Information-Based Asset Pricing: Applications to Equity, Credit, and Interest Rate Products

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Introduction and preliminaries

For asset pricing three ingredients are required:

(A) Specification of cash-flows
(B) Investor preferences
(C) Flow of information available to market participants

Translated into the language of finance theory:

(A*) Model cash-flows as random variables
(B*) Pricing kernel: discounting, risk-neutral measure
(C*) Market filtration

Asset pricing conventionally attaches more weight to (A) and (B) than to (C).

In this presentation, however, we emphasize the importance of (C).

For simplicity we de-emphasize the role of investor preferences (B).

In particular in the first part of the talk, we assume that interest rates are deterministic, and that the pricing measure $\mathbb{Q}$ (the risk-neutral measure) is pre-specified.

Thus, we model the financial markets with the specification of a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ on which we are going to explicitly *construct a filtration* $\{\mathcal{F}_t\}_{0\leq t<\infty}$ representing the flow of information available to market participants.

The markets we consider will, in general, be incomplete.

Absence of arbitrage implies that the deterministic default-free discount bond system $\{P_{tT}\}_{0<t<T<\infty}$ is given by

$$P_{tT} = \frac{P_{0T}}{P_{0t}}, \tag{1}$$

where $\{P_{0t}\}_{0<t<\infty}$ is the initial term structure.

Cash flows and market factors

We consider an asset defined by a set of random cash-flows $\{D_{T_k}\}_{k=1,\dots,n}$ that occur at the pre-specified dates $\{T_k\}_{k=1,\dots,n}$.

The price of the asset that generates the cash flows $\{D_{T_k}\}_{k=1,\dots,n}$ is given by the standard risk-neutral valuation formula:

$$S_t = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} P_{tT_k} \mathbb{E}^{\mathbb{Q}} \left[D_{T_k} | \mathcal{F}_t \right]. \quad (2)$$

We introduce a set of independent random variables $\{X_{T_k}^{\alpha}\}_{k=1,\dots,n}^{\alpha=1,\dots,m_k}$, which we call *market factors*.

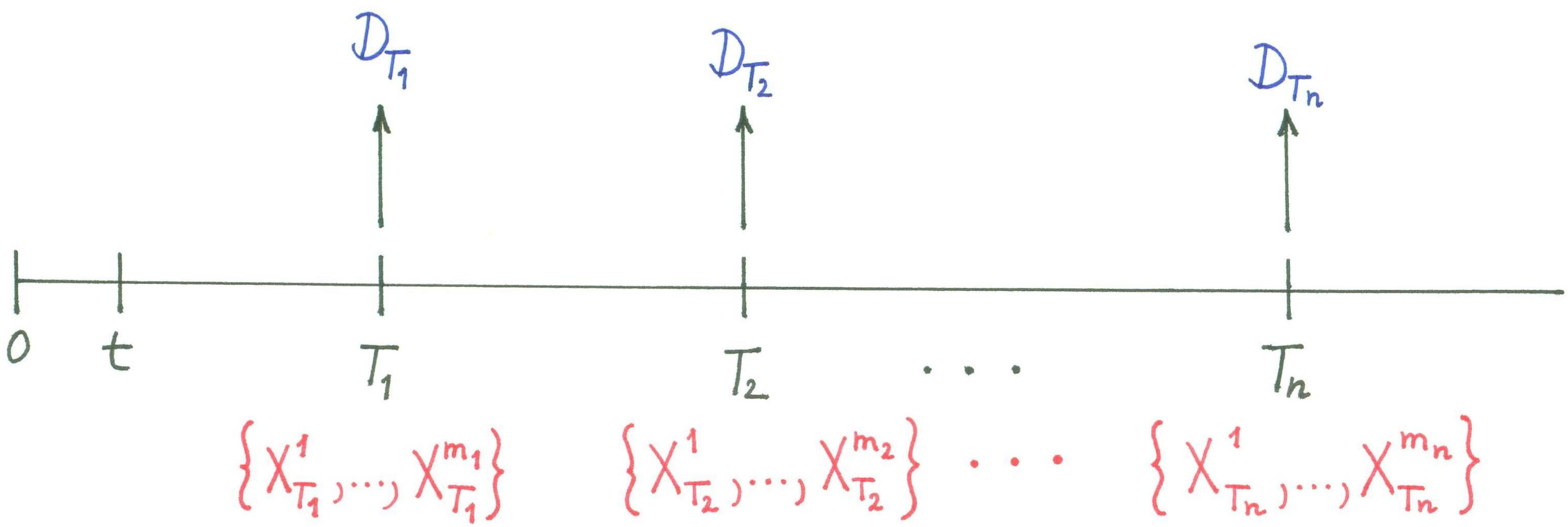
For each k the cash flow D_{T_k} is assumed to depend on the independent market factors $X_{T_1}^{\alpha}, X_{T_2}^{\alpha}, \dots, X_{T_k}^{\alpha}$.

