_{n-1}\frac{1}{\Lambda(qn)+\nu_{\Lambda(qn)}(a,\infty)}.$$

The proof now concludes in the obvious way. □

The following corollary is rather specific but is precisely what is needed later on.

Corollary 2. *Suppose that for $\theta\ge 0$, $\mathbf{e}_{\Lambda(\theta)}$ is an independent exponential random variable with parameter $\Lambda(\theta)$. (In the usual sense we understand this random variable to be equal to infinity with probability one when $\theta=0$). Then for $n=1,2,3,...$, $q>0$ and $\theta\ge 0$,*

$$\mathbf{E}^{\Lambda(\theta)}\left[\left(\int_0^{\mathbf{e}_{\Lambda(\theta)}} e^{-q\xi_t}\mathbf{1}_{(\sup_{s\le t}\Delta\eta_s\le a)}\right)^n\right] = n!\prod_{i=1}^n\frac{1}{\Lambda(\theta+qk)+\nu_{\Lambda(\theta+qk)}(a,\infty)}. \tag{5}$$

Proof. We again keep to similar ideas to those given in Bertoin and Yor (2005). Specifically, in the previous result, replace ξ_t by $\xi'_t=\xi_t+\alpha N_t$ where $N=\{N_t:t\ge 0\}$ is an independent Poisson process with rate $\Lambda(\theta)$ and $\alpha>0$. Then taking account of the fact that under $\mathbf{P}^{\Lambda(\theta)}$, ξ is still a subordinator with Laplace exponent $\Lambda(\theta+q)-\Lambda(q)$, we have from Theorem 3 that

$$\begin{aligned}&\mathbf{E}^{\Lambda(\theta)}\left[\left(\int_0^\infty e^{-q\xi'_t}\mathbf{1}_{(\sup_{s\le t}\Delta\eta_s\le a)}dt\right)^n\right]\\ &= n!\prod_{k=1}^n\frac{1}{\Lambda(\theta+qk)-\Lambda(\theta)+\Lambda(\theta)(1-e^{-\alpha qk})+\nu^{(k)}(a,\infty)},\end{aligned}$$

where $\nu^{(k)}(a,\infty)$ is the jump measure of η under $\mathbf{P}^{(k)}$ where the latter is given by the change of measure

$$\left.\frac{d\mathbf{P}^{(k)}}{d\mathbf{P}^{\Lambda(\theta)}}\right|_{\mathcal{F}_t} = e^{[\Lambda(\theta+qk)-\Lambda(\theta)+\Lambda(\theta)(1-e^{-\alpha qk})]t-q\xi'_t}.$$

Now taking limits as $\alpha\uparrow\infty$ the result follows by monotone convergence. □

5. Proof of Theorem 1

The proof of Theorem 1 follows by combining the series of Lemmas presented below. As well as making use of the conclusions of the previous section, we shall also make use of the following fundamental property associated with the process of excursions from zero of Y which was earlier denoted $\{(t,\epsilon_t) : t\geq 0\}$. Note that the excursion ϵ_t which occurs at local time t is a process in time and hence has a second index referring to the time spent in the excursion. Strictly speaking $\epsilon_t = \{\epsilon_t(s) := X_{L^{-1}_{t-}} - X_{L^{-1}_{t-}+s} : 0 < s\leq L^{-1}_t - L^{-1}_{t-}\}$ whenever $L^{-1}_t - L^{-1}_{t-} > 0$. (See Bertoin (1996) for further discussion). Accordingly we refer to a generic excursion as $\epsilon(\cdot)$ or just ϵ for short as appropriate. Define the set A in the space of excursions to be all excursions whose height is greater than a; that is $A = \{\epsilon\in\mathcal{E} : \overline{\epsilon} > a\}$. Let

$$T_a = \inf\{t > 0 : \epsilon_t\in A\}.$$

Note immediately that $T_a = L_{\sigma_a}$. The thinning property of Poisson point processes tells us that

(i) T_a is a stopping time with respect to the natural filtration of the underlying Poisson point process and is exponentially distributed with parameter $n(\overline{\epsilon} > a)$ where $\overline{\epsilon}$ is the maximum of the generic excursion,

(ii) ϵ_{T_a} is independent of $\{\epsilon_t : t < T_a\}$ and has law given by

$$\frac{n(d\epsilon;\overline{\epsilon} > a)}{n(\overline{\epsilon} > a)},$$

(iii) $\{\epsilon_t : t < T_a\}$ is equal in law to a Poisson point process with intensity $dt\times n(d\epsilon;\overline{\epsilon}\leq a)$ stopped at an independent and exponentially distributed time with parameter $n(\overline{\epsilon} > a)$.

