Operational Risk

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Introduction

Risk management considers four risks:

✠ market (equities, interest rates, fx, commodities)

✠ credit (default)

✠ liquidity (selling pressure)

✠ operational (losses due to failed processes, people and systems).

Market risk & credit risk are well researched.

Liquidity risk has seen recent advances (Jarrow & Protter [2005]).

Operational risk is relatively unstudied.

Purpose: to study operational risk

- ✠ to provide an economic characterization
- ✠ to provide a convenient mathematical characterization
- ✠ to discuss empirical estimation

Economic Characterization

1. Internal versus external markets:

✠ operational risk is *internal* to the firm.

✠ market, credit, liquidity are *external* to the firm.

This implies:

♦ operational risk must be studied in conjunction with the firm's NPV process, and

♦ data availability will be limited.

2. Two types of operational risks:

✠ system risk (failed processes and/or transactions), and

✠ agency cost risk (fraud and/or mismanagement).

This implies:

♦ two distinct stochastic loss processes are necessary to model operational risk.

Mathematical Characterization

1. Use models from reduced form credit risk literature:

✠ doubly stochastic counting process (Cox process), and

✠ conditional independence assumption.

2. Martingale pricing in an incomplete market:

✠ standard valuation formulas, and

✠ formulas simplify - a simple adjustment to the discount rate (analogous to credit risk).

Empirical Estimation

1. If asset values are observable, then possible to estimate parameters using market prices.

2. If asset values are not observable, then need to estimate parameters using data internal to the firm.

Case (2) is the usual situation.

Market Set-up

Continuous trading, finite horizon $[0,T]$.

Filtered probability space $(\Omega, F, \{F_t\}, P)$ with P the statistical probability measure.

$\mathbf{X}_t$ denotes a vector of state variables, $F_t - measurable$. The σ –algebra generated by $\mathbf{X}_t$ is denoted F_t^X.

r_t denotes the default free spot rate of interest, $F_t - measurable$.

S_t^i represent the market value of assets $i \in \{1,\ldots,n\}$, $F_t - measurable$, where these assets have no cash flows.

No Arbitrage

Assume that there exists an equivalent martingale probability measure Q such that

$$S_t^i = E_t^Q\{S_T^i e^{-\int_t^T r_v dv}\} \text{ for all } i.$$

Q need not be unique (incomplete markets possible).

The Firm

Consider a firm operating in this setting, trading assets with prices S_t^i for $i = 1, \ldots, n$.

This activity generates a firm value, denoted V_t, $F_t - measurable$.

Assume the firm's value V_t trades, and that the firm pays no cash flows.

Therefore,

$$V_t = E_t^Q \{ V_T e^{-\int_t^T r_v dv} \}.$$

The Firm (continued)

The firm represents a transformation of market inputs (S_t^i for $i = 1, \ldots, n$) to a market output V_t.

It is this transformation that includes operational risk.

