Fourier Reconstruction from Non-Uniform Spectral Data

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1 Introduction
   ■ Magnetic Resonance Imaging
   ■ Sampling Patterns in MR Imaging
   ■ Challenges in Cartesian Reconstruction
     ■ Spectral Reprojection

2 The Non-Uniform Data Problem
   ■ Problem Formulation
   ■ The Non-harmonic Kernel
   ■ Reconstruction Results using the Non-harmonic Kernel

3 Current Methods
   ■ Reconstruction Methods
   ■ Error Characteristics

4 Alternate Approaches
   ■ Spectral reprojection
   ■ Incorporating Edge Information
   ■ Spectral reprojection for Fourier Frames
MR Imaging Process

- An acquired MR signal can be written as (2D slice)

\[ S(k_x, k_y) = \int \int \rho(x, y) e^{-i(k_x x + k_y y)} dx \, dy \]

\( \rho(x, y) \) is a measure of the concentration of MR relevant nuclei (spins) and

\[ k_x = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau, \quad k_y = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau \]

- \( G_x \) and \( G_y \) are the applied gradient fields
- We denote the signal acquisition space \( k = (k_x, k_y) \) as “k-space”
Sampling Patterns

(c) Cartesian Sampling

(d) Non-Cartesian Sampling – Spiral Imaging

Figure: MR Imaging Sampling Patterns

1Spiral sampling pattern courtesy Dr. Jim Pipe, Barrow Neurological Institute, Phoenix, Arizona
# Advantages and Disadvantages of each Pattern

## Cartesian Imaging

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The Gibbs Phenomenon

- affects the reconstruction of piecewise-smooth functions.
- occurs when data is sampled uniformly or non-uniformly.
- Two important consequences:
  - Gibbs ringing artifact – presence of non-physical oscillations in the vicinity of discontinuities.
  - Reduced rate of convergence to first order even in smooth regions of the reconstruction.
- Why is this important?
  - Oscillations cause post-processing problems in tasks like segmentation, edge detection etc. Filtering oscillations may cause loss of important information.
  - The reduced order of convergence means more Fourier coefficients are needed to obtain a good reconstruction.
The Gibbs Phenomenon – An Example

\[ S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{ikx}, \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx \]

**Figure:** Gibbs Phenomenon, \( N = 32 \)
The Gibbs Phenomenon – An Example

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Figure: Gibbs Phenomenon

(a) Reconstruction

(b) Error does not go away by increasing \( N \)
Shepp Logan MRI
Reconstruction from Fourier Data by Spectral Reprojection (Gottlieb, Shu et. al.)

- Assume that Fourier coefficients \( \{ \hat{f}_k \}_{k=-N}^{N} \) are given for a piecewise analytic function \( f(x) \) in \([-1, 1]\).
- Use edge detection to determine sub-intervals of smoothness \([a, b]\).
- Compute the Fourier partial sum in \([a, b]\): \( S_N f(x) = \sum_{k=-N}^{N} \hat{f}_k e^{i k \pi x} \).
- Reproject \( S_N f(x) \) onto a new basis for \( x \in [a, b] \):
  \[
P_M(S_N f(x)) \rightarrow f(x)
  \]
- Many other algorithms available (Banerjee & Geer, Driscoll & Fornberg, Eckhoff, Jung & Shizgal, Solomonoff, Tadmor & Tanner, more ...)

Anne Gelb
Fourier Reconstruction from Non-Uniform Spectral Data
Goal: approximate a function that is smooth but *not periodic* for $\xi \in [-1, 1]$ (after linear transformation from $x \in [a, b]$)

We *could* use an orthogonal polynomial expansion,

$$P_M f(x(\xi)) = \sum_{l=0}^{M} a_l \psi_l(\xi), \quad a_l = \frac{1}{\gamma_l} < f, \psi_l >$$

but we *have* the Fourier expansion

$$S_N f(x(\xi)) = \sum_{k=-N}^{N} \hat{f}_k e^{ik\pi x(\xi)}$$

Can we use $S_N f$ to approximate $a_l$? What should $P_M$ look like?
Spectral Reprojection (Gottlieb & Shu)