Our sequence of Lemmas now follows.

Lemma 2. *For $n = 1,2,3,...$, $x\in(0,a)$ and $\lambda,\mu,\kappa\geq 0$,*

$$\Theta(x;n,\lambda,\mu,\kappa) = \frac{W^{(\lambda+qn)}(x)}{W^{(\lambda+qn)}(a)}\Theta(a;n,\lambda,\mu,\kappa).$$

Proof. Let

$$\tau^+_a := \inf\{t > 0 : X_t > a\} \text{ and } \tau^-_0 := \inf\{t > 0 : X_t < 0\}.$$

The random variable in the expectation described by $\Theta(x;n,\lambda,\mu,\kappa)$ is identically equal to zero on the event $\{\tau^+_a > \tau^-_0\}$. On the other hand, on the event $\{\tau^+_a < \tau^-_0\}$, factoring out the time spent to reach a in the exponential involving λ and q, the Strong Markov Property and spectral negativity of X imply that

$$\Theta(x;n,\lambda,\mu,\kappa)=\mathbb{E}(e^{-(\lambda+qn)\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)})\Theta(a;n,\lambda,\mu,\kappa).$$

The result now follows from the identity for the two sided exit problem of a spectrally negative Lévy process. See Theorem 8.1 of Kyprianou (2006). □

Lemma 3. *For $n=1,2,3,...$ and $\lambda,\mu,\kappa\geq 0$,*

$$\Theta(a;n,\lambda,\mu,\kappa)=\mathbb{E}_a^{\Phi(\lambda)}\left[e^{-\Phi(\lambda)T_a}\left(\int_0^{T_a}e^{-qL_s^{-1}}ds\right)^n\right]\frac{n(e^{-\mu\rho_a-\kappa\epsilon(\rho_a)};\overline{\epsilon}>a)}{n(\overline{\epsilon}>a)},$$

where $T_a=L_{\sigma_a}$ and $\rho_a=\inf\{s\geq 0:\epsilon(s)>a\}$.

Proof. Using the thinning property (ii) of the Poisson point process of excursions together with exponential change of measure at the stopping time T_a (following from thinning property (i) of Poisson point processes) we have that

$$\begin{aligned}
&\Theta(a;n,\lambda,\mu,\kappa)\\
&=\mathbb{E}_a\left[e^{-\lambda L^{-1}_{T_a}-\mu(\sigma_a-L^{-1}_{T_a})-\kappa Y_{\sigma_a}}\left(\int_0^{L^{-1}_{T_a}}e^{-qt}dL_t\right)^n\right]\\
&=\mathbb{E}_a\left[e^{-\lambda L^{-1}_{T_a}}\left(\int_0^{T_a}e^{-qL_t^{-1}}dt\right)^n\right]\frac{n(e^{-\mu\rho_a-\kappa\epsilon(\rho_a)};\overline{\epsilon}>a)}{n(\overline{\epsilon}>a)}\\
&=\mathbb{E}_a^{\Phi(\lambda)}\left[e^{-\Phi(\lambda)T_a}\left(\int_0^{T_a}e^{-qL_t^{-1}}dt\right)^n\right]\frac{n(e^{-\mu\rho_a-\kappa\epsilon(\rho_a)};\overline{\epsilon}>a)}{n(\overline{\epsilon}>a)}
\end{aligned}$$

and the result follows. □

Lemma 4. *For $\mu,\kappa\geq 0$,*

$$\begin{aligned}
&n(e^{-\mu\rho_a-\kappa\epsilon(\rho_a)};\overline{\epsilon}>a)\\
&=Z_\kappa^{(\mu-\psi(\kappa))}(a)\frac{W_\kappa^{(\mu-\psi(\kappa))\prime}(a)}{W_\kappa^{(\mu-\psi(\kappa))}(a)}-(\mu-\psi(\kappa))W_\kappa^{(\mu-\psi(\kappa))}(a).
\end{aligned}$$