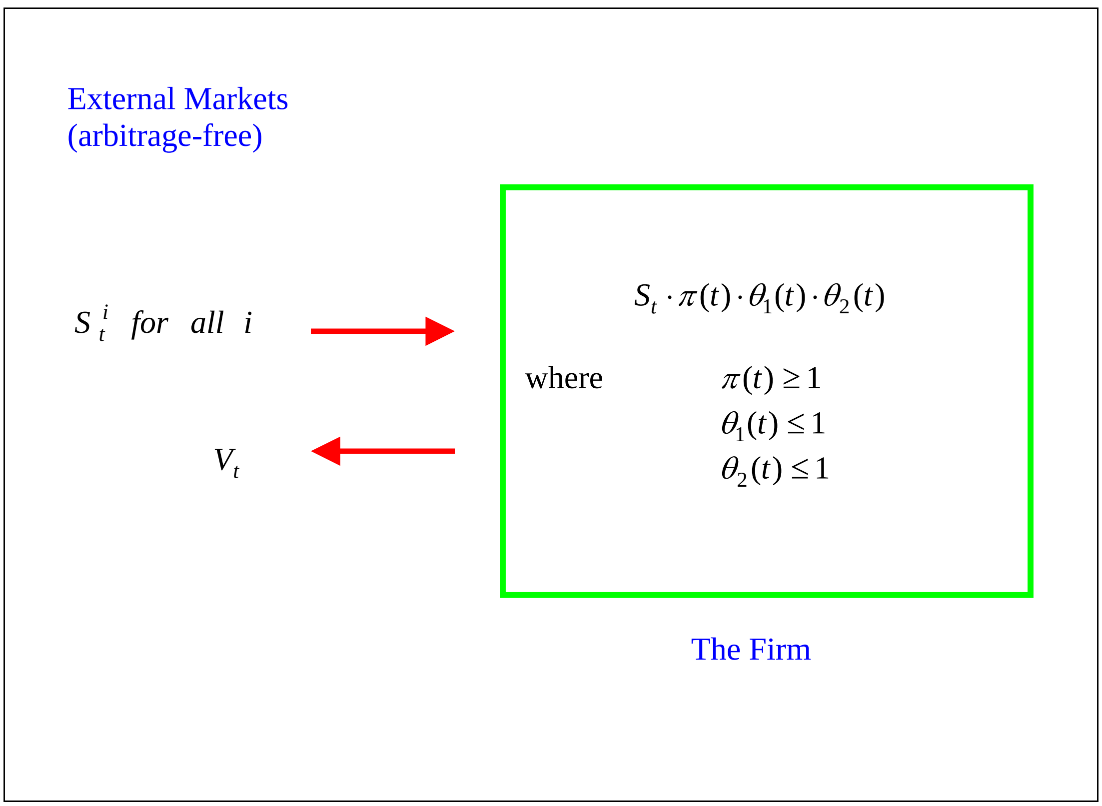
Decompose transformation into three components:

- NPV process.
- System operational risk process.
- Agency cost operational risk process.

<see Figure>

Figure 1: The Economic Setting.

S_t^i are prices of traded assets, S_t represents the aggregate value of the asset portfolio purchased by the firm, V_t is the firm's value, $\pi(t)$ is the proportionate change in the value of the firm's asset portfolio due to the firm's operating technology, $\theta_1(t)$ is the proportionate change in the value of the firm's asset portfolio due to the system operational risk, and $\theta_2(t)$ is the proportionate change in the value of the firm's asset portfolio due to the agency cost operational risk.



NPV Process

Firm takes inputs and transforms these to outputs.

We assume the transformation process adds value.

(Otherwise, as seen below, with the presence of operational risk, a firm would have no economic reason to exist.)

Let S_t denote the aggregate market value of the firm's input portfolio (assets).

The increase in asset value is given by the $F_t - measurable$ stochastic process $\pi(t) \geq 1$ for all t.

System Type Operational Risk

Think of an owner-managed firm (no agency or incentive problems).

Losses due to:

- ♦ failed transactions,
- ♦ an error in a trade,
- ♦ a legal dispute, or
- ♦ errors in judgement.

These losses given by the $F_t - measurable$ stochastic process $\theta_1(t) \leq 1$ for all t.

$\theta_1(t)$ is the accumulated time t ”recovery” value after all of these events have occurred.

Agency Type Operational Risk

Think of an agent-managed firm with agency costs. Management and shareholders' incentives are not perfectly aligned.

Large corporate finance literature related to agency costs - e.g. motivates debt/equity and dividend policy.

Losses due to fraud and/or mismanagement (distinct from system type operational risk losses).

These losses given by the $F_t - measurable$ stochastic process $\theta_2(t) \leq 1$ for all t.

$\theta_2(t)$ is the accumulated time t ”recovery” value after all these events have occurred.

