Robust Replication of Default Contingent Claims

PETER CARR

Head of Quantitative Financial Research, Bloomberg LP, New York
Director of the Masters Program in Math Finance, Courant Institute, NYU

JOINT WORK WITH BJORN FLESAKER, BLOOMBERG

File reference: C:/Texfiles/1Research/RRDCC/RRDCC40minoh.tex

College Park, Md *September 29th, 2006*

Introduction

- On Tuesday September 19, 2006, Bloomberg News reported that the global credit derivatives market has more than doubled in the past year to $26 trillion.

- On August 31, 2006, the Wall Street Journal reported that the underlying notional in credit default swaps (CDS) exceeds $17 trillion.

- As the saying goes, a trillion here, a trillion there, and pretty soon we'll be talking about real money.

Overview

- We show how to replicate the payoffs to a large class of default-contingent claims by taking static positions in a continuum of CDS's of different maturities.

- Although we assume deterministic interest rates and a constant recovery rate on the bond underlying the CDS's, the replication is otherwise robust in that we make no assumptions on how default occurs.

- In particular, we can robustly replicate Heaviside and Dirac payoffs written on the random default time τ.

- As a consequence, we can robustly determine risk-neutral survival probabilities from an arbitrarily given forward rate curve and CDS curve.

Is This New?

- It is frequently asserted that risk-neutral default probabilities can be stripped from the single name CDS market.

- We find that it cannot be done in general; in particular correlated interest and/or recovery rates prevent identification.

- Even under deterministic interest and/or recovery rates, we are unable to find references on replication (as opposed to pricing) and would appreciate them.

Our Mission

- Suppose that the default time τ is random and unspecified.

- Let the target be a claim paying a coupon $c(t)dt$ for each $t \in [0, \tau \wedge T]$. If $\tau < T$, then the target claim also pays a lump sum recovery amount $R(\tau)$ at time τ, where the recovery function $R(t), t \in [0, T]$ is known.

- The objective is to replicate the payoffs to the target claim by taking static positions in a continuum of CDS of all maturities.

- One can also manage a position in a riskfree asset which we call a bank balance. No other assets are used in the replication. As the CDS position is static, the bank balance is determined by what we initially deposit and what goes out and in to finance the cash flows associated with the continuum of CDS positions across maturities.

Our Initial Data

- Although we can handle deterministic recovery from the CDS, this presentation will assume for simplicity that the recovery rate on the bond underlying the CDS is constant.

- It follows that the loss given default is also constant at some $L \in (0, 1]$.

- As we have deterministic interest rates, future short rates are given by the initial forward rate curve $r(t) \equiv f_0(t), t \in [0, T]$, which we assume is C^1.

- In contrast, future CDS rates are stochastic and unmodelled, but we know the initial CDS curve $s_0(u), u \in [0, T]$, which we assume is C^1.

- Our positions in CDS and cash can depend on the initial interest rate curve $r(t), t \in [0, T]$, the initial CDS curve $s_0(u), u \in [0, T]$, and the loss given default $L \in (0, 1]$. No other initial data is needed.

Overview of the Solution Strategy

- Recall how replication occurs in the single period binomial model. One forms a portfolio of two assets where the initial holdings cannot depend on whether the underlying goes up or down.

- To find the two initial holdings, one obtains two linear equations via target matching in the up and down state. One then solves the two equations for the two unknowns.

- Replication in the single period binomial model is static; It only becomes dynamic when there are two or more periods.

Overview of the Solution Strategy (con'd)

- Our credit problem is similar to the single period binomial model in that all of our decisions must be made at time 0.

- We also have two controls, namely:

  1. $M(t)$: the amount of money kept in the money market account if we have survived to time $t \in [0, T]$,

  2. $Q(u)du$: the notional amount of CDS to initially *write* for maturity $u \in [0, T]$.

- The major difference is that the two controls in the credit problem are functions on $[0, T]$. Nonetheless, for each given $t \in [0, T]$, we either have that $\tau = t$ or $\tau \neq t$ and we now explore target matching in these two states.

Matching Recovery if t is the Default Time

- The two control functions $M(t)$ and $Q(t)$ are uniquely determined by two equations. One of these equations arises from equating the portfolio value to the target's recovery value when the candidate time t is the default time:

$$M(t) - L \int_t^T Q(u) du = R(t), \qquad t \in [0, T].$$

- Differentiating w.r.t. t and solving for $Q(t)$ implies:

$$Q(t) = \frac{1}{L} [R'(t) - M'(t)], \qquad t \in [0, T].$$

- Since the loss given default $L \in (0, 1]$ of the CDS and the recovery function $R(t), t \in [0, T]$ of the target are both given, the rate $Q(u)$ at which CDS are written is determined, once we determine the survival-contingent bank balance $M(t)$.

