Equilibrium Asset Pricing: With Non-Gaussian Factors and Exponential Utilities

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Motivation

- Asset Pricing Theory is about explaining differences in required returns across assets in terms of risk compensation when positive or insurance premia when negative.
- We recognize that expected returns are, empirically, near impossible to estimate and are nonetheless an important input into decisions on asset allocation.
- A theoretical determination of these magnitudes from the easier to estimate risk structure is thereby an important contribution to financial decision making.

- The first successful theory in this regard was the Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965) and Mossin (1966).
- This was followed by the factor structure Arbitrage Pricing Theory of Ross (1976) and the pure implications of no arbitrage more generally of Kreps and Harrison (1979), Harrison and Pliska (1981,1983) and Delbaen and Schachermayer (1994).

- These latter theories made fewer assumptions about economic structure and derived less operational results on asset pricing.
  - For example in the Ross structure it is unclear how the factors are to be constructed or identified.
  - It is also unclear what are the required risk premiums on the factor exposures.
    - As a result practical implementations rely on second pass cross sectional average return regressions to infer required returns.
      - This use of average returns to infer expected returns is theoretically objectionable, as the former are essentially a random input into the decision process.
- In the more general results on no arbitrage, apart from existence, the specification of the pricing kernel is totally unspecified.

- We shall here go in the other direction from CAPM and assume a lot more economic structure than that employed in CAPM.
- We shall use this additional structure to implement the asset pricing theory that derives required returns on assets without using average returns in any way other than to demean the data.
- The additional structure we employ has been learned over the last ten years in implementing solutions to derivative pricing problems and we now bring this material to bear on the more traditional questions of asset pricing.

- The important ideas in the structural specification are
  - Independent Components Analysis (ICA) to identify factor structure.
  - The use of self similarity and scaling to represent risks over the longer term.
    * In particular we mention the class of selfdecomposable random variables as possible non-Gaussian limit laws.
  - The use of the law of Lévy processes at unit time to describe risk characteristics beyond the variance.
    * In particular the Variance Gamma (VG) model as synthesizing skewness and kurtosis while possesing some exponential moments.

Outline

- The Economic Structure, Scaling and Self Similarity.
- The Equilibrium Asset Pricing Equation.
- ICA and Factor Detection.
- The VG Factor Laws.
- Closed Form Equations for Factor Risk Premia.
- Results on Required Returns.

The Economic Structure

- We follow Ross (1976) and adopt a factor structure for annual returns and write
$$R = \mu + BX + \varepsilon$$
where R is a vector of asset returns, X is a set of systematic zero mean, unit variance, and orthogonal factors and ε is a noise or idiosyncratic component.
- However we borrow from the literature on signal processing and impose some additional structural hypotheses.
- It is noted in this literature, that among finite variance zero mean random variables maximum uncertainty in the sense of entropy is attained by the Gaussian density.
- This density is therefore a good candidate for modeling noise.

- Informative random variables are possibly closer to delta functions with long necks and fat tails displaying excess kurtosis.
- Our analysis, for example in Carr, Geman, Madan and Yor (2002) finds significant kurtosis both statistically and risk neutrally in index returns.
- Independent Components Analysis seeks to recover signal components of data by performing a PCA (principal components analysis) and then finding rotation matrices to maximise a metric of non-Gaussianity with the view that information and signals are best characterized this way.

- These considerations lead us to postulate that X consists of independent random variables with non-Gaussian kurtotic and skewed densities.
- In implementation we shall use a kurtosis cutoff and treat as noise the lower kurtosis components.
- Additionally we model the noise component as multivariate Gaussian.
- Our model may be compared with Simaan (1993) who used a single, stable α, non-Gaussian component and an elliptical distribution for the noise component.

Investment Horizons

- There is plenty of data on daily and intraday returns, but investment horizons run in years.
- Additionally we have the critique that surely by central limit theorem arguments long horizon returns are Gaussian and the focus on skews and kurtosis is therefore misplaced.

- We make two responses to these views.
  - First the Gaussian distribution is not the only limit law and both Lévy (1937) and Khintchine (1938) characterized all the other limit laws resulting on more general scaling than $\sqrt{n}$ as the class L_0 of self decomposable random variables, a subclass of the infinitely divisible laws, of which the Variance Gamma is an example.

- Second there is evidence that self similarity or scaling is a reasonable model for the longer horizon.
  - Statistically we refer to Peters (1991), Mandelbrot (1997), Shiryaev (1999), Heyde (1999) and Cont (2001).
  - Risk Neutrally we refer to Carr, Geman, Madan and Yor (2006) forthcoming.
    - They show in particular that the scaled VG process at unit time synthesizes the option surface remarkably well with just four parameters, while the associated Lévy process could never do this.

- Under scaling, skewness and kurtosis are independent of the time horizon and remains an important risk concern.
- We take demeaned annual returns as distributed like $\sqrt{252}$ times the daily return.
- It remains to describe the law of the systematic components. For this we employ the centered law at unit time of the Variance Gamma Lévy process.