Firm Value Transformation

The internal value of the firm's asset portfolio at time T:

$$S_T \bullet \pi(T) \bullet \theta_1(T) \bullet \theta_2(T).$$

The market value (external value) of the firm:

$$V_t = E_t^Q\{S_T \bullet \pi(T) \bullet \theta_1(T) \bullet \theta_2(T) \bullet e^{-\int_t^T r_v dv}\}.$$

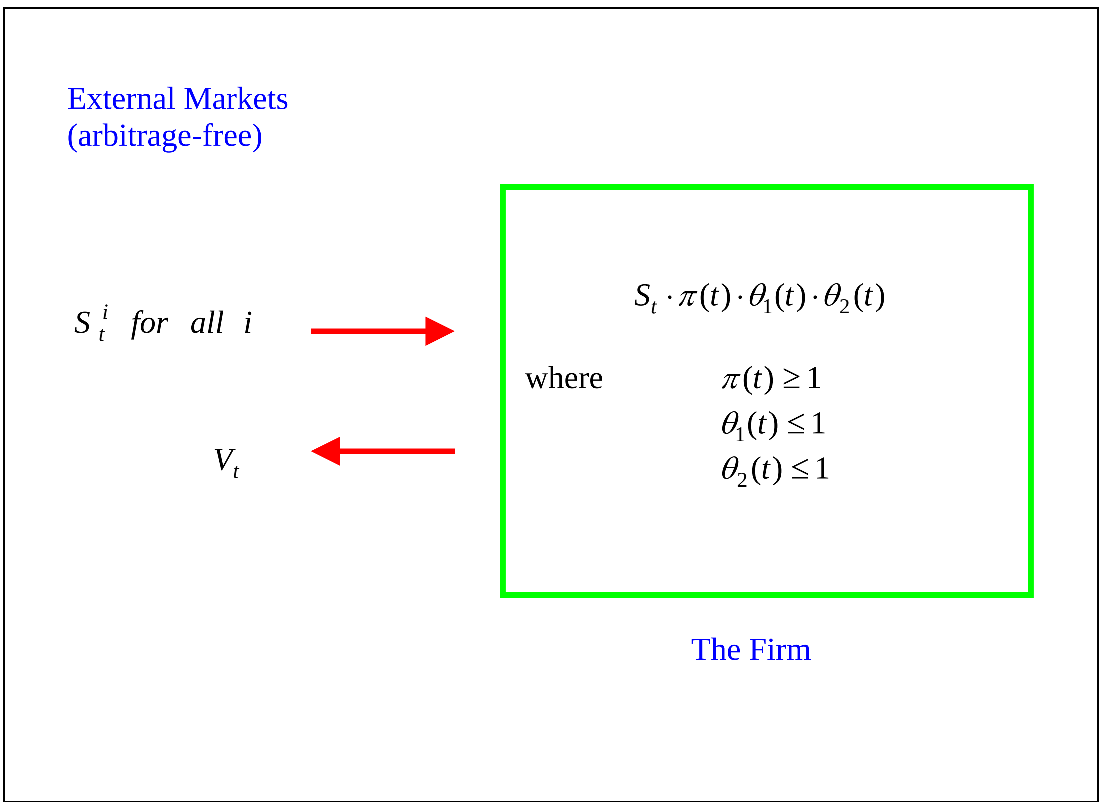
♦ Without the NPV process, $V_t \leq S_t$. No reason for the firm to exist.

♦ Given the firm exists, reasonable to assume that $V_t > S_t$, implies $E_t^Q\{\pi(T) \bullet \theta_1(T) \bullet \theta_2(T)\} > 1$.

♦ Important for revised Basel II. Implies that there is no need for capital to cover expected operational risk losses, only the unexpected.

Figure 1: The Economic Setting.

S_t^i are prices of traded assets, S_t represents the aggregate value of the asset portfolio purchased by the firm, V_t is the firm's value, $\pi(t)$ is the proportionate change in the value of the firm's asset portfolio due to the firm's operating technology, $\theta_1(t)$ is the proportionate change in the value of the firm's asset portfolio due to the system operational risk, and $\theta_2(t)$ is the proportionate change in the value of the firm's asset portfolio due to the agency cost operational risk.



