ITO FORMULAS FOR FRACTIONAL BROWNIAN MOTIONS

Robert J. Elliott

Haskayne School of Business, University of Calgary

ABSTRACT

The difficulty of studying financial models with fractional Brownian motions is partly due to the fact that the stochastic calculus for fractional Brownian motions is not as well developed as for traditional models. Ito formulas play an important role in this stochastic calculus. This paper will review the current state of knowledge of such formulas and present a new and transparent approach to the derivation of these generalized Ito formulas. We will present applications to financial models. This presentation represents joint work with **John van der Hoek** from the University of Adelaide.

What is FBM ?

For each Hurst index $0 < H < 1$

$$B^H = \Big\{B^H(t), t \in \mathsf{R}\Big\}$$

is a Gaussian process with mean 0 for each t and covariance

$$E\Big[B^H(t)B^H(s)\Big] = \frac{1}{2}\Big[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big]$$

We now take $B^H(0) = 0$, $B^{\frac{1}{2}}$ is a standard Brownian motion.

$$E\Big[\Big(B^H(t) - B^H(s)\Big)^2\Big] = |t-s|^{2H}$$

defined on a suitable probability space $(\Omega, \mathcal{F}, P)$, with expectation operator $E[\cdot]$

Write

$$\begin{aligned} \Omega &= \mathcal{S}'(R) = \text{space of tempered distributions} \\ \mathcal{F} &= \text{Borel sets of } \Omega. \end{aligned}$$

The probability measure P exists by Bochner-Minlos Theorem, so that

$$E\Big[e^{i\langle f,\cdot\rangle}\Big] = e^{-\frac{1}{2}\|f\|^2}$$

where

$$f \in \mathcal{S}(\mathbb{R})$$

and

$$\|f\|^2 = \int_{\mathbb{R}} f^2(x)dx$$

This is the *White Noise* framework of Hida *et al*.

$$B(t)(\omega) := \langle \mathbf{I}(0,t), \omega \rangle \quad \text{for} \quad \omega \in \Omega$$

$$(M_H f)(x) = -\frac{d}{dx} C_H \int_{-\infty}^{\infty} (t-x)|t-x|^{H-\frac{3}{2}}\ f(t)\ dt$$

When $f = \mathbf{I}(0,t)$, the characteristic function of $[0,t]$, we will write $M_H(0,t)$ for this $M_H f$. For $H \in (0,1)$ define B^H as the continuous version of the process defined by

$$\widetilde{B}^H(t)(\omega) = \langle M_H(0,t), \omega\rangle \quad \text{for} \quad \omega \in \Omega.$$

The constant C_H is chosen so that

$$E\Big[B^H(t)B^H(s)\Big] = \frac{1}{2}\Big[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big]$$

Hermite Functions and Chaos Expansions

$$\xi_n(x) = \pi^{-\frac{1}{4}}\Big((n-1)!\Big)^{-\frac{1}{2}} h_{n-1}(\sqrt{2}\,x)e^{-\frac{x^2}{2}}, \quad n = 1, 2, \ldots$$

$$h_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad n = 0, 1, 2, \ldots$$

$$B(t)(\omega) := \langle \mathbf{I}(0,t), \omega \rangle = \sum_n \langle \xi_n, \omega \rangle \int_0^t \xi_n(s)\ ds$$

$$B^H(t)(\omega) := \langle M_H(0,t), \omega \rangle = \sum_n \langle \xi_n, \omega \rangle \int_0^t M_H \xi_n(s)\ ds$$

$$W(t)(\omega) := \sum_n \langle \xi_n, \omega \rangle \xi_n(t)$$

$$W^H(t)(\omega) := \sum_n \langle \xi_n, \omega \rangle M_H \xi_n(t)$$

The space $\mathcal{L}^2\Big(\mathcal{S}'(R), \mathcal{F}, P\Big)$ of square integrable random variables on $\mathcal{S}'(R)$. $\mathcal{I}$ denotes the set of all finite multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ for some $n \geq 1$ of non-negative integers. Write $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$.