- The process is defined as Brownian motion $(W(t), t > 0)$ with drift rate θ and variance rate σ^2 evaluated at a random time given by a gamma process $(g(t;\nu), t > 0)$ with mean rate unity and variance rate ν. Hence we may write the centered VG Lévy process $(X(t), t > 0)$ as
$$X(t) = \theta(g(t;\nu) - t) + \sigma W(g(t;\nu))$$
- The cumulant is easily evaluated as
$$\psi(u) = -\frac{1}{\nu}\ln\left(1 - u\theta\nu - \frac{\sigma^2\nu}{2}u^2\right) - u\theta.$$
- In summary we model asset annual returns as scaled daily returns that are linear mixtures of the laws of Lévy processes at unit time plus a multivariate correlated Gaussian noise component.

Investor Risk Preferences

- Investors in the economy are expected utility maximizers with exponential utility functions.
- Investors differ in just their coefficients of absolute risk aversion.
- Hence the utility function of investor i, with final cash flow C_i is
$$U_i(C_i) = 1 - \exp(-A_i C_i)$$
where A_i is the coefficient of absolute risk aversion for investor i.

- These are strong assumptions made for reasons of getting tractable and operational results.
- Nonetheless we mention that as a basic completely monotone marginal utility function, we expect to see aversion to variance and kurtosis and skewness preference.
- For exponential utility and our asset return model the certainty equivalent of the cash flow from financed investment at the interest rate r

$$C_i = \alpha_i'(R - r)$$

- is

$$\begin{aligned}(CE)_i(\alpha_i) &= \alpha_i'(\mu - r) - \frac{A_i}{2}\alpha_i'\Sigma\alpha_i \\ &\quad - \frac{1}{A_i}\sum_{k=1}^{K}\psi_k(-\alpha_i'\beta_k A_i)\end{aligned}$$

  where β_k is the k^{th} column of the factor loading matrix B, Σ is the covariance matrix of ε and ψ_k is the cumulant function of the k^{th} independent factor.

Investment Allocations

- The first order conditions for maximizing certainty equivalents yield the dollar investments by each investor as

$$\alpha_i = \frac{1}{A_i}\Sigma^{-1}(\widetilde{\mu} - r)$$

$$\widetilde{\mu} = \mu + \sum_{k=1}^{K} \psi_k'(-\alpha_i'\beta_k A_i)\beta_k$$

- We may now define the factor exposure of investor i in factor j as

$$y_{ij} = \alpha_i'\beta_j$$

- Evaluating this exposure we see that
$$A_i y_{ij} = (\mu - r)' \Sigma^{-1} \beta_j + \sum_k \psi'_k(-y_{ik} A_i) \beta'_k \Sigma^{-1} \beta_j$$
- We now see that for each i if we define
$$\eta_j = y_{ij} A_i$$
then the magnitudes η_j that may be interpreted as the factor exposure desired by unit risk aversion satisfy the equation system independent of i and given by
$$\eta_j = (\mu - r)' \Sigma^{-1} \beta_j + \sum_k \psi'_k(-\eta_k) \beta'_k \Sigma^{-1} \beta_j$$
- It follows that risk aversion scaled factor exposures are independent of i, the more risk averse holding a proportionately lower exposure.

Aggregate Factor Exposures

- The total exposure to factor j held in the economy is then
$$\sum_i y_{ij} = \eta_j \sum_i \frac{1}{A_i}$$
and we have unit risk aversion exposures given by
$$\eta_j = \overline{A} \sum_i y_{ij}$$
where $\overline{A}$ is the harmonic mean of investor risk aversions.

Pricing Factor Exposures

- We now interpret the terms $\psi_k'(-\eta_k)$ in the adjustment to excess returns in defining $\widetilde{\mu}$ that appears in the final asset allocation equation.
- Differentiating the moment generating function we observe that
$$\exp\left(\psi_k(u)\right)\psi_k'(u) = E\left[\exp\left(uX_k\right)X_k\right]$$
- and hence
$$\psi_k'(-\eta_k) = \frac{E\left[\exp\left(-\eta_k X_k\right)X_k\right]}{E\left[\exp\left(-\eta_k X_k\right)\right]}$$
- Defining by
$$\pi_k(\eta_k) = -\frac{E\left[\exp\left(-\eta_k X_k\right)X_k\right]}{E\left[\exp\left(-\eta_k X_k\right)\right]}$$
- We may write
$$\alpha_i = \frac{1}{A_i}\Sigma^{-1}\left(\mu - r - \sum_{k=1}^{K}\pi_k(\eta_k)\beta_k\right)$$