Matching Coupon if t is not the Default Time

- Recall that the possibility that $\tau = t \Rightarrow Q(t) = \frac{1}{L}[R'(t) - M'(t)], t \in [0, T]$.

- The possibility that $\tau \neq t \Rightarrow$ that the *change* in $M(t)$ at t is:

$$M'(t) = r(t)M(t) + \int_t^T s_0(u)Q(u)du - c(t), \qquad t \in [0, T].$$

- Differentiating w.r.t. t implies:

$$M''(t) = r(t)M'(t) + r'(t)M(t) - s_0(t)Q(t) - c'(t), \qquad t \in [0, T].$$

- Substituting in the top equation and re-arranging implies that $M(t)$ solves:

$$\mathcal{L}M \equiv M''(t) - \left[r(t) + \frac{s_0(t)}{L} \right] M'(t) - r'(t)M(t) = f(t), \qquad t \in [0, T],$$

where the forcing function $f(t)$ is given by:

$$f(t) \equiv -c'(t) - \frac{s_0(t)}{L}R'(t), \qquad t \in [0, T].$$

- Recall that the bank balance $M(t)$ solves the 2nd order linear ODE:

$$\mathcal{L}M \equiv M''(t) - \left[r(t) + \frac{s_0(t)}{L} \right] M'(t) - r'(t)M(t) = f(t), \qquad t \in [0,T],$$

  where the forcing function $f(t) \equiv -c'(t) - \frac{s_0(t)}{L}R'(t), t \in [0,T]$.

- A unique solution for $M(t), t \in (0,T)$ arises from 2 terminal conditions:

$$M(T) = 0, \text{ and } \lim_{t \uparrow T} M'(t) = -c(T).$$

- For arbitrary forward rate curves $r(t)$ and CDS curves $s_0(t)$, one numerically solves the above terminal value problem for $M(t), t \in [0,T)$.

- The scalar $M(0)$ is the initial value of the target claim because all of the CDS used in the replicating portfolio are initially costless.

- Recall again that the rate at which CDS are initially written for each maturity $t \in (0,T)$ is given by $Q(t) = \frac{1}{L}[R'(t) - M'(t)], t \in [0,T)$.

- Thus, we have a complete solution to the problem of replicating a target claim using a money market account and CDS of all maturities up to T.

Example: Defaultable Annuity

- Suppose that the target claim is a defaultable annuity of maturity T.

- Let $M_a(t;T)$ denote the survival-contingent bank balance at $t \in [0,T]$.

- Setting $c(t) = 1_{t \in (0,T)}$ and $R(t) \equiv 0$ in the forcing term $f(t) \equiv -c'(t) - \frac{s_0(t)}{L} R'(t)$ implies that $M_a(t;T)$ solves the homogeneous ODE:

$$\frac{\partial^2}{\partial t^2} M_a(t;T) - \left[r(t) + \frac{s_0(t)}{L} \right] \frac{\partial}{\partial t} M_a(t;T) - r'(t) M_a(t;T) = 0,$$

on the open interval $t \in (0,T)$ s.t. $M_a(T;T) = 0, \lim_{t \uparrow T} \frac{\partial}{\partial t} M_a(t;T) = -1$.

- $M_a(0;T)$ is the initial value of the defaultable annuity maturing at T.

- For each maturity $t \in [0,T]$, CDS are initially written at the rate $Q_a(t;T) \equiv -\frac{\partial}{\partial t} \frac{M_a(t;T)}{L}$, where recall $L \in (0,1]$ is the loss given default.

Green's Function

- Again consider the terminal value problem $\mathcal{L}M(t) = f(t), t \in [0, T]$.

- The diff'l operator $\mathcal{L}$ acts on the survival cont't bank balance $M(t)$ that arises when replicating a claim with coupon rate $c(t)$ and recovery function $R(t)$ entering through the forcing term $f(t) \equiv -c'(t) - \frac{s_0(t)}{L}R'(t)$.

- The general solution of the terminal value problem can be expressed in terms of the Green's function $g(t; u)$, defined as a function solving:

$$\mathcal{L}g(t; u) = \delta(t - u), \qquad t, u \in [0, T].$$

Since $\mathcal{L}$ is 2nd order, we need 2 boundary conditions to uniquely determine a Green's function. We choose $g(T; u) = 0$, and $\lim_{t \uparrow T} \frac{\partial}{\partial t} g(t; u) = 0$, causing $g(t; u)$ to vanish for $t \in (u, T)$.