For $\alpha \in \mathcal{I}$ write

$$H_\alpha(\omega) = \prod_{i=1}^{n} h_{\alpha_i}(\langle\, \xi_i, \omega\rangle).$$

In particular, write $\varepsilon_i = (0, \ldots 0, 1)$ (n components). Then

$$H_{\varepsilon_j}(\omega) = h_1(\langle\, \xi_j, \omega\rangle) = \langle\, \xi_j, \omega\rangle$$

Every $F \in \mathcal{L}^2\Big(\mathcal{S}'(R), \mathcal{F}, P\Big)$ has a unique representation

$$\begin{aligned} F(\omega) &= \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha(\omega) \\ &= \sum_{\alpha} c_\alpha h_{\alpha_1}(\langle\, \xi_1, \omega\rangle) \ldots h_{\alpha_n}(\langle\, \xi_n, \omega\rangle) \end{aligned}$$

and

$$\mathbf{E}[H_\alpha H_\beta] = \alpha! \delta_{\alpha,\beta}$$

$$\|F\|^2 = \sum_{\alpha \in \mathcal{I}} \alpha! c_\alpha^2$$

Consider formal sums

$$F(\omega) = \sum_\alpha c_\alpha H_\alpha(\omega) \qquad G(\omega) = \sum_\alpha d_\alpha H_\alpha(\omega)$$

and define

$$(2N)^\gamma := \prod_{j=1}^m (2j)^{\gamma_j} \quad \text{with} \quad \gamma = (\gamma_1, \ldots, \gamma_m) \in \mathcal{I}$$

We define (Hida) spaces $(S) \subset \mathcal{L} \subset (S)^*$ as follows: $F \in (S)$ if for all k,

$$\sum_\alpha \alpha! c_\alpha^2 (2N)^{k\alpha} < \infty \qquad where \quad (2N)^{k\alpha} \equiv \prod_j (2j)^{k\alpha_j}$$

$G \in (S)^*$ if for some $q > 0$, $\sum_\alpha \alpha! d_\alpha^2 (2N)^{-q\alpha} < \infty$ and the dual pairing is

$$\Big\langle \langle F, G \rangle \Big\rangle = \sum_\alpha \alpha! c_\alpha d_\alpha$$

Suppose $Z : R \to (S)^*$ is such that $\Big\langle \langle Z(t), F \rangle \Big\rangle \in L^1(R)$ for all $F \in (S)$

Then $\int_R Z(t)dt$ is defined to be that unique element of $(S)^*$ such that

$$\Big\langle \langle \int_R Z(t)dt, F \rangle \Big\rangle = \int_R \Big\langle \langle Z(t), F \rangle \Big\rangle dt \quad \text{for all} \quad F \in (S)$$

We then say that $Z(t)$ is integrable in $(S)^*$

We introduce **Wick Products**

Suppose

$$F(\omega) = \sum_{\alpha} c_\alpha H_\alpha(\omega) \qquad G(\omega) = \sum_{\alpha} d_\alpha H_\alpha(\omega)$$

are in $(S)^*$. The their Wick product is

$$\begin{aligned} F \diamond G(\omega) :&= \sum_{\alpha,\beta} a_\alpha b_\beta H_{\alpha+\beta}(\omega) \\ &= \sum_{\gamma} \Big(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \Big) H_\gamma(\omega) \end{aligned}$$

Hitsudu-Skorohod Integral

We have $W^H(t) \in (S)^*$ for all t. Suppose $Y : R \to (S)^*$ is such that $Y(t) \diamond W^H(t)$ is integrable in $(S)^*$. Then we define

$$\int_R Y(t)dB^H(t) := \int_R Y(t) \diamond W^H(t)dt$$

For example, if $Y(t)(\omega) = \sum_\alpha c_\alpha(t)H_\alpha(\omega)$, then formally

$$\begin{aligned}\int_R Y(t)dB^H(t)(\omega) &= \sum_{\alpha,i} H_{\alpha+\epsilon_i}(\omega) \int_R c_\alpha(t)M_H\xi_i(t)dt \\ &= \sum_{\alpha,i} H_{\alpha+\epsilon_i}(\omega) \int_R M_H c_\alpha(t)\xi_i(t)dt\end{aligned}$$

This integral generalises the Itô integral.