- We have that
$$\pi_k(\eta_k) = -\frac{E\left[\exp\left(-\eta_k X_k\right) X_k\right]}{E\left[\exp\left(-\eta_k X_k\right)\right]}$$
- This is the expectation of X_k under an exponentially tilted measure change where the tilt is given by η_k that we know to be aggregate factor exposure times average risk aversion.
- This is the price of factor exposure in our economy and it may be evaluated directly from a knowledge of η_k and the probability law of the factor.
- For the VG we may explicitly evaluate that
$$\pi_{VG}(\eta) = \theta + \frac{\sigma^2\eta - \theta}{1 + \theta\nu\eta - \frac{\sigma^2\nu}{2}\eta^2}.$$

- We now see the investor asset allocations as those of a factor adjusted residual CAPM.
- Define returns net of factor exposure as
$$\widetilde{R} = R - BX$$
- Let the mean returns on these residual returns be set at
$$\widetilde{\mu} = \mu - \sum_{k=1}^{K} \pi_k(\eta_k)\beta_k$$
- Then asset allocations are as in Markowitz and may be written as
$$\alpha_i = y_i \omega$$
- where ω is the tangency portfolio of this residual return economy with
$$\omega = \frac{\Sigma^{-1}\left(\mu - r - \sum_{k=1}^{K} \pi_k(\eta_k)\beta_k\right)}{\mathbf{1}_N' \Sigma^{-1}\left(\mu - r - \sum_{k=1}^{K} \pi_k(\eta_k)\beta_k\right)}.$$

The Market Equilibrium

- Given initial shares outstanding at $\overline{n}$ with asset market prices S the aggregate dollar factor exposure has to be
$$(diag(S)\overline{n})'\beta_j$$
- Defining relative risk aversion by
$$\rho = \overline{A}\,(\mathbf{1}'\Delta(S)\overline{n})$$
- We get that
$$\frac{\eta_j}{\rho} = \omega_M'\beta_j$$
where ω_M is the market portfolio based on proportion of market values.
- We shall use this equation to find unit risk aversion exposure vector η.

- Standard arguments now yield the residual Capital Asset Pricing Model where by

$$\mu = r + \sum_{k=1}^{K} \pi_k(\eta_k)\beta_k + \beta_M \lambda$$

- where β_M is covariance of $\widetilde{R}$ with $\omega'_M \widetilde{R}$ and the price λ of the residual market beta exposure is given by

$$\lambda = \rho \omega'_M \Sigma \omega_M$$

  or risk aversion times the variance of the returns net of factor exposure.

- We shall use this equation to find the residual market beta risk premium.
- The prices of factor exposure come from the closed forms for the VG, that we have already displayed.