Cash flow structure

Thus for each date T_k we introduce a cash-flow function Δ_{T_k} such that

$$D_{T_k} = \Delta_{T_k}(X^{\alpha}_{T_1}, X^{\alpha}_{T_2}, \ldots, X^{\alpha}_{T_k}). \tag{3}$$

For any given asset, it is the job of the "financial analyst" to determine the relevant X-factors, their *a priori* probabilities, and the form of the cash-flow functions.



Examples

I. Simple credit-risky coupon bond. (two coupons, no recovery on default).

$$
\begin{aligned}
D_{T_1} &= \mathbf{c}X_{T_1} && (4)\\
D_{T_2} &= (\mathbf{c}+\mathbf{n})X_{T_1}X_{T_2}. && (5)
\end{aligned}
$$

Here $\mathbf{c}$, $\mathbf{n}$ denote the coupon and principal.

X_{T_1}, X_{T_2} are independent binary random variables taking the values $\{0, 1\}$ with *a priori* probabilities $p_1 = \mathbb{Q}[X_{T_1} = 1]$ and $p_2 = \mathbb{Q}[X_{T_2} = 1]$.

II. Credit-risky coupon bond with recovery.

$$
\begin{aligned}
D_{T_1} &= \mathbf{c}X_{T_1} + R_1(\mathbf{c}+\mathbf{n})(1 - X_{T_1}) && (6)\\
D_{T_2} &= (\mathbf{c}+\mathbf{n})X_{T_1}X_{T_2} + R_2(\mathbf{c}+\mathbf{n})X_{T_1}(1 - X_{T_2}). && (7)
\end{aligned}
$$

Here R_1, R_2 are recovery rates.

Information processes

For $t \leq T_k$, we assume that all information available to market participants about $X^{\alpha}_{T_k}$ is contained in the *information process* $\{\xi^{\alpha}_{tT_k}\}_{0 \leq t \leq T_k}$ defined by

$$\xi^{\alpha}_{tT_k} = \sigma^{\alpha}_{T_k} X^{\alpha}_{T_k} t + \beta^{\alpha}_{tT_k}. \tag{8}$$

The process $\{\beta^{\alpha}_{tT_k}\}$ is a standard Brownian bridge over $[0, T_k]$ with mean zero and variance $t(T_k - t)/T_k$.

The X-factors and the Brownian bridges are all independent.

The Brownian bridges represent "market noise" and only the terms $\sigma^{\alpha}_{T_k} X^{\alpha}_{T_k} t$ contain "true market information".

The true value of $X^{\alpha}_{T_k}$ is "revealed" at time T_k.

The parameter $\sigma^{\alpha}_{T_k}$ can be interpreted as the "information flow rate" for the factor $X^{\alpha}_{T_k}$.

Modelling the market filtration

Now we are in a position to construct the market filtration.

We assume that $\{\mathcal{F}_t\}$ is generated by the collection of market information processes $\{\xi^{\alpha}_{tT_k}\}$. Thus:

$$\mathcal{F}_t = \sigma\left(\left\{\xi^{\alpha}_{sT_k}\right\}^{\alpha=1,\dots,m_k}_{0\le s\le t,\ k=1,\dots,n}\right). \tag{9}$$

The information process $\{\xi^{\alpha}_{tT_k}\}$ is $\{\mathcal{F}_t\}$-adapted, but this is not the case for the Brownian bridge $\{\beta_{tT_k}\}$.