- Determine appropriate orthogonal polynomial basis \( \{ \psi_l(\xi) \}_{l=0}^M \) on \([-1, 1]\).
- Build Fourier approximation inside smooth region \([a, b]\):
  \[
  S_N f(x(\xi)) = \sum_{k=-N}^{N} \hat{f}_k e^{ik\pi x(\xi)}, \quad x(\xi) = \epsilon \xi + \delta
  \]
- Expansion coefficients for \( \psi_l(\xi) \) are approximated by
  \[
  b_l = \frac{1}{\gamma_l} < S_n f, \psi_l > \approx a_l
  \]
- Reprojection in smooth region \([a, b]\):
  \[
  P_m(S_N f(x(\xi))) = \sum_{l=0}^{m} b_l \psi_l(\xi)
  \]
Definition: A **Gibbs Complementary Reprojection Basis**, $\{\psi_i\}_{i=0}^{N}$, has the properties:

1. For an analytic function on $[-1, 1]$, the function’s expansion in the orthogonal reprojection basis is exponentially convergent, $P_m f(x) \to f(x)$ exponentially for smooth $f(x)$.

2. The projection of the high modes in the original basis on the low modes in the new basis is exponentially small.

If these conditions are met, then

$$P_m(S_N f(x(\xi))) \to f(x) \text{ exponentially for } x \in [a, b]$$
Error Analysis For Spectral Reprojection

\[ \text{Err}(m, N, f, \omega) := \|f - P_m(S_N f)\| \leq \|f - P_m f\| + \|P_m(f - S_N f)\| \]

\[ = \text{Trunc}(m, f, \omega) + \text{Proj}(m, N, f, \omega) \]

- \( \text{Trunc}(m, f, \omega) \) measures the convergence properties of the reprojection basis for \( m \) expansion coefficients (reflects first Gibbs complementary basis property)
  - Converges exponentially for \( \omega(x) \geq 0 \)
  - \( m = \beta N \) expansion terms, \( 0 < \beta < 1 \), resolves the function

- \( \text{Proj}(m, N, f, \omega) \) measures the near orthogonality of the reprojection space \( P_m \) and the space containing the information about the function that is not known, \( I - S_N \) (reflects second Gibbs complementary basis property)
The projection error corresponds to the decay rate of the coefficients $b_l$:

$$
\text{Proj}(m, N, f, \omega) = P_m(f - S_N f)
= \sum_{l=0}^{m} \psi_l(\xi) \frac{1}{\gamma_l} \langle f - S_N f, \psi_l \rangle_\omega
= \sum_{l=0}^{m} \psi_l(\xi) \frac{1}{\gamma_l} \int_{-1}^{1} \omega(y) \psi_l(y) (f(x(y)) - S_N f(x(y))) dy
= \sum_{l=0}^{m} \frac{1}{\gamma_l} \sum_{|k| > N} \hat{f}_k \psi_l(\xi) \int_{-1}^{1} e^{i\pi k x(y)} \omega(y) \psi_l(y) dy$$

- Error will be small if the weight function $\omega$ is chosen so that the corresponding integral is very small.
Define the weight function $\omega$ as: $\omega_\lambda(\xi) = (1 - \xi^2)^{\lambda - \frac{1}{2}}$. Large $\lambda$ ensures that $\int_{-1}^{1} e^{i \pi k x(y)} \omega(y) \psi_l(y) dy$ is small. Subsequently, the errors from the boundaries do not enter the approximation.

Corresponding reprojection basis, $\psi_l(\xi)$ for $\omega_\lambda(\xi)$, are the Gegenbauer polynomials, $C_\lambda^l(\xi)$ with $\langle C_\lambda^l, C_\lambda^m \rangle _{\omega_\lambda} = \frac{1}{\gamma_l} \delta_{lm}$.