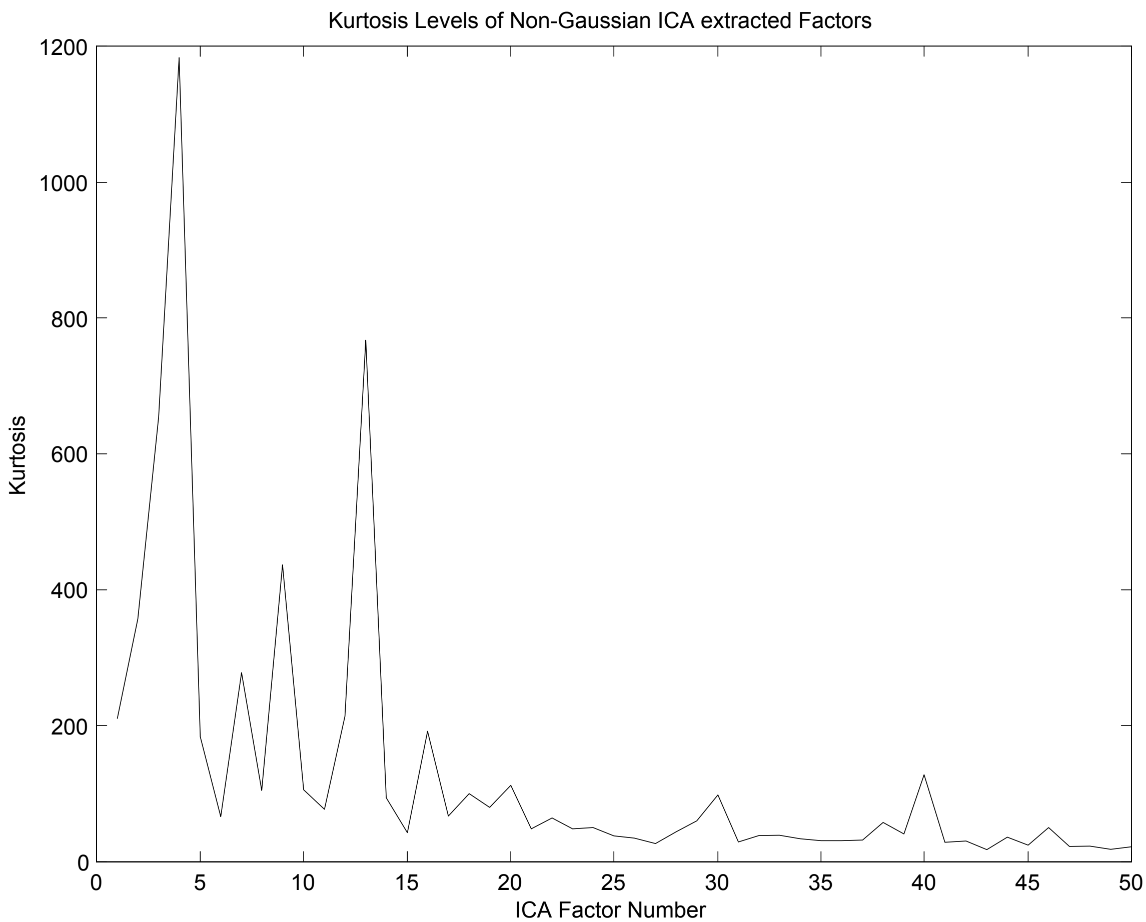
Remarks on Asset Pricing

- The final asset pricing equation is a synthesis of the Ross APT and the CAPM.
- Factor exposures are priced, except here they are explicitly priced by exponential tilting at level η given by risk aversion times aggregate factor exposure in the market portfolio.
- The resiudal risks of ε are also priced but this is done by covariance of residual returns to residual returns of the market portfolio.
- The market price of this residual market beta is risk aversion times residual variance.

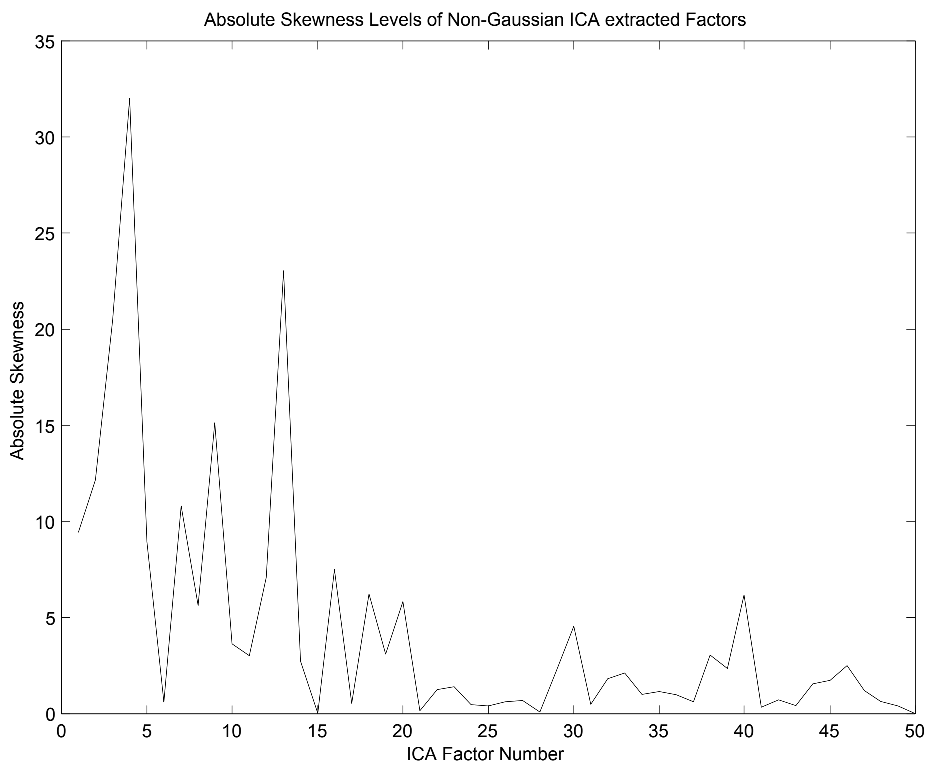
An Illustrative Implementation

- We took data on daily stock returns for 180 stocks that comprise 75% of the US economy as at March 23 2006, for the period January 3 2000 to March 23 2006.
- The first step was to identify the longtailed factors.
- For this we ran the fast ICA algorithm to extract the top 50 factors.
- This algorithm first performs a PCA and then searches in the space of rotation matrices with a view to maximizing the non-Gaussian metric given by the expectation of the logarithm of the hyperbolic cosine.
- This has been found to be a robust criterion for the implementation of ICA.
- We present a graph of the levels of kurtosis and absolute skewness in the first fifty factors.

- We obtain from an application of ICA the times series of data on the first 50 factors. This data is by design of zero mean, unit variance and orthogonal.
- We see the kurtosis levels trailing off as we extract more factors.
- We decided to take as real factors those with a kurtosis level exceeding 50. There were 26 such factors and we next ran a regression of the top 100 asset returns on these 26 factors to estimate the matrix B that is 100 by 26 and the residual covariance matrix Σ.



1.Kurtosis Levels of the top 50 factors extracted by fast ICA



2.Absolute Skewness Levels of the top 50 factors extracted by fast ICA

- To get a positive definite covariance matrix with correlation we performed a factor analysis on the residuals with 50 factors to get a maximum likelihood decomposition in the form

$$\Sigma = \Lambda\Lambda' + J$$

  where J is a 100 by 100 diagonal matrix and Λ is the 100 by 50 factor loading matrix.