Generalized Hida Differentiation

Suppose $F : \mathcal{S}'(R) \to R$ and suppose $\gamma \in \mathcal{S}'(R)$. If

$$D_\gamma^H F(\omega) := \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon M_H \gamma) - F(\omega)}{\varepsilon} \qquad (*)$$

exists in $(S)^*$, we then call $D_\gamma^H F$ the directional M_H-derivative of F in the direction γ. If $\gamma = \delta_t$, then

$$\gamma_t^H(x) \equiv (M_H \delta_t)(x) = -C_H \frac{\partial}{\partial x} \left[(t-x)|t-x|^{H-\frac{3}{2}} \right] \in \mathcal{S}'(R)$$

and we write $\partial_t^H F(\omega)$ for the right hand side of (*).

Remark

Suppose $\gamma \in \mathcal{S}'(R)$ and has compact support. Let ψ be any infinitely differentiable function with compact support and which is identically 1 on the support of γ, then $M_H\gamma$ is a tempered distribution defined by (independent of ψ)

$$\langle M_H\gamma, \phi\rangle \equiv \langle \gamma, \psi M_H\phi\rangle$$

for any $\phi \in \mathcal{S}(R)$.

Examples

(1) If $H > \frac{1}{2}$, then $\gamma_t^H(x) = C_H(H - \frac{1}{2})|t-x|^{H-\frac{3}{2}} \in L^1_{loc}(R)$

(2) If $0 < H < 1$, then

$$\partial_t^H B^H(u) = \begin{cases} H\left[t^{2H-1} - (t-u)^{2H-1}\right] & \text{if } t > u \\ H\left[t^{2H-1} + (u-t)^{2H-1}\right] & \text{if } t < u \end{cases}$$

and $\partial_t^H W^H(u) = H(2H-1)|u-t|^{2H-2}$ for $u \neq t$ and $H \neq \frac{1}{2}$.

(3) If $H > \frac{1}{2}$, $F = \int_R f(u) dB^H(u)$, then

$$\partial_t^H F = H(2H-1)\int_R f(u)|t-u|^{2H-2} du$$

Examples continued

(4) If $H < \frac{1}{2}$, then if

$$F = \int_a^b f(u) dB^H(u) = f(b)B^H(b) - f(a)B^H(a) - \int_a^b f'(u)B^H(u)du$$

then for $a < t < b$

$$\partial_t^H F = f(b)\partial_t^H B^H(b) - f(a)\partial_t^H B^H(a) - \int_a^b f'(u)\partial_t^H B^H(u)du$$

(5) If $H = \frac{1}{2}$ then $\partial_t^H B(u) = 1$ if $u > t$ and $= 0$ if $u < t$

$\partial_t^H W(u) = \delta(t-u)$.

If $F = \int_R f(u) dB(u)$ then $\partial_t^H F = f(t) \equiv \partial_t F$

A Basic Lemma

$$\Phi \int_R \Psi(u) \diamond dB^H(u) = \int_R [\Phi \Psi(u)] \diamond dB^H(u) + \int_R \Psi(u) \partial_u^H \Phi du$$

A Corollary or a Definition

$$\int_R Y(u)dB^H(u) = \int_R Y(u) \diamond dB^H(u) + \int_R \partial_u^H Y(u)du$$

Proof: Let $\Phi = Y(u_i)$ and $\Psi = \mathbf{I}(u_i, u_{i+1}]$ in the basic lemma to gives

$$Y(u_i)\left[B^H(u_{i+1}) - B^H(u_i)\right] = \int_{u_i}^{u_{i+1}} Y(u_i) \diamond dB^H(u) + \int_{u_i}^{u_{i+1}} \partial_u^H Y(u_i)du$$

and so on. ■

First Itô's Lemma for $H > \frac{1}{2}$

Let

$$X(t) = X(0) + \int_0^t u(s)ds + \int_0^t v(s)dB^H(s)$$

and f is sufficiently smooth

$$\begin{aligned} f(X(t)) = f(X(0)) &+ \int_0^t f'(X(s))u(s)ds \\ &+ \int_0^t [f'(X(s))v(s)] \diamond dB^H(s) \\ &+ H(2H-1)\int_0^t f''(X(s))v(s)\left[\int_0^s v(r)(s-r)^{2H-2}dr\right]ds \end{aligned}$$

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$$\begin{aligned} f(X(t)) = f(X(0)) &+ \int_0^t f'(X(s))u(s)ds \\ &+ \int_0^t [f'(X(s))v(s)] \diamond dB^H(s) \\ &+ Hv(0) \int_0^t f''(X(s))v(s)s^{2H-1}ds \\ &+ H \int_0^t f''(X(s))v(s) \left[\int_0^s v'(r)(s-r)^{2H-1}dr \right] ds \end{aligned}$$