- Note that Green's ODE arises from the general ODE by setting $c(t) = 1_{t \in (0, u)}$ and $R(t) \equiv 0$ in $f(t)$.

Green's Function

- The last slide showed the Green's function $g(t;u)$ is connected with the replication of a defaultable annuity maturing at u.

- In fact, a solution to the terminal value problem governing $g(t;u)$ is:
$$g(t;u) = \begin{cases} M_a(t;u) & \text{if } t \in [0,u] \\ 0 & \text{if } t \in (u,T). \end{cases}$$

- Thus, for fixed u, $g(t;u)$ is the survival-contingent bank balance at $t \in [0,u]$ when replicating the u maturity defaultable annuity.

- Once $g(t;u)$ is known as a function of u for some t, then the survival-contingent bank balance which arises when replicating an arbitrary default-contingent claim can be expressed in terms of it:
$$M(t) = \int_t^T f(u)g(t;u)du, \qquad t \in [0,T].$$

14

More Goodies

- The last slide showed that the survival-contingent bank balance which arises when replicating an arbitrary default-contingent claim can be expressed in terms of the Green's function $g(t;u)$. It follows that the rate $Q(t)$ at which CDS are written at maturity t also depends on g:

$$Q(t) = \frac{1}{L}\left[R'(t) - \int_t^T f(u)\frac{\partial}{\partial t}g(t;u)du\right], \qquad t \in [0,T].$$

- Recall that all of the CDS used in the replicating portfolio are initially costless. Hence, evaluating the expression for the bank balance M at $t = 0$ relates the initial value of the target claim to the Green's function:

$$M(0) = \int_0^T f(u)g(0;u)du, \qquad t \in [0,T].$$

- Thus, hedging and pricing of the target reduce to determining $g(0;u), u \in [0,T]$. This looks to be computationally intense, but why be backward?

Forward Equation for the Green's Function

- Recall that when the Green's function $g(t;u)$ is considered as a function of t for fixed u, it solves the ODE $\mathcal{L}g(t;u) = \delta(t-u)$, for $t,u \in [0,T]$ along with two terminal conditions. But when g is considered as a function of its second variable u, then it satisfies the adjoint ODE:

$$\mathcal{L}^* g(t;u) = \delta(t-u),$$

where $\mathcal{L}^*$ is the following linear differential operator:

$$\mathcal{L}^* \equiv \mathcal{D}_u^2 + \left[r(u) + \frac{s_0(u)}{L} \right] \mathcal{D}_u^1 + \frac{s_0'(u)}{L} \mathcal{D}_u^0,$$

and $\mathcal{D}_u$ denotes differentiation w.r.t. u.

- The domain for the adjoint ODE is also the square $(t,u) \in [0,T] \times [0,T]$. In one triangle $t > u$, g vanishes and in the other $t < u$, g is the bank balance when replicating a defaultable annuity.

Forward Equation for Defaultable Annuity Value

- Recall once again that when replicating a defaultable annuity, the survival-contingent bank balance $M_a(t;u)$ that arises is the Green's function $g(t;u)$ for $t \in [0,u]$.

- The results of the last slide imply that M_a solves a forward ODE in its maturity variable u.

- Let $A_0(u) \equiv M_a(0;u)$ denote the initial value of the defaultable annuity maturing at $u \in [0,T]$. It follows that $A_0(u)$ also solves a forward ODE:

$$A_0''(u) + \left[r(u) + \frac{s_0(u)}{L} \right] A_0'(u) + \frac{s_0'(u)}{L} A_0(u) = 0,$$

on the domain $u \in (0,T)$, subject to the initial conditions:

$$A_0(0) = 0, \text{ and } \lim_{u \downarrow 0} A_0'(u) = 1.$$

Determining the Default Time PDF

- Suppose that the default time τ has a probability density function (PDF). Can we learn it from our knowledge of $A_0(u)$ and our initial data?

- The value of the premium leg of a CDS maturing at u is $s_0(u)A_0(u)$.

- By the def'n of $s_0(u)$, $s_0(u)A_0(u)$ is also the value of the protection leg.

- Dividing this value by L produces $\frac{s_0(u)A_0(u)}{L}$, which is the term structure of initial values for a claim paying \$1 at the default time τ if $\tau < u$ and \$0 otherwise.