Note that Chebyshev ($\lambda = 0$) and Legendre ($\lambda = \frac{1}{2}$) do not make good reprojection bases.
If $\lambda = \lambda(N)$ then the reprojection coefficients

$$b_l := \frac{1}{\gamma_l} \int_{-1}^{1} S_N f(x(\xi)) C_l^\lambda(\xi)(1 - \xi^2)^{\lambda(N) - \frac{1}{2}} d\xi,$$

decay exponentially.

- Can rewrite implementation to avoid quadrature (use FFT)
- For many imaging applications, small $m$ and $\lambda$ work well
- Robust with respect to noise in imaging data (Archibald & Gelb)
Shepp Logan Phantom (Archibald & Gelb)
Improvement of image quality for segmentation (Archibald, Chen, Gelb & Renaut)

Gray matter segmented probability maps
- 256 × 256 randomly generated MNI digital brain
- 9% noise level
- non-uniform tissue intensity
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Problem Formulation

- Given these coefficients, can we/how do we reconstruct the function?
- What accuracy can we achieve given a finite (usually small) number of coefficients?
- Computational issue – no FFT available
Problem Formulation

(c) Template Function

(d) Fourier Coefficients, $N = 32$

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Let $f$ be defined on $\mathbb{R}$ and supported in $(-\pi, \pi)$ with Fourier transform

$$\hat{f}(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i\omega x} \, dx, \quad \omega \in \mathbb{R}$$

**Objective**

Recover $f$ given a finite number of its non-harmonic Fourier coefficients,

$$\hat{f}(\omega_k), \quad k = -N, \ldots, N \quad \omega_k \text{ not necessarily } \in \mathbb{Z}$$

- We are particularly interested in sampling patterns with variable sampling density
- The underlying function $f$ is piecewise-smooth
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Sampling Patterns

- **Jittered Sampling**: \( \omega_k = k \pm \tau_k, \quad \tau_k \sim U[0, \theta], \quad k = -N, -(N - 1), \ldots, N \)
- **Log Sampling**: \(|\omega_k|\) is logarithmically distributed between \(10^{-v}\) and \(N\), with \(v > 0\) and \(2N + 1\) being the total number of samples.

**Figure**: Non-uniform Sampling Schemes (right half plane), \(N = 16\)
The Non-harmonic Reconstruction Kernel

- **Standard (harmonic) Fourier reconstruction:**
  \[ S_N f(x) = \sum_{|k| \leq N} \hat{f}(k)e^{ikx} = (f \ast D_N)(x) \]  
  where \( D_N(x) = \sum_{|k| \leq N} e^{ikx} \) is the Dirichlet kernel.

- The non-harmonic Fourier reconstruction:
  \[ S_N \tilde{f}(x) = \sum_{|k| \leq N} \hat{f}(\omega_k)e^{i\omega_kx} = (f \ast A_N)(x) \]  
  where \( A_N(x) = \sum_{|k| \leq N} e^{i\omega_kx} \) is the non-harmonic kernel.

- The non-harmonic kernels do not constitute an orthogonal basis for span \( \{e^{ikx}, |k| \leq N\} \).

**Figure:** The Dirichlet Kernel plotted on the right half plane, \( N = 64 \).
The Non-harmonic Reconstruction Kernel

- Standard (harmonic) Fourier reconstruction:
  \[ S_N f(x) = \sum_{|k| \leq N} \hat{f}(k)e^{ikx} = (f * D_N)(x) \]
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**Figure:** Non-harmonic Kernel, \( N = 64 \)
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Figure: Autocorrelation plot of the kernels
Non-harmonic Kernels

(a) Dirichlet, $N = 32$

(b) Jittered, $N = 32$

(c) Log, $N = 32$

(d) Dirichlet, $N = 128$

(e) Jittered, $N = 128$

(f) Log, $N = 128$

(g) Dirichlet, $N = 256$

(h) Jittered, $N = 256$

(i) Log, $N = 256$
Reconstruction Examples

(a) “Jittered” Sampling

(b) “Log” Sampling

(c) “Spiral” Sampling

Figure: Non-harmonic Fourier sum Reconstruction, $N = 128$
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**Conventional Reconstruction Methods**