Proof. This has already been established in (the proof of) Theorem 1 of Avram, Kyprianou and Pistorius (2004). □

Lemma 5. *For $n=1,2,3,...$ and $\lambda,\mu,\kappa\geq 0$,*

$$\begin{aligned}
&\mathbb{E}_a^{\Phi(\lambda)}\left[e^{-\Phi(\lambda)T_a}\left(\int_0^{T_a}e^{-qL_s^{-1}}ds\right)^n\right]\\
&=\frac{e^{\Phi(\lambda)a}W'_{\Phi(\lambda)}(a)}{W^{(\lambda)\prime}(a)}\mathbb{E}_a^{\Phi(\lambda)}\left[\left(\int_0^{\mathbf{e}_{\Phi(\lambda)}}e^{-qL_t^{-1}}\mathbf{1}_{(t<T_a)}dt\right)^n\right].
\end{aligned}$$

Proof. Using the thinning property (iii) of Poisson point processes mentioned in the proof of the last but one lemma, write $p = n_{\Phi(\lambda)}(\overline{\epsilon} > a)$ where $n_{\Phi(\lambda)}$ is the excursion measure under $\mathbb{P}^{\Phi(\lambda)}$, and

$$\begin{aligned}
&\mathbb{E}_a^{\Phi(\lambda)}\left[e^{-\Phi(\lambda)T_a}\left(\int_0^{T_a} e^{-qL_s^{-1}}ds\right)^n\right] \\
&= \int_0^\infty ds \cdot pe^{-(\Phi(\lambda)+p)s}\mathbb{E}_a^{\Phi(\lambda)}\left[\left(\int_0^s e^{-qS_t}dt\right)^n\right],
\end{aligned}$$

where S is the subordinator with the same drift as L^{-1} but whose jump measure is given by $n(\zeta(\epsilon) \in dx; \overline{\epsilon} \le a)$ and $\zeta(\epsilon)$ is the length of the generic excursion ϵ. Continuing we have,

$$\begin{aligned}
&\mathbb{E}_a^{\Phi(\lambda)}\left[e^{-\Phi(\lambda)T_a}\left(\int_0^{T_a} e^{-qL_s^{-1}}ds\right)^n\right] \\
&= \frac{p}{p+\Phi(\lambda)}\mathbb{E}_a^{\Phi(\lambda)}\left[\left(\int_0^{\mathbf{e}_{\Phi(\lambda)}\wedge\mathbf{e}_p} e^{-qS_t}dt\right)^n\right] \\
&= \frac{p}{p+\Phi(\lambda)}\mathbb{E}_a^{\Phi(\lambda)}\left[\left(\int_0^{\mathbf{e}_{\Phi(\lambda)}} e^{-qL_t^{-1}}\mathbf{1}_{(t<T_a)}dt\right)^n\right].
\end{aligned}$$

Note that both exponential random variables $\mathbf{e}_{\Phi(\lambda)}$ and $\mathbf{e}_p$ are independent in the above calculations.

Recalling from Lambert (2000) that $n(\overline{\epsilon} > a) = W'(a)/W(a)$ and from Avram et al. (2004) that $W^{(q)}(x) = e^{\Phi(q)x}W_{\Phi(q)}(x)$ we may now note that

$$\begin{aligned}
\frac{p}{p+\Phi(\lambda)} &= \frac{W'_{\Phi(\lambda)}(a)}{W_{\Phi(\lambda)}(a)\Phi(\lambda) + W'_{\Phi(\lambda)}(a)} \\
&= \frac{e^{\Phi(\lambda)a}W'_{\Phi(\lambda)}(a)}{W^{(\lambda)\prime}(a)}
\end{aligned}$$

thus concluding the proof. □

Lemma 6. *For $n = 1, 2, 3, \ldots$ and $\lambda \ge 0$ we have*

$$\mathbb{E}_a^{\Phi(\lambda)}\left[\left(\int_0^{\mathbf{e}_{\Phi(\lambda)}} e^{-qL_t^{-1}}\mathbf{1}_{(t<T_a)}dt\right)^n\right] = n!\prod_{k=1}^n \frac{W^{(\lambda+qk)}(a)}{W^{(\lambda+qk)\prime}(a)}.$$

