CHARACTERISATION OF $b(M)$ FOR CONTINUOUS BMO-MARTINGALES.

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(1) $(\Omega, (\mathcal{F}_t)_{0\le t\le\infty}, \mathbf{P})$, usual assumptions

(2) all martingales are continuous, all stopping times are predictable

(3) M is a (real) BMO-martingale

(4) $\mathcal{E}(M) = \exp\left(M - \frac{1}{2}\langle M, M\rangle\right)$, $d\mathbf{Q} = \mathcal{E}(M)_\infty \, d\mathbf{P}$

(5) For $\lambda \in \mathbb{C}$: $\mathcal{E}(\lambda M) = \exp\left(\lambda M - \frac{1}{2}\lambda^2\langle M, M\rangle\right)$

(6) $\mathcal{E}(iM) = \exp\left(iM + \frac{1}{2}\langle M, M\rangle\right)$

$$b(M) = \sup \left\{ b \;\middle|\; \begin{array}{c} \text{there is } K \text{ so that for every stopping time } T: \\ \mathbb{E}\left[\exp\left(\frac{b^2}{2}\left(\langle M, M\rangle_\infty - \langle M, M\rangle_T\right)\right) \mid \mathcal{F}_T\right] \le K \end{array} \right\}$$

Kazamaki asked if there is a relation between $b(M)$ and the BMO-distance of M to $H^\infty = \{N \mid \langle N, N\rangle \in L^\infty\}$

Grandits answered the question by "no"

Schachermayer characterised the closure of H^∞ in BMO

What is the meaning of $b(M)$?

The Föllmer-Schweizer decomposition.

Let $S = M - \langle M, M\rangle + N$ be a price process, here $d\langle N, N\rangle \perp d\langle M, M\rangle$. All price processes can be written in such a way by adapting the "drift related part" of the stochastic driver.

We say that a Föllmer-Schweizer decomposition exists if each random variable $f \in (H^p)^*$ can be written as

$$f = \pi(f) + \int_0^\infty H_u \, dS_u + W_\infty, \text{ where } W \perp S, W \in (H^p)^*.$$

This problem was solved

(1) D., Monat, Schachermayer, Schweizer, Stricker for $p = 2$
(2) Grandits and Krawczyk for $1 < p < \infty$
(3) $p = 1$ i.e. FS holds in BMO (D. and Tang)

Theorem. *The necessary and sufficient condition for FS to hold in* $(H^p)^*$ *is that* $\mathcal{E}(M)$ *satisfies the reverse Hölder condition for* p*, "*R_p*", i.e. there is a constant* K *such that for every stopping time* T

$$\mathbb{E}\left[\left(\frac{\mathcal{E}(M)_\infty}{\mathcal{E}(M)_T}\right)^p \mid \mathcal{F}_T\right] \leq K.$$

We need the operator that maps

f to the element $f - \langle M, Z \rangle_\infty$

where Z is the martingale $Z_t = \mathbb{E}_{\mathbf{P}}[f \mid \mathcal{F}_t]$.

For $M \in BMO$ this operator maps H^p into L^p, where $1 \leq p \leq \infty$. (for $p = \infty$ we use BMO instead of H^∞, L^∞).

But the risk neutral measure involved is

$$d\mathbf{Q}/d\mathbf{P} = \mathcal{E}(M).$$

Basic ingredient is the following

Lemma (Yor). *Suppose* $p > 1$, *suppose* $\mathcal{E}(M)$ *satisfies* R_p. *Let* $Z \in L^\infty$, *let* $Z'_t = \mathbb{E}_{\mathbf{Q}}[Z_\infty \mid \mathcal{F}_t]$, *then* $\|\langle Z', Z' \rangle^{1/2}\|_q \leq c\|Z\|_q$.

For $p = 1$ we need the following.