Remark 1 The formula for $H < \frac{1}{2}$ also holds for $H > \frac{1}{2}$ and agrees with the usual formula when $H = \frac{1}{2}$.

$$
\begin{aligned}
f(X(t)) &= f(X(0)) + \int_0^t f'(X(s))u(s)ds \\
&+ \int_0^t [f'(X(s))v(s)] \diamond dB(s) \\
&+ \frac{1}{2} v(0) \int_0^t f''(X(s))v(s)ds \\
&+ \frac{1}{2} \int_0^t f''(X(s))v(s) \left[\int_0^s v'(r)dr \right] ds \\
&= f(X(0)) + \int_0^t f'(X(s))u(s)ds \\
&+ \int_0^t [f'(X(s))v(s)] \diamond dB(s) + \int_0^t \tfrac{1}{2} f''(X(s))v(s)^2 ds
\end{aligned}
$$

Remark 2

When $X(t) = B^H(t)$, it is now well known that

$$f(B^H(t)) = f(0) + \int_0^t f'(B^H(s)) \diamond dB^H(s) + H \int_0^t f''(B^H(s)) s^{2H-1} ds$$

which again agrees with the usual formula when $H = \frac{1}{2}$.

General Itô Formula for $H > \frac{1}{2}$

$$X(t) = X(0) + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s)) \diamond dB^H(s)$$

implies

$$\begin{aligned} f(X(t)) = f(X(0)) &+ \int_0^t f'(X(s))\mu(X(s))ds \\ &+ \int_0^t [f'(X(s))\sigma(X(s))] \diamond dB^H(s) \\ &+ \int_0^t f''(X(s))\sigma(X(s))\partial_s^H X(s)ds \end{aligned}$$

$$\begin{aligned} \partial_s^H X(t) = &\int_0^t \mu'(X(r))\partial_s^H X(r)dr + \int_0^t [\sigma'(X(r))\partial_s^H X(r)] \diamond dB^H(s) \\ &+ H(2H-1)\int_0^t \sigma(X(r))|s-r|^{2H-2}dr \end{aligned}$$

Outline of Proof for $H > \frac{1}{2}$

Let $t_j = j\Delta t$, $j = 0, 1, 2, ..., n$ with $t_n = t$

$$f(X(t)) - f(X(0)) = \sum_{j=0}^{n-1} f(X(t_{j+1})) - f(X(t_j))$$

$$= \sum_{j=0}^{n-1} \left[\int_0^1 f'(\theta X(t_{j+1}) + (1-\theta)X(t_j))d\theta\right][X(t_{j+1}) - X(t_j)]$$

$$= \sum_{j=0}^{n-1} \Phi_j \left[\int_{t_j}^{t_{j+1}} u(s)ds + \int_{t_j}^{t_{j+1}} v(s)dB^H(s)\right]$$

$$= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Phi_j u(s)ds + \int_{t_j}^{t_{j+1}} \left[\Phi_j v(s)\right] \diamond dB^H(s) + \int_{t_j}^{t_{j+1}} v(s)[\partial_s^H \Phi_j]ds$$

$$= I_n^1 + I_n^2 + I_n^3$$

Outline of Proof for $H > \frac{1}{2}$ continued

$$\begin{aligned}\partial_s^H \Phi_j &= \int_0^1 f''(\theta X(t_{j+1}) + (1-\theta)X(t_j)) \left[\theta \partial_s^H X(t_{j+1}) + (1-\theta)\partial_s^H X(t_j)\right] d\theta \\ &= \int_0^1 \theta f''(\theta X(t_{j+1}) + (1-\theta)X(t_j)) d\theta \quad \partial_s^H [X(t_{j+1}) - X(t_j)] \\ &+ \int_0^1 f''(\theta X(t_{j+1}) + (1-\theta)X(t_j)) d\theta \quad \partial_s^H X(t_j)\end{aligned}$$

$$I_n^3 = I_n^4 + I_n^5$$

$$I_n^4 = \sum_{j=0}^{n-1} \left[\int_0^1 \theta f''(\ldots) d\theta\right] \int_{t_j}^{t_{j+1}} v(s) \partial_s^H [X(t_{j+1}) - X(t_j)] ds$$

$$I_n^5 = \sum_{j=0}^{n-1} \left[\int_0^1 f''(\ldots) d\theta\right] \int_{t_j}^{t_{j+1}} v(s) \partial_s^H X(t_j) ds$$