Clearly, the market factor $X^{\alpha}_{T_k}$ is $\mathcal{F}_{T_k}$-measurable.

A calculation shows that the information processes $\{\xi^{\alpha}_{tT_k}\}$ satisfy the Markov property.

Example

III. Single-dividend paying asset, with a single market factor.

Let the cash-flow function of a single-dividend paying asset be given by $D_T = \Delta_T(X_T)$.

Here the market factor X_T is a continuous non-negative random variable with *a priori* probability density $p(x)$.

The price process of the single-dividend paying asset is therefore

$$\begin{aligned} S_t &= \mathbf{1}_{\{t<T\}} P_{tT} \mathbb{E}^{\mathbb{Q}} \left[\Delta_T | \xi_t \right] &(10)\\ &= \mathbf{1}_{\{t<T\}} P_{tT} \int_0^\infty \Delta_T(x)\, \pi_t(x)\, \mathrm{d}x. &(11)\end{aligned}$$

Making use of the Bayes formula we can show that the conditional probability density $\pi_t(x)$ appearing here is given by

$$\pi_t(x) = \frac{p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right]}{\int_0^\infty p(x) \exp\left[\frac{T}{T-t}(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t)\right] \mathrm{d}x}. \qquad (12)$$

Example

IV. Single-dividend paying asset, with multiple market factors.

Let $D_T = \Delta_T\left(X_T^1, X_T^2, \ldots, X_T^m\right)$.

Write $p^\alpha(x)$ for the *a priori* probability density of the market factor X_T^α.

Then owing to the independence of the information processes ξ_{tT}^α associated with the market factors X_T^α $(\alpha = 1, \ldots, m)$ we find that:

$$\begin{aligned} S_t \;=\; & \mathbf{1}_{\{t<T\}} P_{tT} \int_0^\infty \cdots \int_0^\infty \Delta_T(x_1, x_2, \ldots, x_m) \\ \times\; & \pi_t^1(x_1)\pi_t^2(x_2)\cdots\pi_t^m(x_m)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_m. \end{aligned} \tag{13}$$

Example

V. Single-dividend paying asset: stochastic volatility

We obtain the dynamics of the asset price by applying Ito's lemma to the risk-neutral valuation formula. The result is:

$$\begin{aligned} \mathrm{d}S_t \;=\;& r_t\, S_t\, \mathrm{d}t + \sum_{\alpha=1}^{m} \mathbf{1}_{\{t<T\}} \frac{\sigma^\alpha T}{T-t}\, P_{tT}\, \mathrm{Cov}\left[\Delta_T, X_T^\alpha \middle| \mathcal{F}_t\right] \mathrm{d}W_t^\alpha \\ & + \Delta_T \mathrm{d}\mathbf{1}_{\{t<T\}}. \end{aligned} \tag{14}$$

Here the $\{\mathcal{F}_t\}$-adapted Brownian motions $\{W_t^\alpha\}_{\alpha=1,\ldots,m}$ driving the asset price are defined by

$$W_t^\alpha = \xi_t^\alpha - \int_0^t \frac{1}{T-s}(\sigma^\alpha\, T\, \mathbb{E}_s\left[D_T\right] - \xi_s^\alpha)\mathrm{d}s. \tag{15}$$

The volatility of the price process depends on more than one Brownian motion.

As a consequence, the model gives rise in a natural way to *"unhedgeable" stochastic volatility*.

Example

VI. Defaultable discount bond, with recovery.

Let H_T denote the terminal value of a defaultable discount bond that can take the values $H_T = h_i$ $(i = 0, 1, \ldots, n)$ at maturity.

We assume that $h_0 > h_1 > \cdots > h_n$, and write p_i for $\mathbb{Q}(H_T = h_i)$.

