Weakly Rearrangement Invariant Spaces And Approximation By Largest Element

Björn Jawerth

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Acknowledgement

Joint work with Mario Milman

BJ and Mario Milman, “ Weakly rearrangment invariant spaces and approximation by largest elements”

[HJLY94] C. Hsiao, BJ, B. Lucier, and M. X. Yu, “Near optimal compression of orthonormal wavelet expansions”

X (quasi-)normed sequence space over I

$x \in X \quad x = \{x_i\}_{i \in I} \quad I$ countable

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Problem. *Characterize X for which $x^{0,N}$ of the N largest elements of x is near optimal:*

$$\sigma_N(x) \approx \|x - x^{0,N}\|_X, \quad N \ge 0$$

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Definition. X ***weakly rearrangement invariant*** *if*

(WRI) $\phi_X(\Omega_1) \le C\phi_X(\Omega_2)$ *whenever* $\#\Omega_1 \le \#\Omega_2$.

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(i) X is a weakly rearrangement invariant space
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$$\sigma_N(x) = \inf_{\#\Omega \leq N} \|x - x^{0,\Omega}\|_X ,$$

$$x^{0,\Omega} = \{x_i^{0,\Omega}\}_{i \in I} \qquad x_i^{0,\Omega} = \begin{cases} x_i & \text{if } i \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

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For some Ω_N with $\#\Omega_N \leq N$

$$x^{0,\Omega_N} = x^{0,N}.$$

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$$\|x - x^{\Omega_N}\|_X = \|\{x_i\}_{i \in \Omega_N^c}\|_X = \|\{x_i\}_{i \in \Omega_N^c \cap \Omega^c} + \{x_i\}_{i \in \Omega_N^c \cap \Omega}\|_X$$

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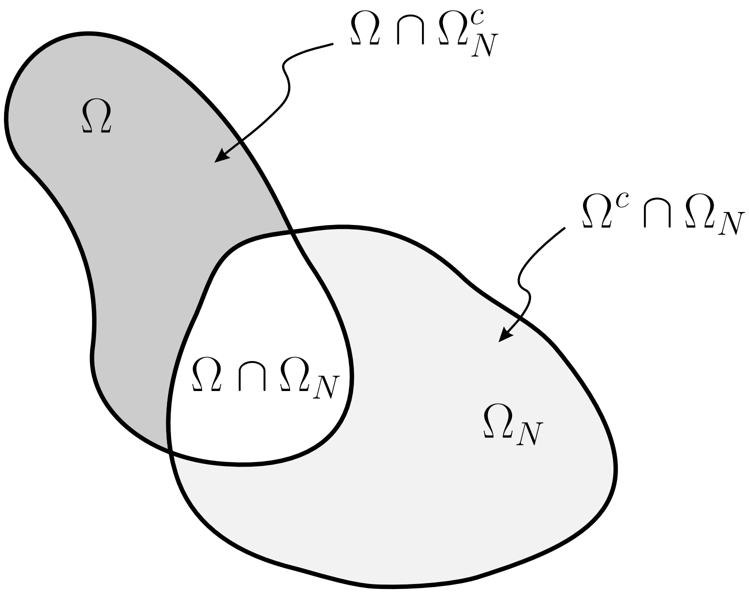

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(i) $\Rightarrow$ (ii): Argue by contraction. $\square$

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Some Examples

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Function Spaces

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"discrete Triebel-Lizorkin spaces"

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Via wavelet transform (for example)

$\dot{f}_p^{\alpha,q} \approx \dot{F}_p^{\alpha,q}$

$\dot{F}_p^{\alpha,2} \approx H_p^{\alpha}$ Sobolev spaces order α

$\dot{F}_p^{0,2} \approx H_p$ Hardy spaces $0 < p < +\infty$

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$$G^{\alpha,q}(1_\Omega)(x) = \Big(\sum_{Q\in\Omega} (|Q|^{-1/p}\chi_Q(x))^q\Big)^{1/q}.$$

Claim. $G^{\alpha,q_0}(1_\Omega)(x) \approx G^{\alpha,q_1}(1_\Omega)(x), \quad 0 < q_0, q_1 \le +\infty$

Corollary (HJLY94). $\dot{f}_p^{\alpha,q}$ *is weakly rearrangement invariant and* $\|1_\Omega\|_{\dot{f}_p^{\alpha,q}} \approx \|1_\Omega\|_{\dot{f}_p^{\alpha,p}} = \#\Omega^{1/p}$.

Proof of claim: Q_x smallest cube containing x, $l(Q_x) = 2^{-\nu_x}$

$$G^{\alpha,q}(1_\Omega)(x) = \Big(\sum_{Q\in\Omega} (|Q|^{-1/p}\chi_Q(x))^q\Big)^{1/q} \le \Big(\sum_{\nu\le\nu_x} 2^{qn\nu/p}\Big)^{1/q}$$
$$\le C|Q_x|^{-1/p} \le CG^{\alpha,\infty}(1_\Omega)(x).$$