Outline of Proof for $H > \frac{1}{2}$ continued

$$\partial_s^H X(t) = H(2H-1)\int_0^t v(r)|s-r|^{2H-2}dr$$

$$\partial_s^H X(t_j) = H(2H-1)\int_0^{t_j} v(r)(s-r)^{2H-2}dr \qquad \text{if} \quad t_j < s < t_{j+1}$$

$$I_n^5 \to H(2H-1)\int_0^t v(s)f''(X(s))\left[\int_0^s v(r)(s-r)^{2H-2}dr\right]ds$$

$$\int_{t_j}^{t_{j+1}} v(s)\partial_s^H[X(t_{j+1}) - X(t_j)]ds \approx v(t_j)^2(t_{j+1}-t_j)^{2H} \quad \text{and} \quad 2H > 1$$

$$I_n^4 \to 0$$

Remark

This proof also provides a new proof for establishing the usual Itô formula for standard Brownian Motion using Hitsuda-Skorohod integrals along the way.

The use of generalized Hida derivatives provides a transparent approach for finding general Itô formulas. Results are equivalent to those of Duncan, Hu & Pasic-Duncan (2000), Sottinen & Valkeila (2001), Biagini & Øksendal (2004) and so on by other approaches.

Now some applications in finance

We use FBM to model long-range dependence.

A second order stationary process $X = \{X(t)|t \geq 0\}$ has Hurst index defined by

$$H = \inf\{h : \limsup_{t\to\infty} t^{-2h} \operatorname{var}(X(t) - X(0)) < \infty\} .$$

B^H has Hurst index H, but so does $B^H + B^K$ when $H > K$ [!]

Fractional Black and Scholes Market - 1

Stock price S_1

$$dS_1(t) = \mu\, S_1(t)\, dt + \sigma\, S_1(t) \diamond dB^H(t) \quad \text{(or } B_M\text{)}$$

$$S_1(0) = s > 0$$

then

$$S_1(t) = s \exp\Big[\mu\, t + \sigma\, B^H(t) - \tfrac{1}{2}\,\sigma^2\, t^{2H}\Big] \equiv f(t, B^H(t))$$

for an appropriate $f = f(t, x)$.

Proof: By the FBM Itô formula:

$$
\begin{aligned}
S_1(t) = s &+ \int_0^t f_t(s, B^H(s))ds + \int_0^t f_x(s, B^H(s)) \diamond dB^H(s) \\
&+ H \int_0^t f_{xx}(s, B^H(s))s^{2H-1}ds \\
= s &+ \int_0^t S_1(s)(\mu - H\sigma^2 s^{2H-1})ds \\
&+ \int_0^t \sigma S_1(s) \diamond dB^H(s) + H \int_0^t S_1(s)\sigma^2 s^{2H-1}ds \\
= s &+ \int_0^t \mu S_1(s)ds + \int_0^t \sigma S_1(s) \diamond dB^H(s) \qquad \blacksquare
\end{aligned}
$$

Fractional Black and Scholes Market - 2

Stock price S_1 (and $H > \frac{1}{2}$)

$$dS_1(t) = \mu\, S_1(t)\, dt + \sigma\, S_1(t) dB^H(t)$$

$$S_1(0) = s > 0$$

then

$$S_1(t) = s \exp\Big[\mu\, t + \sigma\, B^H(t)\Big] \equiv g(t, B^H(t))$$

for an appropriate $g = g(t, x)$.

Proof: We first observe (for $H > \frac{1}{2}$) that

$$\partial_t^H S_1(t) = \sigma S_1(t) \partial_t^H B^H(t) = \sigma S_1(t) H t^{2H-1}$$

and by the FBM Itô formula:

$$\begin{aligned}
S_1(t) &= s + \int_0^t g_t(s, B^H(s))ds + \int_0^t g_x(s, B^H(s)) \diamond dB^H(s) \\
&\quad + H \int_0^t g_{xx}(s, B^H(s)) s^{2H-1} ds \\
&= s + \int_0^t S_1(s)\mu ds + \int_0^t \sigma S_1(s) \diamond dB^H(s) + H \int_0^t S_1(s)\sigma^2 s^{2H-1} ds \\
&= s + \int_0^t \mu S_1(s) ds + \int_0^t \sigma S_1(s) dB^H(s) - \int_0^t \partial_s^H S_1(s) ds \\
&\quad + H \int_0^t S_1(s)\sigma^2 s^{2H-1} ds \\
&= s + \int_0^t \mu S_1(s) ds + \int_0^t \sigma S_1(s) dB^H(s) \qquad \blacksquare
\end{aligned}$$