In this example we assume that the information process takes the following form:

$$\xi_t = \sigma H_T t + \beta_{tT}. \tag{16}$$

In other words, H_T itself is taken to be the only market factor.

The price process $\{B_{tT}\}_{0 \leq t \leq T}$ of the credit-risky discount bond is thus given by

$$B_{tT} = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[H_T | \mathcal{F}_t\right], \tag{17}$$

where $\mathcal{F}_t$ is generated by $\{\xi_t\}_{0 \leq s \leq t}$.

The conditioning with respect to $\mathcal{F}_t$ can therefore be replaced by conditioning with respect to ξ_t:

$$B_{tT} = P_{tT}\mathbb{E}^{\mathbb{Q}}\left[H_T | \xi_t\right]. \tag{18}$$

The bond price can then be expressed in the form

$$B_{tT} = P_{tT} \sum_i h_i \pi_{it}, \tag{19}$$

where $\pi_{it} = \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{\{H_T = h_i\}} | \xi_t]$.

The conditional probability π_{it} is given by:

$$\pi_{it} = \frac{p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_{i=0}^{n} p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}. \tag{20}$$

For the credit-risky bond price we thus obtain:

$$B_{tT} = P_{tT} \frac{\sum_{i=0}^{n} h_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_{i=0}^{n} p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}. \tag{21}$$

One of the attractive features of this formula is that since B_{tT} is expressed explicitly as a function of the information process, simulation of the dynamics of the bond price process can be carried out in an efficient manner.