Theorem.

$$\|H \cdot M\|_1 \leq c\|H \cdot M - (H \cdot M) \cdot M\|_1.$$

Proof. Based on Fefferman's inequality, stochastic analysis and BMO-theory.

$$dX_t = (X_t + g_t)\, dM_t, \quad X_0 = 0$$

Its solution is (put $M^{\mathbf{Q}} = M - \langle M, M \rangle$):

$$X_t = \mathcal{E}(M)_t \int_0^t \mathcal{E}(M)_s^{-1} g_s \, dM_s^{\mathbf{Q}}$$

How good is X if $g \cdot M \in H^p$, $1 \le p < \infty$?

Known Theorem. *The operator φ defined as $\varphi(X) = X \cdot M$ maps the H^p-martingales into H^p-martingales and this for all $1 \le p < \infty$. ($L^\infty \to BMO$)*

The theorem remains true for complex martingale spaces and $\|\varphi^n\|_{\mathbb{C}} \le 2\|\varphi^n\|_{\mathbb{R}}$, $\|\varphi^n\|_{\mathbb{R}}$ depending on p.

The dual of φ on $(H^p)^* \sim L^q$ (for $p > 1$) and $(H^1)^* = BMO$ for $p = 1$ is given by

$$\varphi^*(Z_\infty) = \langle Z, M \rangle_\infty$$

It is defined on the random variables Z_∞ and returns a random variable.

Theorem. *For $1 \leq p < \infty$, the SDE satisfies*

$\|X\|_p \leq c\|g \cdot M\|_p$ *if and only if*

(1) $Y - \varphi(Y)$ *is an isomorphism of H^p or equivalently*

(2) $Z_\infty - \langle Z, M \rangle_\infty$ *defines an isomorphism of $(H^p)^*$.*

The equation can now be written as the equation

$$X - \varphi(X) = g \cdot M.$$

Theorem. *For* $1 \leq p < \infty$, *the following are equivalent:*

1. *the SDE satisfies* $\|X\|_p \leq c\|g{\cdot}M\|_p$ *(H^p norms).*
2. $Y - \varphi(Y)$ *is an isomorphism of* H^p *or equivalently*
3. $Z_\infty - \langle Z, M\rangle_\infty$ *defines an isomorphism of* $(H^p)^*$.
4. *The FS decomposition holds in* $(H^p)^*$.
5. $\mathcal{E}(M)$ *satisfies the reverse Hölder condition* R_p.

Remark. $\mathcal{E}(M)$ satisfies R_p for some $p > 1$.

Study of the operator φ

It is better to study the operator on the complex H^p spaces. In the case of complex λ, the exponential $\mathcal{E}(\lambda M)$ is no longer a uniformly integrable martingale and the case $p = 1$ becomes similar to the case $p > 1$. The operator $Id - \lambda\varphi$, $\lambda \neq 0$ is an isomorphism if and only if λ^{-1} is not in the spectrum of φ. (seen on complex H^p space).

Theorem. *Let r_p be the spectral radius of φ on H^p. Then we have*

$$\frac{1}{b(M)}\sqrt{p} \leq r_p \leq \frac{1}{b(M)}\sqrt{p(2p-1)}$$

Theorem. *Are equivalent*

1. *for all* $\lambda \in \mathbb{C}$ *and all (some)* $1 \leq p < \infty$, $Id - \lambda\varphi$ *is an isomorphism on complex* H^p
2. *for all* $\lambda \in \mathbb{C}$ *and all (some)* $1 \leq p < \infty$ *there is* K *so that for all stopping times* $T \leq \tau$:

$$\mathbb{E}\left[\left|\frac{\mathcal{E}(\lambda M)_\tau}{\mathcal{E}(\lambda M)_T}\right|^p \mid \mathcal{F}_T\right] \leq K.$$

3. $b(M) = \infty$
4. φ *is quasi-nilpotent on all (some)* H^p *(*$1 \leq p < \infty$*), i.e.* $r_p = 0$
5. $\lim \|\varphi^n\|^{1/n} = 0$ *on all (some)* H^p *(*$1 \leq p < \infty$*).*