Several approaches available to perform reconstruction

- **Convolutional Gridding** – most popular
- **Uniform Resampling**
- **Iterative Methods**

"Fix" the quadrature rule while evaluating the non-harmonic sum

\[
S_N \tilde{f}(x) = \sum_{k=-N}^{N} \alpha_k \tilde{f}(\omega_k)e^{i\omega_kx}
\]

- \(\alpha_k\) are density compensation factors
  - e.g., \(\alpha_k = \omega_{k+1} - \omega_k\)
- Evaluated using a "non-uniform" FFT

Although there are distinct differences in methodology and computational cost, reconstruction accuracy is similar in most schemes. We will look at convolutional gridding and uniform resampling to obtain an intuitive understanding of the problems in reconstruction.

*Figure*: Evaluating the non-uniform Fourier sum
Several approaches available to perform reconstruction

- Convolutional Gridding – most popular
- Uniform Resampling
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**Figure**: Iterative Reconstruction

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Convolutional Gridding

Instead of approximating $f$ by $\sum_{k=-N}^{N} \alpha_k \hat{f}(\omega_k) e^{i\omega_k x}$, we compute

$$S_N \tilde{g}(x) = \sum_{k=-N}^{N} \hat{g}(k) e^{ikx}$$

1. Map the non-uniform modes to a uniform grid via convolution. The new coefficients on the uniform grid are therefore given by

$$\hat{g}(k) = \hat{f} \ast \hat{\phi} \bigg|_{\omega = k} \approx \sum_{m \text{ st. } |k - \omega_m| \le q} \alpha_m \hat{f}(\omega_m) \hat{\phi}(k - \omega_m)$$

2. Compute a (filtered) Fourier partial sum.
3. “Compensate” for the mapping operation (divide by $\phi(x)$).
4. Choose the interpolating function $\phi$ to be essentially bandlimited, i.e.,

$$\hat{\phi}(\omega) \approx 0 \quad |\omega| > q, \ q \in \mathbb{R}, \text{small}$$
$$\phi(x) \approx 0 \quad |x| > \pi$$
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Convolutional Gridding

\[ (\hat{f} \ast \hat{\phi})(\omega) = \hat{g} \]

**Figure:** Gridding: \( \hat{g} = \hat{f} \ast \hat{\phi} \)
Convolutional Gridding

Figure: Fourier Reconstruction of \( g(x) = f(x)\phi(x) \)
Convolutional Gridding

**Figure:** Compensation $f(x) = g(x)/\phi(x)$
Reconstruction Examples – Convolutional Gridding

(a) Test Function reconstruction

(b) Cross-section of a brain scan

Figure: Gridding reconstruction, $N = 128$ (processed by a fourth order exponential filter)
Uniform Resampling

Reconstruction is accomplished in two steps:

1. recover equispaced coefficients \( \hat{f}(k) \)
2. partial Fourier reconstruction using the FFT algorithm

Since \( f \) is compactly supported, we use the sampling theorem to relate \( \hat{f}(\omega_k) \) and \( \hat{f}(k) \).

\[
\hat{f}(\omega) = \sum_{p=-\infty}^{\infty} \hat{f}(p) \text{sinc}(\omega - p), \quad \omega \in \mathbb{R}, p \in \mathbb{N}
\]

- To recover \( \hat{f}(k) \), we have to invert the above system, i.e., solve

\[
Ax = b, \quad A_{ij} = \text{sinc}(\omega_i - j), \quad b = \left\{ \hat{f}(\omega_k) \right\}_{k=-N}^{N}, \quad x = \left\{ \hat{f}(p) \right\}_{p=-M}^{M}
\]

- Any number of methods to do so - iterative methods, pseudoinverse-based methods with regularization ...