Mathematical Characterization - NPV process

Let $N_0(t)$ be a doubly stochastic (Cox) counting process, initialized at zero ($N_0(0) = 0$).

$N_0(t)$ counts the number of NPV events between time 0 and t.

Assume that this counting process is F_t adapted with intensity $\lambda_0(\mathbf{X}_t) \geq 0$, $F_t^X - measurable$.

Given F_T^X, we assume that $N_0(t)$ is independent of S_t.

A NPV event at time t causes a percentage increase in firm value equal to $\alpha(\mathbf{X}_t) > 0$, $F_t^X - measurable$.

NPV Process (continued)

The NPV process is:

$$\pi(t) = \prod_{i=0}^{N_0(t)} (1 + \alpha_{T_i})$$

where $\pi(0) = 1$, and

T_i for $i = 0, 1, 2, \ldots$ are the jump times of $N_0(t)$ with $T_0 = \alpha_{T_0} = 0$.

♦ $\pi(t)$ is correlated to S_t due to their dependence on $\mathbf{X}_t$.

♦ $\pi(t)$ could require a market risk premium if these NPV jump risks are not diversifiable.

Under Q, the counting process has the intensity $\lambda_0(s)\mu_0(s)$ where $\mu_0(s) > 0$ is the risk premium.

Mathematical Characterization - System Operational Risk Process

Let $N_1(t)$ be another doubly stochastic (Cox) counting process, initialized at zero ($N_1(0) = 0$).

$N_1(t)$ counts the number of system operational risk events between time 0 and t.

Assume that this counting process is F_t adapted with intensity $\lambda_1(\mathbf{X}_s) \geq 0$, $F_s^X - measurable$.

Given F_T^X, we assume that $N_1(s)$ is independent of S_t and $N_0(t)$.

A operational risk event at time t causes a percentage reduction in firm value equal to $-1 < \delta_1(\mathbf{X}_t) < 0$, $F_t^X - measurable$.

System Operational Risk Process (continued)

The system operational risk process is:

$$\theta_1(t) = \prod_{i=0}^{N_1(t)} (1 + \delta_1(T_i))$$

where $\theta_1(0) = 1$, and

T_i for $i = 0, 1, 2, \ldots$ are the jump times of $N_1(t)$ with $T_0 = \delta_1(T_0) = 0$.

Under Q, the counting process has the intensity $\lambda_1(s)\mu_1(s)$ where $\mu_1(s) > 0$ is the risk premium.

Mathematical Characterization - Agency Cost Operational Risk Process

Using superscripts ”2” to indicate an agency cost operational risk event, we have

$$\theta_2(t) = \prod_{i=0}^{N_2(t)}(1 + \delta_2(T_i))$$

where $\theta_2(0) = 1$, and

T_i for $i = 0, 1, 2, \ldots$ are the jump times of $N_2(t)$, and $T_0 = \delta_2(T_0) = 0$.

$-1 < \delta_2(\mathbf{X}_t) < 0$, $F_t^X - measurable.$

$N_2(t)$ is assumed to be conditionally independent of $N_0(t), N_1(t), S_t$.

Under Q, the counting process has the intensity $\lambda_2(s)\mu_2(s)$ where $\mu_2(s) > 0$ is the risk premium.

Agency Cost Operational Risk Process (continued)

♦ $N_2(t)$ is correlated with $\pi(t)$ and $N_1(t)$ through the state variables $\mathbf{X}_t$.

♦ This is important because as the value of the firm's asset portfolio declines, agency cost operational risk losses may increase due to the firm's managers trying to increase their performance and save their jobs.

♦ Most likely, $|\delta_2| >> |\delta_1|$ and $\lambda_2 << \lambda_1$,i.e. agency cost risk results in a larger loss, but is less likely to occur.