Simulated bond price process for various information flow rates

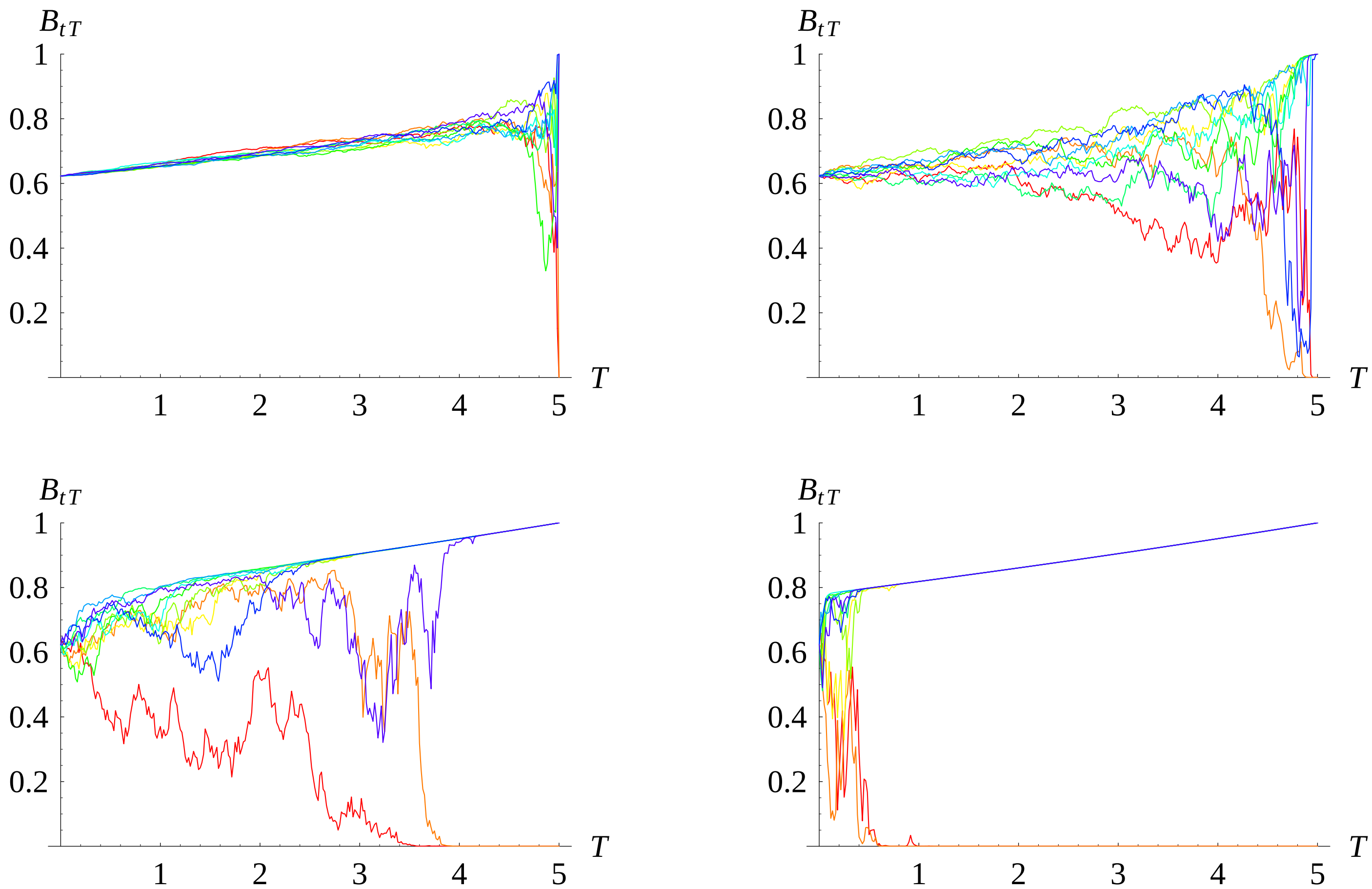


Figure 1: *Bond price processes for various information-release rates*. The parameter σ governs the rate at which information is released to market participants concerning the payout of a defaultable discount bond. Four values of σ are illustrated, given by .04, .2, 1, and 5. The bond has a maturity of five years, and the default-free interest-rate system has a constant short rate given by $r = 5\%$. The *a priori* probability of default is taken to be 20%. For low values of σ, collapse of the bond price occurs only 'at the last minute'.

Example

VII. Options on defaultable bonds.

The value of the information flow rate parameter σ can be inferred from option price data.

The value at time 0 of an option exercisable at time $t > 0$ on a defaultable bond maturing at time $T > t$ is

$$C_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[(B_{tT} - K)^+\right]. \tag{22}$$

Inserting the price B_{tT} for the defaultable bond in (22) we obtain

$$\begin{aligned} C_0 &= P_{0t}\,\mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{i=0}^{n} P_{tT}h_i\pi_{it} - K\right)^+\right] \\ &= P_{0t}\,\mathbb{E}^{\mathbb{Q}}\left[\left(P_{tT}\frac{\sum_i h_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]} - K\right)^+\right]. \end{aligned} \tag{23}$$

The call option price can thus be viewed as an "exotic derivative" with payoff $f(\xi_t)$ at maturity t.

The price of such an *information derivative* is given by

$$V_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[f(\xi_t)\right]. \tag{24}$$

We first look at an abstract security expiring at time t with the payoff

$$f(\xi_t) = \delta(\xi_t - x). \tag{25}$$

To work out the price in this case we use the "Fourier representation"

$$\delta(\xi_t - x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\xi_t - x)\kappa}\,\mathrm{d}\kappa. \tag{26}$$

The value of a general information derivative can then be expressed as a weighted integral of elementary information securities:

$$V_0 = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[f(\xi_t)\right] = P_{0t}\mathbb{E}^{\mathbb{Q}}\left[\int_{-\infty}^{\infty}\delta(\xi_t - x)f(x)\mathrm{d}x\right] = P_{0t}\int_{-\infty}^{\infty}A_{0t}(x)f(x)\mathrm{d}x. \tag{27}$$

A calculation shows that the value of the "elementary information security" is

$$A_{0t}(x) = P_{0t} \sum_{j=0}^{n} p_j \sqrt{\frac{T}{2\pi t(T-t)}} \exp\left[-\frac{1}{2}\frac{(\sigma h_j t - x)^2 T}{t(T-t)}\right]. \tag{28}$$