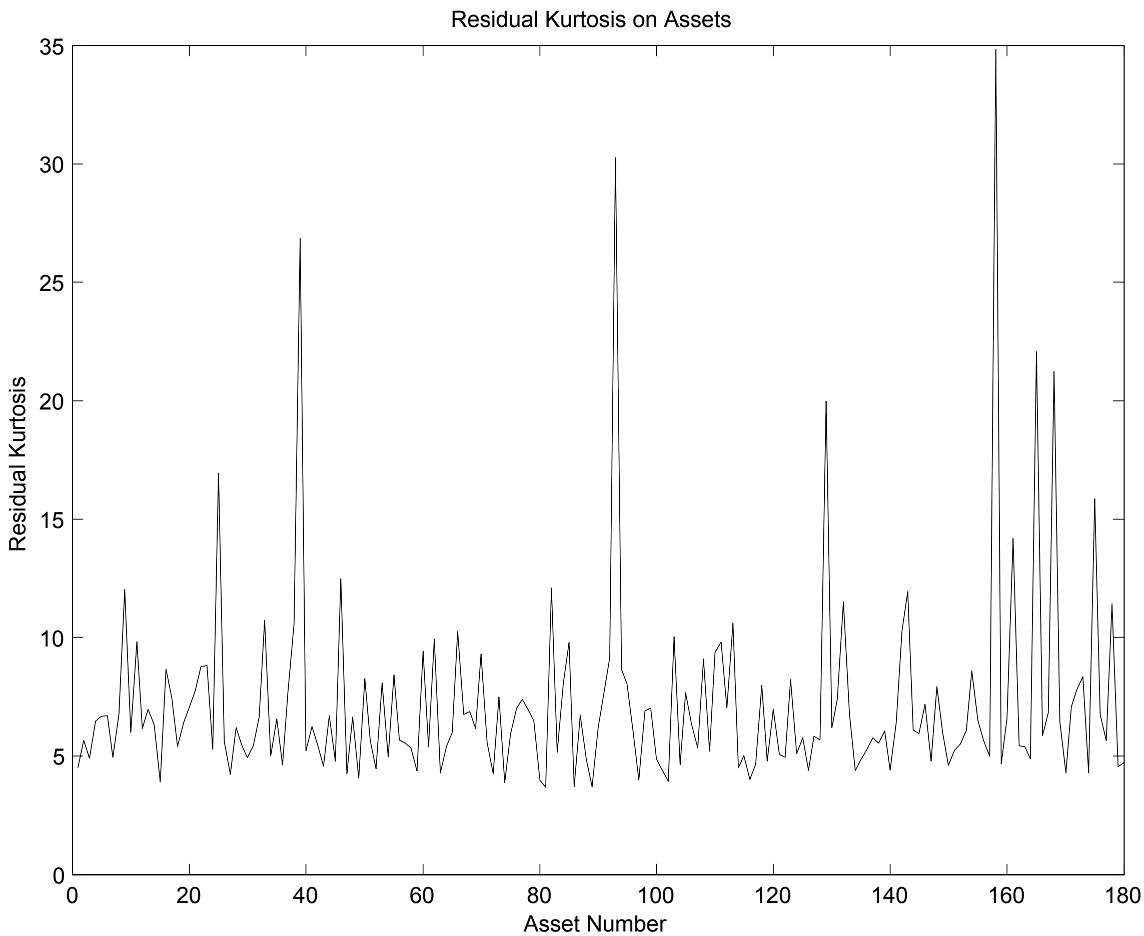
- These operations give us the B matrix, the residual covariance matrix Σ, and the times series data on the 26 selected longtailed factors.

- We present the graphs of kurtosis and absolute skewness of the residuals.