Firm's Internal Value Process

$$S_t \bullet \pi(t) \bullet \theta_1(t) \bullet \theta_2(t)$$

$$= S_t \prod_{i=0}^{N_0(t)} (1 + \alpha_{T_i}) \prod_{j=1}^{2} [\prod_{i=0}^{N_j(t)} (1 + \delta_j(T_i))]$$

where $\{N_0(t),\ N_1(t),\ N_2(t)\}$ have the intensities:

$\{\lambda_0(t),\ \lambda_1(t),\ \lambda_2(t)\}$ under P, and

$\{\lambda_0(t)\mu_0(t),\ \lambda_1(t)\mu_1(t),\ \lambda_2(t)\mu_2(t)\}$ under Q.

Firm's Internal Value Process (continued)

The market value of the firm

$$V_t = E_t^Q\{S_T e^{-\int_t^T r_v dv} \prod_{i=0}^{N_0(T)}(1+\alpha_{T_i}) \prod_{j=1}^{2} \times [\prod_{i=0}^{N_j(T)}(1+\delta_j(T_i))]\}.$$

Can show:

$$V_t = E_t^Q\{S_T e^{-\int_t^T [r_s - \alpha_s \lambda_0(s)\mu_0(s) - \delta_1(s)\lambda_1(s)\mu_1(s) - \delta_2(s)\lambda_2(s)\mu_2(s)]ds}\}.$$

♦ S_T is discounted with an adjusted spot rate process.

♦ Results from the term structure of interest rate literature apply (analogous to credit risk).

***Example 1**: Constant Parameters*

Let

$$\alpha_t\lambda_0(t)\mu_0(t),\ \delta_1(t)\lambda_1(t)\mu_1(t),\ \delta_2(t)\lambda_2(t)\mu_2(t)$$

be constants.

Then,

$$V_t = S_t(1+\alpha)^{N_0(t)}(1+\delta_1)^{N_1(t)}(1+\delta_2)^{N_2(t)} \times e^{[\alpha\lambda_0\mu_0+\delta_1\lambda_1\mu_1+\delta_2\lambda_2\mu_2](T-t)}.$$

Easy to use for risk management (e.g. computing VaR).

Example 2: Two Factor Gaussian Model

Term Structure of Interest Rates

The state variables:

$$dX_1(t) = -\phi_1 X_1(t)dt + dW_1(t)$$

$$dX_2(t) = -\phi_{21} X_1(t)dt - \phi_2 X_2(t) + dW_2(t)$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions under Q, and $\phi_1 > 0$, $\phi_2 > 0$, ϕ_{21} are constants.

Spot rate follow an affine process:

$$r_t = a_0 + a_1 X_1(t) + a_2 X_2(t).$$

Example 2 (continued)

The value of a default free zero-coupon bond (where $S_T = 1$ with probability one) is:

$$S_t = E_t^Q\{1 \cdot e^{-\int_t^T r(u)du}\}$$
$$= e^{-X_1(t)C_1(T-t)-X_2(t)C_2(T-t)-A(T-t)}$$

where $C_1(0) = C_2(0) = A(0) = 0$.

If $a_1 \neq a_2$,

$$C_1(\tau) = \frac{1}{\phi_1}\left(a_1 - \frac{\phi_{21}a_2}{a_1}\right)(1 - e^{-\phi_1\tau})$$
$$+ \frac{\phi_{21}a_2}{a_2(a_1 - a_2)}(e^{-\phi_2\tau} - e^{-\phi_1\tau}),$$

$$C_2(\tau) = \frac{a_2}{\phi_2}(1 - e^{-\phi_2\tau}),$$

$$A(\tau) = \int_0^\tau\left(-\frac{1}{2}C_1^2(u) - \frac{1}{2}C_2^2(u) + a_0\right)du.$$

Example 2 (continued)