Proof. This result follows as a direct consequence of Corollary 2. Note in that case the term $\nu_{\Lambda(\theta+qk)}(a,\infty)$ is played by the role of

$$\begin{aligned}
n_{\Phi(\lambda+qk)}(\overline{\epsilon} > a) &= \frac{W'_{\Phi(\lambda+qk)}(a)}{W_{\Phi(\lambda+qk)}(a)} \\
&= \frac{W^{(\lambda+qk)\prime}(a)}{W^{(\lambda+qk)}(a)} - \Phi(\lambda+qk).
\end{aligned}$$

and a little algebra is necessary on the right hand side of (5). □

6. Proof of Theorem 2

First we should note that the Spitzer-Doney condition implies that $\Phi(q) = q^\rho L(q)$ where L is a slowly varying function; see Theorem VI.14 of Bertoin (1996). Following this observation, the proof now mimics very closely the proof of Proposition 2 of Rivero (2003) at the heart of which is an application of Kashara's Tauberian Theorem (cf. Theorem 4.12 of Bingham et al. (1987)). Essentially the arguments go though identically except in our case, it is now necessary to verify the following asymptotics

(i) $\lim_{q\uparrow\infty} n_{\Phi(q)}(\overline{\epsilon} > a) = 0$,

(ii) $\limsup_{n\uparrow\infty} n\log n/\log(1/m_n) = 1/\rho$,

(iii) $\lim_{n\uparrow\infty} m_n^{1/n}\Phi(n) = e^\rho$,

where

$$m_n = \frac{W^{(qn)}(x)}{W^{(qn)}(a)}\prod_{k=1}^n \frac{W^{(qk)}(a)}{W^{(qk)\prime}(a)} = e^{\Phi(qn)(x-a)}\frac{W_{\Phi(qn)}(x)}{W_{\Phi(qn)}(a)}\prod_{k=1}^n \frac{1}{\Phi(qk) + n_{\Phi(qk)}(\overline{\epsilon} > a)}.$$

Note in the last equality we have used the fact that $W^{(q)}(x) = e^{\Phi(q)x}W_{\Phi(q)}(x)$ and that $n_{\Phi(q)}(\overline{\epsilon} > a) = W'_{\Phi(q)}(a)/W_{\Phi(q)}(a)$. In fact even the proofs of (ii) and (iii) go through as in Rivero (2003) once we prove (i).

To this end, we recall from (8.15), (8.3) and the discussion below it in Chapter 8 of Kyprianou (2006) that for fixed $a > 0$,

$$0 \le 1 - \psi'(\Phi(q))W_{\Phi(q)}(a) = \mathbb{P}_a^{\Phi(q)}(\tau_0^- < \infty) = \mathbb{E}_a(e^{\Phi(q)(X_{\tau_0^-}-a)-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}).$$

It is now clear taking limits as $q \uparrow \infty$ that $\psi'(\Phi(q))W_{\Phi(q)}(a) \to 1$. Next note also with the help of equation (7.7) in Chapter 7 of Kyprianou (2006) that we also have that

$$\psi'(\Phi(q))U_{\Phi(q)}(a,\infty) = 1 - \psi'(\Phi(q))W_{\Phi(q)}(a), \tag{6}$$

where $U_{\Phi(q)}$ is the potential measure associated with the descending ladder height process of X under the measure $\mathbb{P}^{\Phi(q)}$. It follows from (6) that $U_{\Phi(q)}$ has a density which satisfies on $x > 0$