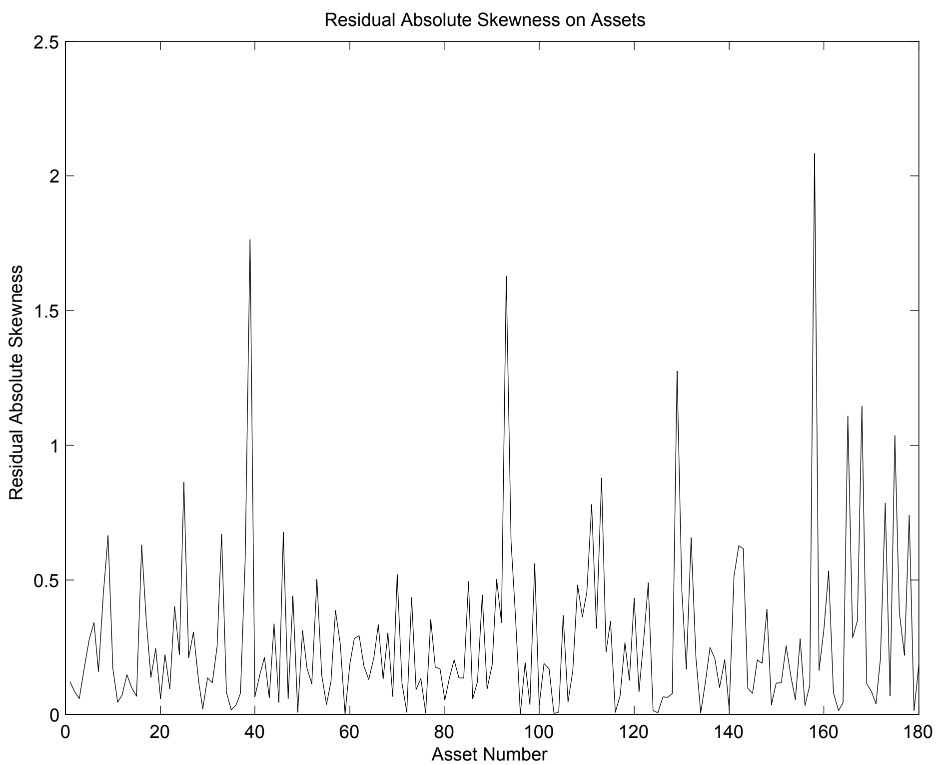
- We observe that though there remains some kurtosis and skewness in the residuals it is substantially below the levels observed in the extracted factors.

The Probability Laws of the Independent Components

- For each of the 26 factors we estimate the parameters of the centered VG law by maximum likelihood.
- To reduce computation times we first bin the data on the factor time series into 100 bins and then we maximize the likelihood of the binned data.
- The VG density is available in closed form in terms of the modified Bessel function (Madan, Carr and Chang (1998)) and we use this representation in the maximum likelihood estimations.
- We present a graph of the fitted VG densities to six of the more long tailed factors.