□

Corollary (HJLY94). L_p, $1 < p < +\infty$ *is isomorphic to* $\dot{f}_p^{\alpha,2}$, $\alpha/d = 1/p - 1/2$ *with*

$$l_{p,\min(p,2)} \subset \dot{f}_p^{\alpha,2} \subset l_{p,\max(p,2)}$$

Conversely, if x is rearrangement invariant space based on countably infinite measure space, normalized so that the mass of each point is one, and L_p is isomorphic to x, then

$$l_{p,\min(p,2)} \subset x \subset l_{p,\max(p,2)}.$$

Balance two competing objectives – error/cost

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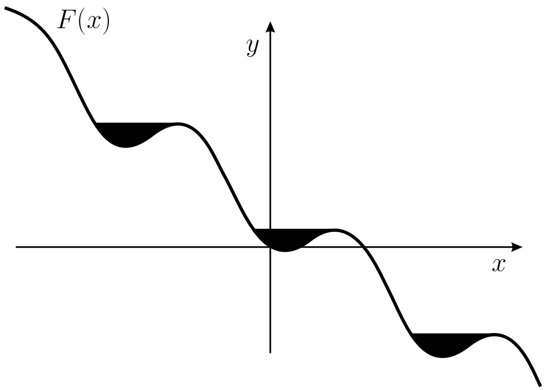

F. Riesz's Rising Sun Lemma

$$F(x) = \int_{-\infty}^{x} f(u)du - tx$$

$$F(a_i) = F(b_i) \Leftrightarrow \int_{-\infty}^{a_i} f(u)du - ta_i = \int_{-\infty}^{b_i} f(u)du - tb_i$$

$$\Leftrightarrow \frac{1}{b_i - a_i} \int_{a_i}^{b_i} f(u)du = t$$

Balance two competing objectives – error/cost

Peetre's K-functional

$$K(t, f; X_0, X_1) = \inf_{f=f_0+f_1} (\|f_0\|_{X_0} + t\|f_1\|_{X_1})$$

Balance two competing objectives – error/cost

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Best approximation functional

$$E(t, f; X_0, X_1) = \inf_{\|f_0\| \leq t} \|f - f_0\|_{X_1}$$

Balance two competing objectives – error/cost

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Best approximation functional

$$E(t, f; X_0, X_1) = \inf_{\|f_0\|\leq t} \|f - f_0\|_{X_1}$$

$$K(t, f; X_0, X_1) = \sup_{s} (s + tE(s, f; X_0, X_1))$$

The 0/1 Knapsack Problem

$O = \{o_j\}_{j=1}^{J}$ objects

$P = \{p_j\}_{j=1}^{J}$ profits

$C = \{c_j\}_{j=1}^{J}$ sizes

$p_i = p_j(o_j) \in \mathbb{N}$ and $c_j = c_j(o_j) \in \mathbb{N}$

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$p_i = p_j(o_j) \in \mathbb{N}$ and $c_j = c_j(o_j) \in \mathbb{N}$

(KP) $\max \left\{ \sum_{j=1}^{J} p_j x_j : \sum_{j=1}^{J} c_j x_j \leq t, \quad x_j \in \{0,1\}, j = 1, \ldots, J \right\}$ $t > 0$ fixed

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(KP) $$\max\left\{\sum_{j=1}^{J} p_j x_j : \sum_{j=1}^{J} c_j x_j \leq t, \quad x_j \in \{0,1\}, j = 1, \ldots, J\right\}$$ $t > 0$ fixed

$$p_{\max} = \sum_{j=1}^{J} p_j \qquad \sum_{j=1}^{J} p_j x_j = p_{\max} - \sum_{j=1}^{J} p_j (1 - x_j)$$

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$$p_{\max} = \sum_{j=1}^{J} p_j \qquad \sum_{j=1}^{J} p_j x_j = p_{\max} - \sum_{j=1}^{J} p_j (1 - x_j)$$

(KP) $\Longleftrightarrow E(t, 1; l^0(C), l^1(P)) = \inf_{\|x\|_{l^0(C)} \leq t} \|1 - x\|_{l^1(P)}$

General Knapsack Problem

$N \geq 0, i \in I$

$$E_i(N) \geq 0$$

$$C_i(N) \geq 0, \quad C_i(0) = 0$$

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$$E(\bar{N}) = \sum_{i \in I} E_i(N_i)$$

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General Knapsack Problem

$$\mathcal{E}(t) = \mathcal{E}(t; C, E) = \inf_{C(\bar{N}) \leq t} E(\bar{N})$$

$$= \inf_{\bar{N} = \{N_i\}} \left\{ \sum_{i \in I} E_i(N_i) : \sum_{i \in I} C_i(N_i) \leq t \right\}.$$

$$\delta E_i(N) = E_i(N+1) - E_i(N), \quad N \geq 0, i \in I;$$

$$\delta C_i(N) = C_i(N+1) - C_i(N), \quad N \geq 0, i \in I;$$

$$\left(\frac{\delta E_i}{\delta C_i}\right)(N) = \frac{\delta E_i(N)}{\delta C_i(N)}, \quad N \geq 0, i \in I.$$

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$$\Big(\frac{\delta E_i}{\delta C_i}\Big)(N) = \frac{\delta E_i(N)}{\delta C_i(N)}, \quad N \geq 0, i \in I.$$

Theorem. *Suppose* $\{E_i\}_{i\in I}$ *is a sequence of non-increasing, convex functions, and suppose* $\{C_i\}_{i\in I}$ *is a sequence of non-decreasing, convex functions. Then*

$$\mathcal{E}(t; C, f) = E\Big(t, \{-\frac{\delta E_i}{\delta C_i}(N)\}_{i\in I, N\in \mathbf{N}}; \ell^0(\delta C), \ell^1(\delta C)\Big)$$
$$= E\Big(t, 1; \ell^0(\delta C), \ell^1(-\delta E)\Big),$$

where 1 *denotes the constant sequence* $1 = \{1\}_{i\in I, N\in\mathbb{N}}$.

Non-convex: through Gagliardo discretization