Firm Valuation

Let the NPV process and operational risk event intensities, under Q, also satisfy an affine process:

$$\alpha_t \lambda_0(t)\mu_0(t) = b_0 + b_1 X_1(t) + b_2 X_2(t),$$

$$\delta_1 \lambda_1(t)\mu_1(t) = c_0 + c_1 X_1(t) + c_2 X_2(t),$$

$$\delta_2 \lambda_2(t)\mu_2(t) = d_0 + d_1 X_1(t) + d_2 X_2(t).$$

Define new parameters by

$$\psi_0 = a_0 - b_0 - c_0 - d_0,$$

$$\psi_1 = a_1 - b_1 - c_1 - d_1,$$

$$\psi_2 = a_2 - b_2 - c_2 - d_2,$$

and an adjusted spot rate process by

$$\begin{aligned} R(t) &= r_t - \alpha_t \lambda_0(t)\mu_0(t) - \delta_1 \lambda_1(t)\mu_1(t) - \delta_2 \lambda_2(t)\mu_2(t) \\ &= \psi_0 + \psi_1 X_1(t) + \psi_2 X_2(t). \end{aligned}$$

Example 2 (continued)

The firm value (of a traded Treasury zero-coupon bond) is:

$$V_t = E_t^Q\{1 \cdot e^{-\int_t^T R(u)du}\}$$
$$= e^{-X_1(t)\widetilde{C}_1(T-t)-X_2(t)\widetilde{C}_2(T-t)-\widetilde{A}(T-t)}$$

where $\widetilde{C}_1(0) = \widetilde{C}_2(0) = \widetilde{A}(0) = 0$. If $\psi_1 \neq \psi_2$, then

$$\widetilde{C}_1(\tau) = \frac{1}{\phi_1}\left(\psi_1 - \frac{\phi_{21}\psi_2}{\psi_1}\right)(1 - e^{-\phi_1\tau})$$
$$+ \frac{\phi_{21}\psi_2}{\psi_2(\psi_1 - \psi_2)}(e^{-\phi_2\tau} - e^{-\phi_1\tau}),$$

$$\widetilde{C}_2(\tau) = \frac{\psi_2}{\phi_2}(1 - e^{-\phi_2\tau}),$$

$$\widetilde{A}(\tau) = \int_0^\tau\left(-\frac{1}{2}\widetilde{C}_1^2(u) - \frac{1}{2}\widetilde{C}_2^2(u) + \psi_0\right)du.$$

Values for the default free zero-coupon bond and the firm differ.

Estimation

Observe V_t and S_t.

Consider

$$\frac{V_t}{S_t} = (1+\alpha)^{N_0(t)}(1+\delta_1)^{N_1(t)}(1+\delta_2)^{N_2(t)} \times e^{[\alpha\lambda_0\mu_0+\delta_1\lambda_1\mu_1+\delta_2\lambda_2\mu_2](T-t)}.$$

♦ Left side represent changes in market prices.

♦ When $\frac{V_t}{S_t}$ jumps, it is due to one of the counting processes $\{N_0(t),\ N_1(t),\ N_2(t)\}$ changing, and the percentage change in $\frac{V_t}{S_t}$ is due to the amplitude of the relevant jump process: $\{\alpha,\ \delta_1,\ \delta_2\}$

♦ Time series observations of left side enables estimation of the parameters (e.g. using maximum likelihood).

Estimation (continued)

But, in most cases, do not observe V_t.

Analogous to criticism of structural approach to credit risk.

Then, internal data can be used to estimate processes parameters, using standard hazard rate estimation procedures.

Estimation (continued)

Agency cost risk events, if large enough to be publicly reported, implies $N_2(t)$ is observed externally to the firm.

Then, observing $\frac{V_t}{S_t}$ enables one to estimate (λ_2, δ_2) directly.

This approach has some difficulties:

♦ does not include estimates for the NPV and system type operational risk parameters $(\alpha, \lambda_0,\ \delta_1, \lambda_1)$,

♦ does not include agency cost operational loses not significant enough to be reported in the financial press,

♦ procedure still requires an estimate of the change in V_t when the agency cost event occurs.