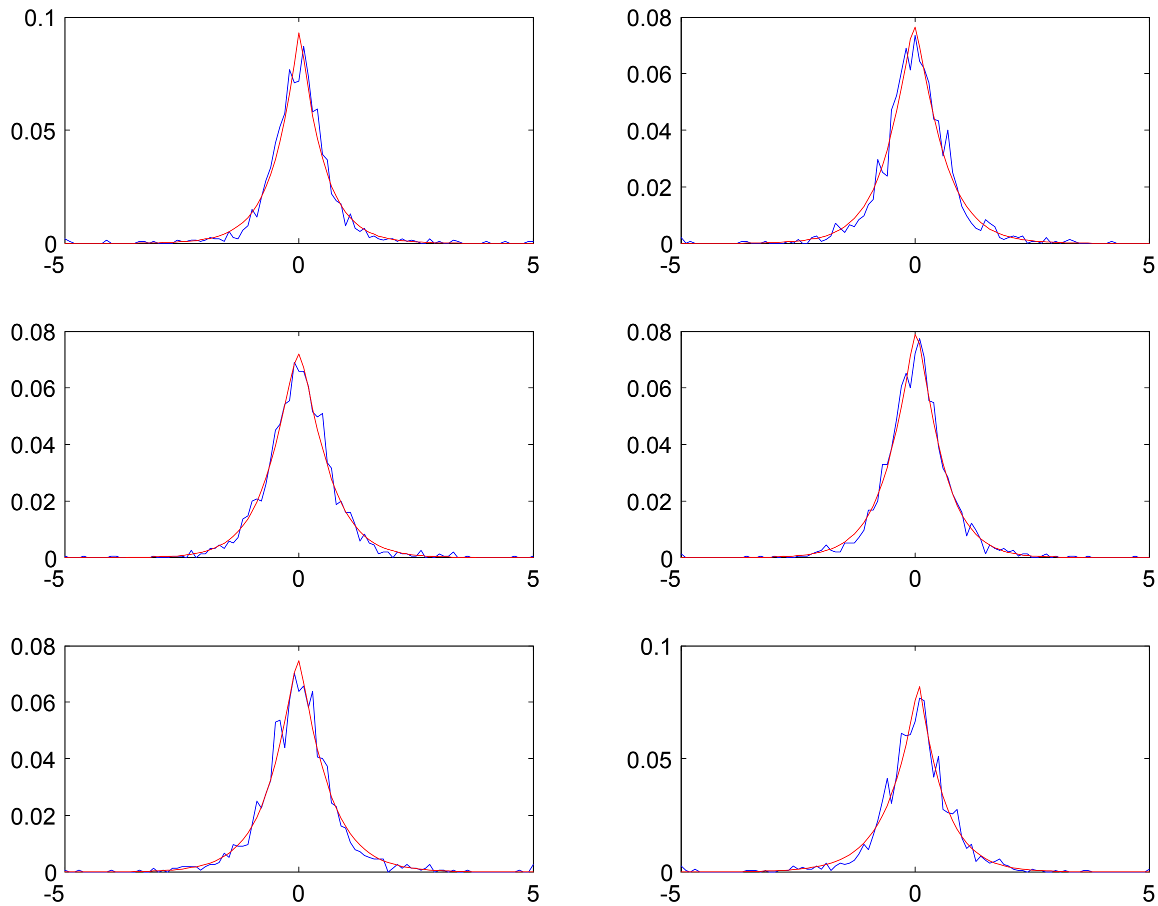
3.Kurtosis Levels of Residual Returns



4.Absolute Skewness of Residual Returns

Factor Pricing

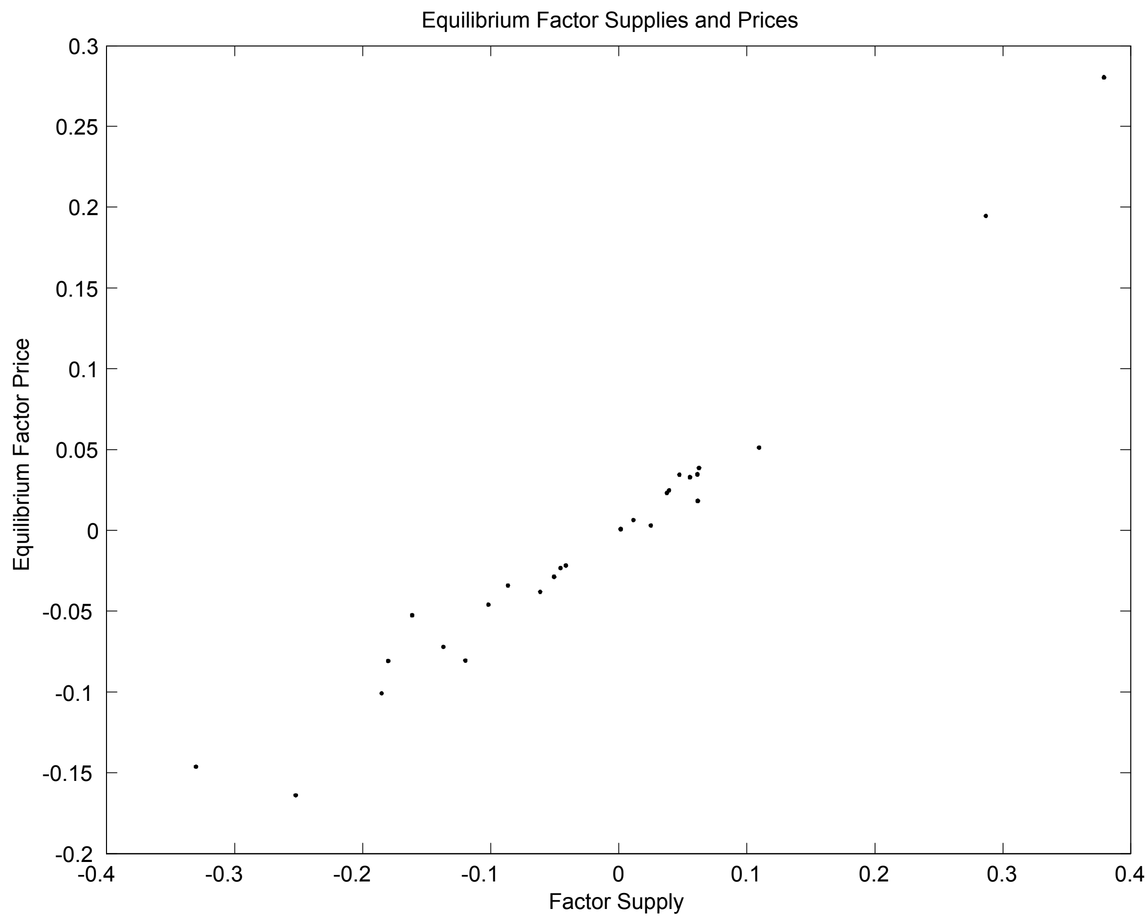
- For factor pricing we first extract the factor exposure in the market portfolio as
$$\omega'_M \beta_j$$
- Here ω_M is the vector of relative capitalizations of the first 100 stocks by market cap and β_j is the j^{th} column of the regression estimated B matrix annualized on multiplication by $\sqrt{252}$.
- For the value of η_j we used a risk aversion of 10 and defined
$$\eta_j = 10\omega'_M \beta_j.$$
- The prices of factor exposure are given by the VG factor pricing closed form.
- We present a graph of the equilibrium factor price against η_j the factor supply adjusted by risk aversion.
- We observe that factor prices have to increase to induce higher levels of factor holdings.