Fractional Black and Scholes Market - 3

Stock price S_1 (and $H > \frac{1}{2}$), B is Brownian motion independent of B^H

$$dS_1(t) = (\mu + \tfrac{1}{2}a^2)\, S_1(t)\, dt + aS_1(t)dB(t) + \sigma\, S_1(t)dB^H(t)$$

$$S_1(0) = s > 0$$

then

$$S_1(t) = s \exp\Big[\mu\, t + a\, B(t) + \sigma\, B^H(t)\Big] \equiv g(t, B(t), B^H(t))$$

for an appropriate $g = g(t, x, y)$.

Self Financing Portfolios

Portfolio replication in continuous time:

$$\theta = \Big\{\theta(t)\Big\}$$

$\theta(t) = \Big(\alpha(t), \beta(t)\Big)$ is a pair of (suitably) adapted processes which give the number of units of S_1 and S_0 held at time t.

$V^{\theta} = \Big\{V^{\theta}(t)\Big\}$ is a **wealth process** and with $S_0(t) \equiv \exp(rt)$,

$$V^{\theta}(t) = \alpha(t)\, S_1(t) + \beta(t)\, S_0(t)$$

θ satisfies certain technical conditions and must be self-financing:

$$dV^{\theta}(t) = \alpha(t)\, dS_1(t) + \beta(t)\, dS_0(t)$$

Changes in the wealth process come from changes in the values of the underlying assets S_0 and S_1.

$$V^{\theta}(T) = \Big[S_1(T) - K\Big]^{+}$$

for a European call option, say.

With

$$dS_1(t) = rS_1(t)dt + \sigma S_1(t) \diamond dB^H(t)$$

we can show

$$dV^\theta(t) = rV^\theta(t)dt + \sigma[\alpha(t)S_1(t)] \diamond dB^H(t) + \sigma S_1(t)\partial_t^H \alpha(t)dt$$

and

$$e^{-rT}V^\theta(T) = V^\theta(0) + \int_0^T \sigma e^{-ru}[\alpha(u)S_1(u)] \diamond dB^H(u) + \int_0^T \sigma e^{-ru} S_1(u)\partial_u^H \alpha(u)du$$

As equation (1) makes sense for $0.5 \leq H \leq 1$ and (2) for all $0 < H < 1$, the latter can be taken as the self-financing requirement, and could be established from discrete approximations.

Now in general

$$V^{\theta}(0) \neq \mathbf{E}\left[e^{-rT}V^{\theta}(T)\right]$$

But when $H = 1/2$, then $\partial_t^H \alpha(t) \equiv 0$.

Girsanov Transformation

Suppose that under probability $\widehat{P}$

$$\widehat{B}^H(t) = B^H(t) - \int_0^t \phi(u)\ du$$

then

$$e^{-rT}V^\theta(T) = V^\theta(0) + \int_0^T \sigma e^{-rt}\left[\alpha(t)S(t)\right] \diamond d\widehat{B}^H(t) \\ + \int_0^T \sigma e^{-rt}\Big(S(t)\partial_t^H \alpha(t) + \left[\alpha(t)S(t)\right] \diamond \phi(t)\Big)\, dt$$

whence we are led to seek ϕ so that

$$S(t)\partial_t^H \alpha(t) + \left[\alpha(t)S(t)\right] \diamond \phi(t) = 0 \qquad (*)$$

for all t.

A Lemma

If for each self-financing strategy θ, there exists a probability measure change (there exists a suitable solution of (*)), then

$$V^{\theta}(0) = \hat{\mathbf{E}}\left[e^{-rT}V^{\theta}(T)\right]$$

and there are no-arbitrage opportunities in this market

When $H = \frac{1}{2}$, $\phi \equiv 0$ solves (*) for all α.