- The condition number of the sinc matrix depends on the sampling pattern’s deviation from equispaced nodes.
Representative Results – Uniform Resampling

(a) Recovered Fourier coefficients

(b) Filtered reconstruction

Figure: URS solution, \( N = 128 \)

- Solved a square \( 128 \times 128 \) system
- Inverted the system by computing the pseudoinverse
- Pseudoinverse was computed using TSVD, with a SVD threshold of \( 10^{-5} \)
Theorem (Convolutional Gridding Error (Viswanathan, 2009))

Let \( \hat{g} = \hat{f} * \hat{\phi} \) denote the true gridding coefficients and \( \hat{\tilde{g}} \) denote the approximate gridding coefficients. Let \( \Delta_k \) be the maximum distance between sampling points and \( d_k := \frac{1}{\Delta_k} \) be the minimum sample density in the \( q \)-vicinity of \( k \). Then, the gridding error at mode \( k \) is bounded by

\[
e(k) \leq C \cdot \frac{1}{d_k^2}, \quad k = -N, \ldots, N
\]

for some positive constant \( C \).
Convolutional Gridding Error

Physical space reconstruction error

\[ e(x) \approx g(x) - S_N \tilde{g}(x) = g(x) - S_N g(x) + S_N g(x) - S_N \tilde{g}(x) \]

\[ = \sum_{|k| > N} \hat{g}(k) e^{ikx} + \sum_{|k| \leq N} \left( \hat{g}(k) - \hat{\tilde{g}}(k) \right) e^{ikx} \]

standard Fourier truncation  gridding

- \( S_N g \) suffers from Gibbs; the maximum error occurs in the vicinity of a jump (\( \approx 1.09 \) of the jump value). There is also a reduced rate of convergence with \( \| g - S_N g \| = O(N) \).
- Gridding (sampling) error

\[
|S_N g(x) - S_N \tilde{g}(x)| = \left| \sum_{|k| \leq N} \left( \hat{g}(k) - \hat{\tilde{g}}(k) \right) e^{ikx} \right| \\
\leq \sum_{|k| \leq N} \left| \hat{g}(k) - \hat{\tilde{g}}(k) \right| \\
\leq C \sum_{|k| \leq N} \frac{1}{d_k^2}
\]
Error Plots

(a) Error bound, log sampling

(b) $|S_N g(x) - S_N \tilde{g}(x)|$ vs $N$

Figure: Error Plots
Error Plots

(a) Fourier coefficients – High modes

(b) $|S_N g(x) - S_N \tilde{g}(x)|$ vs $N$

Figure: Error Plots
The reconstruction error is

\[ e(x) \approx \sum_{|k| > N} \hat{g}(k)e^{ikx} + \sum_{|k| \leq N} \left( \hat{g}(k) - \hat{\tilde{g}}(k) \right) e^{ikx} \]

- 1\textsuperscript{st} term decreases as \( N \) increases
- 2\textsuperscript{nd} term increases as \( N \) increases

For a given sampling trajectory and function, there is a critical value \( N_{\text{crit}} \) beyond which adding coefficients does not improve the accuracy. While filtering decreases the error, the underlying problem is not solved.
The reconstruction error is

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**Figure:** Error in uniform re-sampling

For a given sampling trajectory and function, there is a critical value \( N_{\text{crit}} \) beyond which adding coefficients does not improve the accuracy. While filtering decreases the error, the underlying problem is not solved.
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\[ e(x) \approx \sum_{|k| > N} \hat{g}(k)e^{ikx} + \sum_{|k| \leq N} \left( \hat{g}(k) - \hat{\hat{g}}(k) \right) e^{ikx} \]

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For a given sampling trajectory and function, there is a critical value \( N_{\text{crit}} \) beyond which adding coefficients does not improve the accuracy. While filtering decreases the error, the underlying problem is not solved.
Error in Recovered Coefficients – Uniform Resampling

(a) Recovered Fourier coefficients

(b) Error in recovered coefficients

Figure: URS solution, $N = 128$

- The condition number of the resampling matrix is directly related to the error in the coefficients.
Outline