5.Empirical Density in Blue and the Fitted VG in red for six of the 26 ICA extracted factors

Required Returns and Required Factor Exposure Compensation

- We evaluate the variance of residual market protfolio returns by
$$\omega'_M \Sigma \omega_M$$
- The market price of residual beta risk is
$$\lambda = 10\omega'_M \Sigma \omega_M$$
- The residual market betas are evaluated by
$$\beta_M = \Sigma \omega_M$$

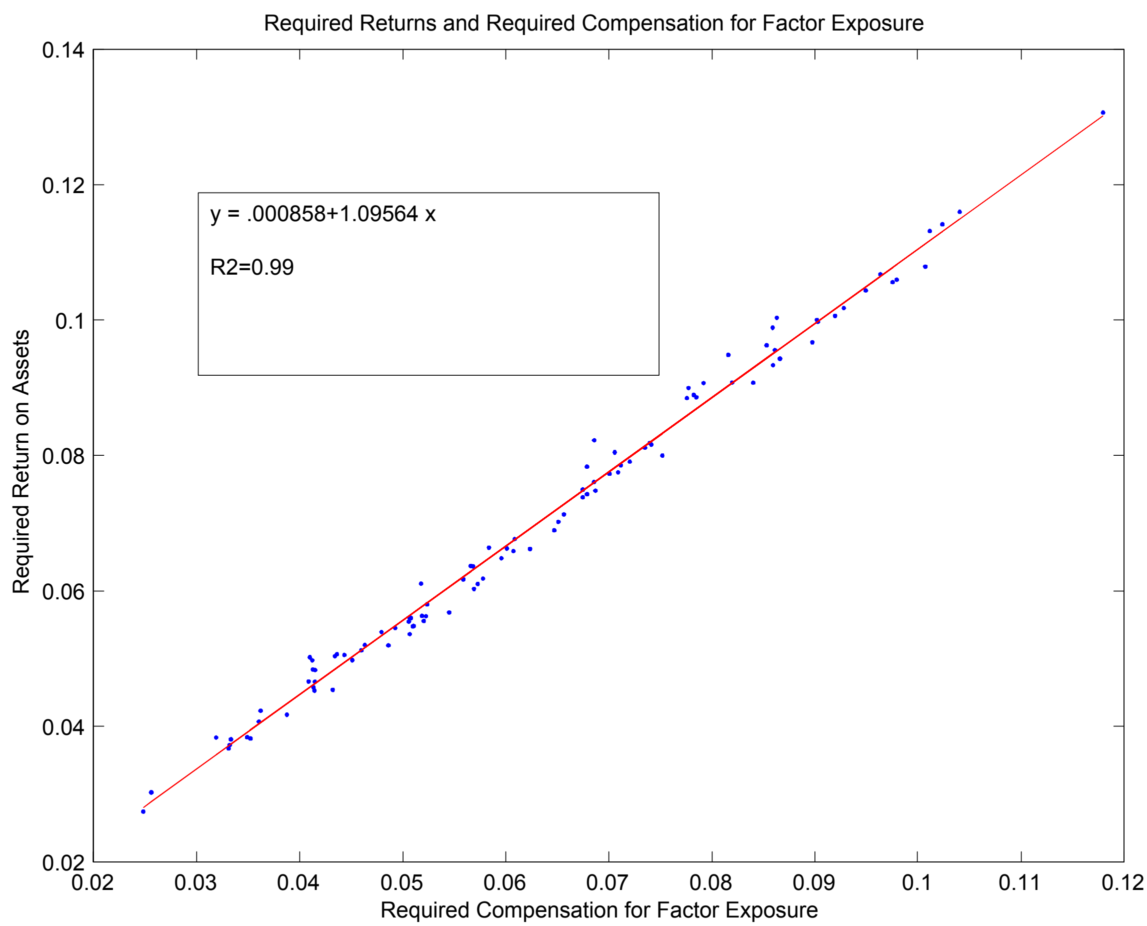


6.VG exponentially tilted Factor prices vs Factor holdings by unit risk aversion

- We may then evaluate both the required compensation for factor exposure
$$RCFE = r + \sum_{k=1}^{K} \pi_{VG}(\eta_k)\beta_k$$
and the total required return as
$$RR = RCFE + \beta_M \lambda$$
- We present a graph of RR against $RCFE$ and the regression line of RR on $RCFE$.
- We observe that the bulk of the asset pricing variation is due to compensation for factor exposure accounted for by exponential tilting.

Conclusion

- We model annual asset returns as scaled daily returns.
- Daily returns are linear mixtures of independent skewed and kurtotic informative (in the sense of entropy) random variables plus correlated multivariate Gaussian noise.
- Investors have exponential utility displaying skewness preference and kurtosis aversion.



7.Required Returns on the top 100 stocks by market cap vs Required Compensation for Factor Exposure

- In equilibrium exposures to the factors are priced by exponential tilting.
- The degree of tilt depends on risk aversion and the exposure of the market portfolio to the factor.
- The residual noise risk is also priced.
- The residual noise risk is priced by covariation of this risk with the noise risk in the market portfolio.
- The market price of noise risk is depends on its variance in the market portfolio and risk aversion.

- The results are illustrated on data for the US economy.
- Factors are identified using fast ICA.
- Factor probability laws are estimated in closed form as the law of the VG process at unit time.
- Factor risk prices are given in closed form for VG exponential tilting.