We now calculate the value C_0 of the call option by rewriting (23) in terms of $A_{0t}(x)$, that is:

$$\begin{aligned} C_0 &= P_{0t}\,\mathbb{E}^{\mathbb{Q}}\left[\left(P_{tT}\frac{\sum_i p_i h_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i \xi_t - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]} - K\right)^+\right] && (29) \\ &= \int_{-\infty}^{\infty} A_{0t}(x)\,\left(P_{tT}\frac{\sum_i p_i h_i \exp\left[\frac{T}{T-t}\left(\sigma h_i x - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]}{\sum_i p_i \exp\left[\frac{T}{T-t}\left(\sigma h_i x - \frac{1}{2}\sigma^2 h_i^2 t\right)\right]} - K\right)^+ \mathrm{d}x. && (30) \end{aligned}$$

In the case of a binary defaultable bond paying either h_0 or h_1 at maturity we obtain a closed-form expression for the price of the call option.

The result for the option price is very similar in form to the Black-Scholes formula:

$$C_0 = P_{0t}\Big[p_1(P_{tT}h_1 - K)N(d^+) - p_0(K - P_{tT}h_0)N(d^-)\Big]. \tag{31}$$

The argument of the normal distribution function here is defined by

$$d^{\pm} = \frac{\ln\left[\frac{p_1(P_{tT}h_1 - K)}{p_0(K - P_{tT}h_0)}\right] \pm \frac{1}{2}\sigma^2\tau(h_1 - h_0)^2}{\sigma\sqrt{\tau}(h_1 - h_0)}, \tag{32}$$

where $\tau = tT/(T-t)$.

The information flow rate σ thus plays a role very similar to that of the volatility parameter in the Black-Scholes formula.

Monotonicity of option price with respect to σ.

How does the value of the a call option change when we change the value of the information flow rate σ?

Let us define the option vega in this model by:

$$\mathcal{V} = \frac{\partial C_0}{\partial \sigma}. \tag{33}$$

For the vega we then obtain a positive expression:

$$\mathcal{V} = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-rt-\frac{1}{2}A}(h_1 - h_0)\sqrt{\tau p_0 p_1 (P_{tT}h_1 - K)(K - P_{tT}h_0)}, \tag{34}$$

where

$$A = \frac{1}{\sigma^2\tau(h_1-h_0)^2} \ln^2\left[\frac{p_1(P_{tT}h_1 - K)}{p_0(K - P_{tT}h_0)}\right] + \frac{1}{4}\sigma^2\tau(h_1-h_0)^2. \tag{35}$$

Thus C_0 is an increasing function of the information flow rate σ.

In other words, the more rapidly information regarding the ‘true’ value of the bond payoff is released, the higher the premium of the call option.

Another conclusion is that bond option prices can be used to recover an *implied* value for the information flow rate σ, and hence to calibrate the model.

Correlation in a multiname structure

Correlation results if assets share one or more X-factors in common.

Consider two credit-risky zero-coupon bonds.

The first ZCB is defined by the cash flow D_{T_1} at T_1, and the second ZCB is defined by the cash flow D_{T_2} at time T_2, where $T_2 > T_1$.

The cash flow structure is given by:

$$\begin{aligned} D_{T_1} &= \mathbf{n}_1 X_{T_1} + R_1 \mathbf{n}_1 (1 - X_{T_1}) && (36) \\ D_{T_2} &= \mathbf{n}_2 X_{T_1} X_{T_2} + R_2^a \mathbf{n}_2 (1 - X_{T_1}) X_{T_2} \\ &+ R_2^b \mathbf{n}_2 X_{T_1} (1 - X_{T_2}) + R_2^c \mathbf{n}_2 (1 - X_{T_1})(1 - X_{T_2}). && (37) \end{aligned}$$

Here, $\mathbf{n}_1$ and $\mathbf{n}_2$ denote the bond principals.

X_{T_1} and X_{T_2} are independent binary random variables taking the values $\{0, 1\}$.

R_1, R_2^a, R_2^b, and R_2^c denote possible recovery rates in the case of default.

Interest rates and information

To model interest rates in an information-based framework we work in a discrete-time setting.

For the time-line we write $\{t_i\}_{i=0,1,2,\ldots}$.