$$\begin{aligned}\psi'(\Phi(q))W'_{\Phi(q)}(x)dx &= \int_0^\infty dt\cdot\psi'(\Phi(q))\mathbb{P}^{\Phi(q)}(\widehat{H}_t \in dx)\\ &= \int_0^\infty dt\cdot\psi'(\Phi(q))e^{-\Phi(q)x}\mathbb{E}(e^{-q\widehat{L}_t^{-1}}\mathbf{1}_{(\widehat{H}_t\in dx)}),\end{aligned}$$

where now $\{(\widehat{L}_t^{-1},\widehat{H}_t) : t \ge 0\}$ is the descending ladder process. Since $\Phi(q)$ is concave (cf. Section 5.5 of Kyprianou (2006)) and regularly varying with index ρ it follows that ψ' is regularly varying with index $\rho^{-1} - 1$. From this we deduce that the the measure $\psi'(\Phi(q))W'_{\Phi(q)}(x)dx$ converges weakly to the zero measure as $q \uparrow \infty$. Consequently, since $W'_{\Phi(q)}(x)$ is known to be continuous in both its arguments (cf. Section 8.3 of Kyprianou (2006)), we conclude that it also tends to zero in the limit as $q \uparrow \infty$.

To conclude the proof of (i) and hence the proof of the theorem, simply recall that that

$$n_{\Phi(q)}(\bar{\epsilon} > a) = \frac{\psi'(\Phi(q))W'_{\Phi(q)}(a)}{\psi'(\Phi(q))W_{\Phi(q)}(a)}$$

and take limits as $q \uparrow \infty$. □

7. Concluding remarks

We conclude this paper with some remarks on techniques and results pertaining to some special cases handled by other authors. In what follows we shall write for short $\Theta(x;n)$ in place of $\Theta(x;n,0,0,0)$.

7.1. Special examples

There are few examples for which the scale function is known in explicit form (although Rogers (2000) and Surya (2006) have developed methods to numerically invert the Laplace transform (2)). One of them is the case that X is a Cramér-Lundberg process with exponentially distributed jump. In that case, if $c^{CL} > 0$ is the premium rate, $\lambda^{CL} > 0$ is the it is rate of arrival of claims and μ^{CL} is the parameter of the exponentially distributed claims, then it is known that

$$W^{(q)}(x) = \frac{1}{c^{CL}}(k_+(q)e^{r_+(q)x} - k_-(q)e^{r_-(q)x}), \tag{7}$$

where $k_\pm(q) = (\mu^{CL} + r_\pm(q))/(r_+(q) - r_-(q))$ and

$$r_\pm(q) = \frac{q + \lambda^{CL} - \mu^{CL}c^{CL} \pm \sqrt{(q + \lambda^{CL} - \mu^{CL}c^{CL})^2 + 4c^{CL}q\mu^{CL}}}{2c^{CL}}.$$

One may then develop the expression for $\Theta(x;n)$ given in Corollary 1 and see consistency with the expression given in (2.7) of Dickson and Waters (2004). Note also that in this case the deficit at the time of ruin is exponentially distributed independent of the time of ruin σ_a and

$$\mathbb{E}_x\left[e^{-\mu\sigma_a}\right] = Z^{(\mu)}(x) - C_{0,\mu}(a)W^{(\mu)}(x)$$

with $W^{(\mu)}$ given in (7) which agrees with (4.4) of Dickson and Waters (2004).

In fact, the scale function could be derived in an explicit form for any phase-type distribution F of the claim size in the Cramér-Lundberg model. That is, let

$$1 - F(x) = \alpha e^{\mathbf{T}x}\mathbf{1},$$

where $\mathbf{T}$ is a subintensity matrix of some killed finite-state Markov process, α is its initial probability vector and $\mathbf{1}$ denotes a column vector of ones. Then

$$W(x) = \frac{1}{\psi'(0+)}\left(1 - \sum_{j\in\mathcal{I}} A_j e^{\varrho_j x}\right), \tag{8}$$

where $\mathcal{I}$ is set of all roots ϱ_j with negative real part (we assume that they are distinct) solving Cramér-Lundberg equation