- Differentiating w.r.t. u implies that $\frac{\partial}{\partial u}\frac{s_0(u)A_0(u)}{L}$ is the spot value of a claim with the Dirac payoff $\delta(\tau - u)$ at u. Future valuing this spot value gives the default time PDF:

$$\mathbb{Q}\{\tau \in du\} = e^{\int_0^u r(v)dv}\frac{\partial}{\partial u}\frac{s_0(u)A_0(u)}{L}.$$

Determining the Default Time PDF Again

- Recall that $A_0(u)$ denotes the initial value of defaultable annuity, which can be obtained by solving an initial value problem.

- So, $A_0'(u)$ is the spot value of a survival claim, i.e. a claim that pays \$1 at u if $\tau \geq u$. Thus, the forward price $e^{\int_0^u r(v)dv} A_0'(u) = \mathbb{Q}\{\tau \geq u\}$.

- Differentiating both sides w.r.t. u and negating implies that:
$$\mathbb{Q}\{\tau \in du\} = -r(u)e^{\int_0^u r(v)dv} A_0'(u) - e^{\int_0^u r(v)dv} A_0''(u).$$

- But recall from the last slide that $\mathbb{Q}\{\tau \in du\} = e^{\int_0^u r(v)dv} \frac{\partial}{\partial u} \frac{s_0(u)A_0(u)}{L}$.

- Equating the 2 RHS's yields the forward ODE governing $A_0(u)$:
$$A_0''(u) + \left[r(u) + \frac{s_0(u)}{L}\right] A_0'(u) + \frac{s_0'(u)}{L} A_0(u) = 0, \qquad u \in [0, T].$$
Imposing $A_0(0) = 0$, and $\lim_{u \downarrow 0} A_0'(u) = 1$ uniquely determines $A_0(u)$.

Pricing Claims on the Default Time

- The last two slides amount to a *direct* derivation of the initial value problem governing the initial value of the defaultable annuity.

- Once $A_0(u)$ is known, either of the equations for the risk-neutral PDF of τ. given on the last slide can be used to determine it.

- It follows that any claim that pays $f(\tau)$ at time τ can be uniquely priced relative to the initial yield curve and the initial CDS curve.

- For example, one can price the payoff $(\tau - K)^+$ paid at τ.

- Our results imply that that any claim written on the default time can also be hedged.

Similarities to Breeden & Litzenberger (1978)

- Again recall the 2 formulas linking the default time PDF to $A_0(u)$:

$$\mathbb{Q}\{\tau \in du\} = e^{\int_0^u r(v)dv} \frac{\partial}{\partial u} \frac{s_0(u)A_0(u)}{L}.$$

$$\mathbb{Q}\{\tau \in du\} = -r(u)e^{\int_0^u r(v)dv}A_0'(u) - e^{\int_0^u r(v)dv}A_0''(u).$$

- These are the analogs to BL's result: $\mathbb{Q}\{s_u \in dK\} = e^{\int_0^u r(v)dv} \frac{\partial^2}{\partial u^2} C_0(K, u)$.

- All 3 results link the RN PDF to partial derivatives of observables.

- While we assumed deterministic interest rates and a constant recovery rate on the CDS, our determination of the default time PDF is otherwise robust in that no assumptions were made on how default occurs.

- Inversely, starting from known $\mathbb{Q}\{\tau \in du\}$ and $r(v), v \in [0, u]$, one can explictly solve the 2nd equation for the defaultable annuity price $A_0(u)$, and then explictly solve the top equation for the CDS spread $s_0(u)$.

Differences from Breeden & Litzenberger (1978)

- When European options trade outright (eg. for stock indices or FX), then BL's call prices are directly observable, while our defaultable annuity values $A_0(u)$ only become observable once an initial value problem is solved.

- While our result is not presently as explicit as the BL result, we show that our results can be made completely explicit if either the CDS curve or yield curve is initially flat.

- The paper also explicitly shows how to hedge and price:

  1. unit recovery claims: i.e. a claim paying \$1 at τ if $\tau < T$ and \$0 otherwise.
  2. survival claims: a claim paying \$1 at T if $\tau \geq T$ and \$0 otherwise.

Summary

- We showed how to replicate the payoffs to a large class of default-contingent claims by taking static positions in a continuum of CDS of different maturities.

- Although we assumed deterministic interest rates and a constant recovery rate on the CDS, the replication was otherwise robust in that we made no assumptions on how default occurs. In particular, our results are consistent with both reduced form and structural models of default.

- Our results imply that we can robustly replicate Heaviside and Dirac payoffs on the default time τ.

- As a consequence, we were able to robustly determine risk-neutral survival probabilities from an arbitrarily given forward rate curve and CDS curve. The extension to correlated recovery and/or interest rates is open.