Associated with each date t_i we have a set of X-factors, denoted X_i^α $(\alpha = 1, \ldots, m_i)$.

Let us then write Y_i for the collection

$$Y_i = \{X_1^\alpha, X_2^\alpha, \ldots, X_i^\alpha\}. \tag{38}$$

We assume the existence of a *pricing kernel* process $\{\pi_i\}_{i\geq 0}$, where

$$\pi_i = K(Y_i). \tag{39}$$

Thus π_i depends on all of the X-factors revealed up to time i.

We assume that $\{\pi_i\}$ is a positive supermartingale, and that $\lim_{i\to\infty} \mathbb{E}[\pi_i] = 0$.

The associated discount-bond system $\{P_{ij}\}_{i<j}$ is given by

$$P_{ij} = \frac{\mathbb{E}_i[\pi_j]}{\pi_i}. \tag{40}$$

For the money market account we have

$$B_i = (1+r_1)(1+r_2)\cdots(1+r_i) \tag{41}$$

where

$$1 + r_n = \frac{\pi_{n-1}}{\mathbb{E}_{n-1}[\pi_n]}. \tag{42}$$

The filtration is generated by the information processes associated with the given system of X-factors.

To model inflation we need appropriate expressions for the *real* pricing kernel $\{\pi_i^R\}$, and the consumer price index $\{C_i\}$.

We take a dynamic macroeconomic approach to inflation modelling, introducing processes for aggregate consumption $\{k_i\}$ and the money supply $\{M_i\}$.

We also introduce a liquidity benefit process $\{\lambda_i\}$, with the property that $\lambda_i M_i$ represents the nominal benefit conveyed by the money supply at time t_i.

The corresponding real liquidity benefit is given by

$$l_i = \frac{\lambda_i M_i}{C_i}. \tag{43}$$

We introduce a bivariate utility $U(x, y)$ and assume that agents optimise the value of

$$\mathbb{E}\left[\sum_{n=0}^{N} \mathrm{e}^{-\gamma t_n} U(k_n, l_n)\right] \tag{44}$$

subject to the budget constraint

$$w = \mathbb{E}\left[\sum_{n=0}^{N} \pi_n (C_n k_n + \lambda_n M_n)\right]. \tag{45}$$

Here w is the total wealth available for consumption over the relevant period.

The first-order conditions are:

$$U_x(k_n, l_n) = \mu\, \mathrm{e}^{-\gamma t_n} \pi_n C_n \tag{46}$$

$$U_y(k_n, l_n) = \mu\, \mathrm{e}^{-\gamma t_n} \pi_n C_n. \tag{47}$$

As a consequence we obtain the fundamental relation

$$U_x\left(k_n, \lambda_n M_n / C_n\right) = U_y(k_n, \lambda_n M_n / C_n). \tag{48}$$

This relation allows us to express C_n in terms of k_n and $\lambda_n M_n$.

We also therefore obtain expressions for π_n and π_n^R in terms of k_n and $\lambda_n M_n$.

For example in the case of log-utility we have

$$U(x, y) = A \ln(x) + B \ln(y). \tag{49}$$

The fundamental relation becomes:

$$k_n C_n = \frac{A}{B} \lambda_n M_n, \tag{50}$$

which shows the relation between the price level and the money supply in this model.

In the log-utility case we then have

$$\pi_n = \frac{B\mathrm{e}^{-\gamma t_n}}{\mu \lambda_n M_n}. \tag{51}$$

Hence in the log-utility theory we obtain a relation between the nominal money supply and the term structure of interest rates.

Consider now any contingent claim with random nominal payoff H_j at time t_j.

The value of the claim at time t_0 in the log-utility model is given by

$$H_0 = \lambda_0 \, M_0 \, \mathrm{e}^{-\gamma t_j} \, \mathbb{E}\left[\frac{H_j}{\lambda_j M_j}\right]. \tag{52}$$

The model is completed with the specification of $\{k_i\}$ and $\{\lambda_n M_n\}$ as functions of the X-factors.

Thus we obtain a framework for interest rates and inflation within the information-based approach.

Conclusions

We present a new information-based framework for asset pricing.