Some Issues

1. Given a (B, S) market, with S driven by FBM or by **multiple** FBMs, and a class of admissible self-financing strategies, does there exist for each α in this class, a suitable choice of ϕ for that (*) holds.

2. Given ϕ as in the previous item, is there $\widehat{P}$ so that $\widehat{B}^H(t) = B^H(t) - \int_0^t \phi(u)du$ is FBM under $\widehat{P}$ (or a suitable generalization for multiple FBMs). One needs to use an anticipative Girsanov Theorem.

In the (B,S) market, Cheridito(2000) claimed that there are arbitrage opportunities, so one may try to construct α for which there is no ϕ. However for some "mixture" models there are no-arbitrage opportunities (e.g. Kuznetsov(1999), Cheridito(2000), Androshchuk + Mishura(2005)). For FBM markets, models for S have to be specified carefully, and the specification of admissible trading strategies.

Our machinery can be adapted to multiple FBM models and used to study these questions. This is on-going research.

Fractional Black and Scholes Market - 3 again

Bank account $S_0 = e^{rt}$, stock $S_1(t) = s\,\text{exp}\Big[\mu\, t + a\, B(t) + \sigma\, B^H(t)\Big]$
Let

$$\phi_t(t,x) + \frac{1}{2}a^2x^2\phi_{xx}(t,x) + rx\phi_x(t,x) - r\phi(t,x) = 0$$
$$\phi(T,x) = g(x)$$

that is

$$\phi(t,x) = \frac{1}{\sqrt{2\pi}}\int_R g(x\,\text{exp}[(r - \tfrac{1}{2}a^2)(T - t) + |a|z\sqrt{T-t}\,])exp(-\tfrac{1}{2}z^2)dz$$

when $a \neq 0$. Let

$$\alpha(t) = \phi_x(t, S_1(t))$$
$$\beta(t) = e^{-rt}(\phi(t, S_1(t)) - S_1(t)\phi_x(t, S_1(t)))$$

Then $V^\theta(t) \equiv \alpha(t)S_1(t) + \beta(t)S_0(t)$ is self-financing and $V^\theta(T) = g(S_1(T))$. With $a \neq 0$ it is possible to construct a measure change so that $V^\theta(0) = \tilde{E}[e^{-rT}V^\theta(T)] = \tilde{E}[e^{-rT}g(S_1(T))]$ In fact

$$\begin{aligned} e^{-rT}V^\theta(T) = V^\theta(0) &+ \int_0^T e^{-rt} a S_1(t)\phi_x(t, S_1(t)) dB(t) \\ &+ \int_0^T e^{-rt}\sigma[S_1(t)\phi_x(t, S_1(t))] \diamond dB^H(t) \\ &+ \int_0^T e^{-rt} S_1(t)\phi_x(t, S_1(t))\lambda(t, S_1(t)) dt \end{aligned}$$

and now we perform a Girsanov shift on B. A call price on S_1 could in principle then be computed either as $V^\theta(0) = \phi(0, s)$ or by computing an appropriate expectation.

This shows that when we restrict self-financing strategies to ones which are of Markov-type $[\theta(t) \equiv \theta(t, S_1(t))]$, then there are no arbitrage opportunities.

But when $a = 0$

$$\phi(t, x) = e^{-r(T-t)} g(x \cdot e^{r(T-t)})$$

and with the choice $g(x) = (x - s \cdot e^{rT})^2$ say, we have $V^\theta(0) = 0$ and $V^\theta(T) \geq 0$ with $P[V^\theta(T) > 0] > 0$, an arbitrage.

This example was stimulated by the study of Androshchuk and Mishura(2005).

When $g(x) \equiv (x-K)^+$, then $\phi(S(t),t)$ gives the price of the European call option. That is

$$C_t = \phi(S(t),t) = S(t)\mathcal{N}(d_1(t)) - Ke^{r(T-t)}\mathcal{N}(d_2(t))$$

with $d_1(t) = \frac{\ln\left[\frac{S(t)}{K}\right] + (r+\frac{1}{2}a^2)(T-t)}{|a|\sqrt{T-t}}$ and $d_2(t) = d_1(t) - |a|\sqrt{T-t}$.

This agrees with the usual Black-Scholes formula when $\sigma \to 0+$.

But when $a \to 0+$, the expression C_t converges to

$$e^{-r(T-t)}\left[S(t)e^{r(T-t)} - K\right]^+$$