1. Introduction
   - Magnetic Resonance Imaging
   - Sampling Patterns in MR Imaging
   - Challenges in Cartesian Reconstruction
     - Spectral Reprojection

2. The Non-Uniform Data Problem
   - Problem Formulation
   - The Non-harmonic Kernel
   - Reconstruction Results using the Non-harmonic Kernel

3. Current Methods
   - Reconstruction Methods
   - Error Characteristics

4. Alternate Approaches
   - Spectral reprojection
   - Incorporating Edge Information
   - Spectral reprojection for Fourier Frames

Anne Gelb
Fourier Reconstruction from Non-Uniform Spectral Data
We still have the “Gibbs phenomenon” – non-physical oscillations at discontinuities, and a reduced rate of convergence (first order). Hence, we require a large number of coefficients to get acceptable reconstructions.

However, by formulation of the sampling scheme and recovery procedure, the coefficients recovered at large $\omega$ are inaccurate.

$\implies$ we need more coefficients, but the coefficients we get are inaccurate!

Spectral Reprojection – High frequency modes of $f$ in the original basis have exponentially small contributions on the low modes in the new basis.
Reducing the Impact of the High Mode Coefficients using Spectral Reprojection (Viswanathan, Cochran, Gelb, & Renaut, 2010)

Spectral reprojection expansion coefficients:

\[
\frac{1}{\gamma^\lambda} < S_N \tilde{g}, C_i^\lambda > \omega(\lambda) = \frac{1}{\gamma^\lambda} \int_{-1}^{1} (1 - \eta^2)^{\lambda - 1/2} C_i^\lambda(\eta) \sum_{|k| \leq N} \hat{g}(k)e^{i\pi kn} d\eta
\]

\[\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & -5 \\
0 & -0.5 & -1 & -1.5 & -2 & -2.5 & -3 & -3.5 & -4 & -4.5 & -5 & -5.5 \\
\end{array}\]

**Figure:** Decay of Gegenbauer coefficients
Spectral Reprojection Error of Gridded Coefficients

- \[ \text{Err}(m, N, g, \omega) := \| g - P_m(S_N \tilde{g}) \| \leq \| g - P_m g \| + \| P_m (g - S_N g) \| + \| P_m S_N g - P_m S_N \tilde{g} \| \]
- \[ \| P_m S_N g - P_m S_N \tilde{g} \|_\infty \leq \text{Const} \sum_{l=0}^{m} \sum_{|k| \leq N} \frac{1}{d_k^2} \left| \frac{C_l^\lambda(1)}{\gamma_l^\lambda} \int_{-1}^{1} (1 - \eta^2)^{\lambda - 1/2} C_l^\lambda(\eta) e^{i\pi k \eta} d\eta \right| \]
- \[ \left| \frac{C_l^\lambda(1)}{\gamma_l^\lambda} \int_{-1}^{1} (1 - \eta^2)^{\lambda - 1/2} C_l^\lambda(\eta) e^{i\pi k \eta} d\eta \right| \leq \frac{\Gamma(\lambda)(l + \lambda)\Gamma(l + 2\lambda)}{l!\Gamma(2\lambda)} \left( \frac{2}{\pi |k|} \right)^\lambda \]
- The corresponding gridding error does not increase rapidly with \( N \)
Spectral Reprojection – Contribution from Gridding Error

(a) Gridding Error $\lambda = 2$

(b) Gridding Error $\lambda = 3$

(c) Gridding Error $\lambda = 4$

(d) Gridding Error $\lambda = 8$

Figure: Contribution from Gridding Error in Spectral Reprojection
Gegenbauer Reconstruction - Results

Figure: Gegenbauer reconstruction from log spectral samples

- Filtered Fourier (second-order exponential) reconstruction uses 256 coefficients
- Gegenbauer reconstruction uses \(2N + 1 = 121\) coefficients
- Edges detected using the concentration method (Gelb & Tadmor) with thresholding and non-linear post-processing.
- Parameters \((m, \lambda)\) chosen proportional to size of reconstruction interval.
Figure: Error Plots – Filtered and Gegenbauer Reconstruction
Obtain edge information by