In this approach the market filtration is modelled explicitly.

We have shown how closed-form solutions for the price processes of assets can be obtained in a number of different examples, and that complex cash-flow structures can be modelled efficiently with a good deal of flexibility.

The new framework offers a natural explanation for the origin of stochastic volatility, and leads to a specific model for stochastic volatility.

The price process of a defaultable zero-coupon bond is given by a closed-form expression that leads to an efficient simulation methodology.

In the case where a binary defaultable bond is the underlying, the price of a call option can be exactly computed, and turns out to be of the Black-Scholes form.

The option price has a positive vega and thus is an increasing function of the information flow rate σ.

Option positions can be hedged with credit-risky bonds.

The information-based model for binary defaultable bonds can be calibrated by use of bond options.

Correlation structures can be incorporated in a natural way

A natural extension of the framework to the macroeconomic domain leads to new models for interest rates and inflation.

A variety of specific applications are envisaged for equity, credit, interest rates, inflation and insurance.

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Credit default swaps

We consider a CDS written on a credit-risky coupon bond.

The seller of protection receives a series of premium payments, each of the amount g, at some pre-designated dates.

The payments continue until a credit event occurs, which here we model as the failure of a coupon payment in the reference bond.

In the event of a default a lump sum is paid to the buyer of protection at the default time equal to the principal minus the effective recovery value of the reference bond at that time.

As an illustration we consider the case when the reference bond has two outstanding coupon payments due, at the times T_1 and T_2.

The cash-flow structure of the coupon bond is given by

$$H_{T_1} = \mathbf{c}X_{T_1} + R_1(\mathbf{c}+\mathbf{n})(1 - X_{T_1}) \tag{53}$$
$$H_{T_2} = (\mathbf{c}+\mathbf{n})X_{T_1}X_{T_2} + R_2(\mathbf{c}+\mathbf{n})X_{T_1}(1 - X_{T_2}). \tag{54}$$

Here $\mathbf{n}$ is the principal of the reference bond, $\mathbf{c}$ is the coupon, and R_1, R_2 are the effective recovery rates.

X_{T_1}, X_{T_2} are independent binary random variables taking the values $\{0, 1\}$ with *a priori* probabilities $p_1 = \mathbb{Q}[X_{T_1} = 1]$ and $p_2 = \mathbb{Q}[X_{T_2} = 1]$.

For simplicity we assume that the default-swap premium payments are made immediately after the bond coupon dates.

From the perspective of the seller of protection, the value of the CDS is given by:

$$\begin{aligned} V_t &= \mathbf{g} P_{tT_1} \mathbb{E}\left[X_{T_1} | \xi_t^{(1)}\right] \\ &- (\mathbf{n} - R_1(\mathbf{c}+\mathbf{n})) P_{tT_1} \left(1 - \mathbb{E}\left[X_{T_1} | \xi_t^{(1)}\right]\right) \\ &+ \mathbf{g} P_{tT_2} \mathbb{E}\left[X_{T_1} | \xi_t^{(1)}\right] \mathbb{E}\left[X_{T_2} | \xi_t^{(2)}\right] \\ &- (\mathbf{n} - R_2(\mathbf{c}+\mathbf{n})) P_{tT_2} \mathbb{E}\left[X_{T_1} | \xi_t^{(1)}\right] \left(1 - \mathbb{E}\left[X_{T_2} | \xi_t^{(2)}\right]\right). \end{aligned} \tag{55}$$

We note that the expectations in the above equation are given by

$$\mathbb{E}\left[X_{T_i} \middle| \xi_t^{(i)}\right] = \frac{p_1^{(i)} \exp\left[\frac{T_i}{T_i - t}\left(\sigma_i \xi_t^{(i)} - \frac{1}{2}\sigma_i^2 t\right)\right]}{p_0^{(i)} + p_1^{(i)} \exp\left[\frac{T_i}{T_i - t}\left(\sigma_i \xi_t^{(i)} - \frac{1}{2}\sigma_i^2 t\right)\right]} \qquad (i = 1, 2). \tag{56}$$