$$\psi(\varrho) = 0$$

and

$$A_j = \lim_{s \to \varrho_j} \frac{\psi'(0+)(\varrho_j - s)}{\psi(s)}$$

with $\psi(\theta) = c^{CL}\theta + \lambda^{CL}\left(\alpha(\theta\mathbf{I} - \mathbf{T})^{-1}\mathbf{t} - 1\right)$ where $\mathbf{t} = -\mathbf{T}\mathbf{1}$; for details see Lemma 1 of Asmussen et al. (2004) and (13) of Kyprianou and Palmowski (2004). We recall also that $W^{(q)}(x) = e^{\Phi(q)x}W_{\Phi(q)}(x)$. Using Lemma 1 the expression (8) and straightforward differentiation allows to calculate

$$\mathbb{E}_x\left[e^{-\mu\sigma_a}(Y_{\sigma_a} - a)^n\right] \tag{9}$$

in the case of phase-type distribution of the claim size. This will generalize Lin and Willmot (2000) and Lin et al. (2003) where in particular combinations of exponentials and mixtures of Erlangs are considered.

Another example is the case of an α-stable process where $\alpha \in (1, 2)$. In this case we may take $\psi(\theta) = \theta^\alpha$. It is known for this class of processes that the positivity parameter $\rho = \alpha^{-1}$ and that

$$W^{(q)}(x) = \alpha x^{\alpha-1}E'_\alpha(qx^\alpha),$$

where E_α is the Mittag-Leffler function of index α:

$$E_\alpha(x) = \sum_{n=0}^\infty \frac{x^n}{\Gamma(1 + \alpha n)}$$

for $x \in \mathbb{R}$. Note that such stable processes do not fulfil the condition $\lim_{t\uparrow\infty} X_t = \infty$ but instead oscillate. This puts them within the context of Theorem 2 which now says that

$$-\log \mathbb{P}_x(I_q > y) \sim \left(\frac{\alpha - 1}{\alpha}\right) y^{1/(\alpha-1)}.$$

as $y \uparrow \infty$.

7.2. Integro-differential equations

In the case that X is the classical Cramér-Lundberg process with linear drift coefficient c^{CL} and negative jumps arriving at rate $\lambda^{CL} > 0$ having distribution function F, by conditioning on the first jump, Dickson and Waters (2004) show that $\Theta(x; n)$ solves the integro-differential equation

$$c^{CL}\Theta'(x; n) + \lambda^{CL}\int_{(0,\infty)}\{\Theta(x - y; n) - \Theta(x; n)\}F(dy) - qn\Theta(x; n) = 0 \tag{10}$$

on $(0, a)$ with boundary condition $\Theta'(a-; n) = n\Theta(a; n-1)$ for $n = 1, 2, 3, ...$ (see their Theorem 2.1 to make the connection with the above equation noting in particular that $\Theta(x; n) = 0$ for $x < 0$). In the language of extended generators, (10) can be written more simply as

$$(\Gamma - qn)\Theta(x; n) = 0, \tag{11}$$

where Γ is the extended generator of X.

It is now interesting to note from the conclusion of Lemma 1 that, in the general case, up to a constant which depends on a, q and n we have that $\Theta(x; n) \propto W^{(qn)}(x)$.

It is known (cf. Avram et al. (2004)) that $e^{-qnt}W^{(qn)}(X_t)$ is a martingale and hence $W^{(qn)}$ is in the domain of the extended generator and it solves the generator equation $(\Gamma - qn)W^{(qn)}(x) = 0$ on $(0,\infty)$ (and in particular on $(0,a)$). Therefore, under such circumstances, it also follows that (11) holds. We also note that from the conclusion of Corollary 1 that

$$\Theta'(a-;n) = n!\frac{W^{(qn)\prime}(a)}{W^{(qn)}(a)}\prod_{i=1}^{n}\frac{W^{(qk)}(a)}{W^{(qk)\prime}(a)} = n(n-1)!\prod_{i=1}^{n-1}\frac{W^{(qk)}(a)}{W^{(qk)\prime}(a)} = n\Theta(a;n-1).$$

In fact Dickson and Waters (2004) consider equation (11) in the context of infinitesimal generator Γ for sufficiently smooth function $\Theta(x;n)$. 'Sufficiently smooth' here means that the aforementioned generator equation is mathematically well defined. Note that the fact that scale functions are not known to be necessarily smooth enough to use in conjunction with the infinitesimal generator of the associated Lévy process implies that the method of Dickson and Waters (2004) could not be easily be implemented for the general case considered here.

7.3. The case that $q = 0$

In Section 3 of Dickson and Waters (2004) it was noted that when $q = 0$, $\int_0^{\sigma_a} e^{-qt}dL_t = L_{\sigma_a}$ (i.e. the total dividends paid until ruin) is equal in distribution to a mixture of an atom at zero and an exponential distribution. Again we remark that their method of proof cannot be applied in the general Lévy process setting as it relies heavily on the fact that in the Cramér-Lundberg model the support of the local time measure dL consists of the union of a finite number of closed intervals.

None the less their result is still true even in the more general context here. Taking account of whether the process Y hits zero before exceeding level a we see that for $x > 0$

$$\begin{aligned}
\mathbb{P}_x(L_{\sigma_a} \in dz) &= \delta_0(dz)\mathbb{P}_x(\tau_a^+ > \tau_0^-) + \mathbb{P}_a(L_{\sigma_a} \in dz)\mathbb{P}_x(\tau_a^+ < \tau_0^-) \\
&= \delta_0(dz)\left(1 - \frac{W(x)}{W(a)}\right) + \frac{W(x)}{W(a)}\mathbb{P}_a(L_{\sigma_a} \in dz) \\
&= \delta_0(dz)\left(1 - \frac{W(x)}{W(a)}\right) + \frac{W(x)}{W(a)}n(\overline{\epsilon} > a)e^{-n(\overline{\epsilon}>a)z}dz,
\end{aligned}$$

where the last equality follows by virtue of the fact that under $\mathbb{P}_a$, $L_{\sigma_a} = T_a$ which, following the thinning property of Poisson point processes (i) is exponentially distributed. Finally recall that $n(\overline{\epsilon} > a) = W'(a)/W(a)$ so that in conclusion

$$\mathbb{P}_x(L_{\sigma_a} \in dz) = \delta_0(dz)\left(1 - \frac{W(x)}{W(a)}\right) + \frac{W(x)W'(a)}{W(a)^2}e^{-zW'(a)/W(a)}dz.$$

7.4. The red period

Denoting the distribution F_x of the deficit $Y_{\sigma_a} - a$ under $\mathbb{P}_x$, one can also find the Laplace transform of so-called 'red period' r, that is the duration of the first negative surplus (see Dos Reis (1993) and Dickson and Dos Reis (1996)). Indeed we have for $s \geq 0$,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-sr}\right] &= \int_0^\infty \mathbb{E}\left[e^{-s\tau_y^+}\right] F_x(dy) = \int_0^\infty e^{-\Phi(s)y} F_x(dy) \\
&= \mathbb{E}_x\left[e^{-\Phi(s)(Y_{\sigma_a}-a)}\right] = e^{\Phi(s)x}\left(Z_{\Phi(s)}^{(-s)}(x) - C_{\Phi(s),0}(a)W_{\Phi(s)}^{(-s)}(x)\right),
\end{aligned}$$

where Theorem 1 of Kyprianou and Palmowski (2004) has been used in the second equality and Lemma 1 has been used in the last equality. The last expression can be developed further when one takes account of the relations $W^{(-s)}_{\Phi(s)}(x) = e^{-\Phi(s)x}W(x)$ and $Z^{(-s)}_{\Phi(s)}(x) = 1 + s\int_0^x e^{-\Phi(s)y}W(y)dy$, which is a consequence of Lemma 8.4 of Kyprianou (2006). In conclusion,

$$\mathbb{E}_x\left[e^{-sr}\right] = e^{\Phi(s)x} + s\int_0^x e^{\Phi(s)(x-y)}W(y)dy - C_{\Phi(s),0}W(x)$$

where now we can write

$$C_{\Phi(s),0} = \frac{\Phi(s)e^{\Phi(s)a} + s\Phi(s)\int_0^a e^{\Phi(s)(a-y)}W(y)dy - sW(a)}{W'(a)}.$$

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References

[1] Asmussen, S., Avram, F. and Pistorius, M. (2004) Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.* **109(1)**, 79–111.

[2] Avram, F., Kyprianou, A.E. and Pistorius, M.R. (2004) Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.* **14**, 215–238.

[3] Avram, F., Palmowski, Z. and Pistorius, M. R. (2006) On the optimal dividend problem for a spectrally negative Lévy process. *To appear in Ann. Appl.Probab.*

[4] Bertoin, J. (1996) *Lévy processes.* Cambridge University Press.

[5] Bertoin, J. and Yor, M. Exponential functionals of Lévy processes. *Probab. Surveys* **2**, 191–212.

[6] Bingham, N., Goldie, C. and Teugels, J. (1987) *Regular variation.* Cambridge University Press.

[7] De Finetti, B. (1957) Su un'impostazion alternativa dell teoria collecttiva del rischio. *Transactions of the XVth International Congress of Actuaries* **2**, 433–443.

[8] Dickson, D. C. M. and Waters, H. R. (2004) Some optimal dividends problems. *Astin Bull.* **34**, 49–74.

[9] Dickson, D. C. M. and Dos Reis, A. E. (1996) On the distribution of the duration of negative surplus. *Scand. Actuar. J.* **2**, 148–164.

[10] Doney, R.A. (1995) Spitzer's condition and ladder variables in random walks. *Probab. Theory Related Fields* **101**, 577–580.

[11] Doney, R. A. and Kyprianou, A. E. (2006) Overshoots and undershoots of Lévy processes. *Ann. Appl. Probab.* **16**, 91–106.

[12] Dos Reis, A. E. (1993) How long is the surplus below zero? *Insurance Math. Econom.* **12(1)**, 23–38.

[13] Furrer, H. (1998). Risk processes perturbed by α-stable Lévy motion. *Scand. Actuar. J.* 5974.

[14] Huzak, M., Perman, M., Šikić, H. and Vondraček, Z. (2004a) Ruin probabilities and decompositions for general perturbed risk processes. *Ann. Appl. Probab.* **14**, 1378–1397.

[15] Huzak, M., Perman, M., Šikić, H. and Vondraček, Z. (2004b) Ruin probabilities for competing claim processes. *J. Appl. Probab.* **41**, 679–690.

[16] Klüppelberg, C., Kyprianou, A. E. and Maller, R. A. (2004) Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.* **14**, 1766–1801.

[17] Klüppelberg, C. and Kyprianou, A. E. (2006) On extreme ruinous behaviour of Lévy insurance risk processes. *J. Appl. Probab.* **43(2)**, 594–598.

[18] Kyprianou, A. E. (2006) *Introductory lectures on fluctuations on fluctuations of Lévy processes with applications.* Springer.

[19] Kyprianou, A. E. and Palmowski, Z. (2004) A martingale review of some fluctuation theory for spectrally negative Lévy processes. *Séminaire de Probabilités XXXVIII*, 16–29.

[20] Lin, X. S. and Willmot, G. E. (2000) The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. *Insurance Math. Econom.* **27(1)**, 19–44.

[21] Lin, X. S., Willmot, G. E. and Drekic, S. (2003) The classical risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function. *Insurance Math. Econom.* **33(3)**, 551–566.

[22] Rivero, V. (2003) A law of iterated logarithm for increasing self-similar Markov processes. *Stoch. Stochast Rep.*, **75**, 443-472.

[23] Rogers, L. C. G. (2000) Evaluating first-passage probabilities for spectrally one-sided Lévy processes. *J. Appl. Probab.* **37**, 1173–1180.

[24] Surya, B.A. (2006) Evaluating Scale Functions of Spectrally Negative Lévy Processes. *Preprint.*

[25] Willmot, G.E. (2000) On evaluation of the conditional distribution of the deficit at the time of ruin. *Scand. Actuarial J.* **1**, 63–79.