- Using the concentration method on the recovered coefficients

\[ S_N^\sigma [\tilde{f}] (x) = i \sum_{k=-N}^{N} \hat{f}(k) \text{sgn}(k) \sigma \left( \frac{|k|}{N} \right) e^{ikx} \]

- Solving for the jump function directly from the non-harmonic Fourier data

Let \( \tilde{f}(\omega_k) = i \text{sgn}(\omega_k) \hat{f}(\omega_k) \)

\[ \min_P \| \mathcal{F}(WP) \big|_{\omega_k} - \tilde{f}(\omega_k) \|^2 + \lambda \| P \|_1 \]

**Figure:** Edge Detection using the iterative formulation
Methods Incorporating Edge Information

- Compute the high frequency modes using the relation

\[ \hat{f}(k) = \sum_{p \in \mathcal{P}} [f](\zeta_p) \frac{e^{-ik\zeta_p}}{2\pi i k} \]

(a) Reconstruction - Using edge information

(b) The high modes - Using edge information

**Figure**: Reconstruction of a test function using edge information
Spectral reprojection for Fourier Frames (Gelb & Hines, 2010)

Finite frame approximation: Assume \( \{e^{i\omega_n x}\}_{n \in \mathbb{Z}} \) is a frame for \( L^2(I) \).

- Given frame coefficients \( \{\hat{f}(\omega_n)\}_{n=-N}^{N} = \{ \langle f(x), e^{i\omega_n x} \rangle \}_{n=-N}^{N} \)
- Compute \( T_N f = \sum_{n=-N}^{N} \hat{f}(\omega_n) S^{-1} e^{i\omega_n x} \)
- Filtering does not improve convergence:
  \[
  T_{N}^{\sigma} f = \sum_{n=-N}^{N} \sigma(\omega_n) \hat{f}(\omega_n) S^{-1} e^{i\omega_n x} \not\to f
  \]
- Spectral reprojection yields exponential convergence.

**Theorem** (Gelb and Hines (2010)): Let \( \{e^{i\omega_n x}\}_{n \in \mathbb{Z}} \) be a frame generated by a balanced sampling sequence. Suppose we are given the first \( 2N + 1 \) frame coefficients of \( f \in L^2[-1, 1] \). If \( \lambda = \alpha N \) and \( M = \beta N \) for \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \), then there exists a constant \( 0 < q < 1 \) such that

\[
\|P^\lambda_{M}(f - T_N f)\|_\infty \leq cN^2 q^N
\]
Spectral reprojection for Fourier Frames (Gelb & Hines, 2010)

(a) Reconstruction with Gegenbauer frame algorithm

(b) Pointwise error with Gegenbauer frame algorithm

Figure: Gegenbauer frame reconstruction
Spectral reprojection for Fourier Frames (Gelb & Hines, 2010)

Figure: Gegenbauer frame reconstruction
Spectral reprojection for Fourier Frames (Gelb & Hines, 2010)

(a) pointwise error with frame reconstruction

(b) pointwise error with Gegenbauer frame algorithm

Figure: Gegenbauer frame reconstruction
Summary

- Problem: Fourier reconstruction of piecewise smooth functions from non-uniform coefficients – MRI, SAR
- Conventional reconstruction methods from MRI community (density compensation, uniform resampling, iterative methods)
- Error analysis and characteristics related to sampling density
- *Spectral reprojection* mitigates the impact of the error from the high frequency coefficients
- Fourier based edge information can help to obtain better approximation to high frequency coefficients
- Fourier frames may provide a better alternative in reconstruction since no interpolation is needed
- Special thanks to Adityavikram Viswanathan, ASU EE PhD 2010, currently a postdoctoral fellow at ACM. California Institute of Technology.
Special thanks

Special thanks to the organizers and to my sons (for letting me come)

Figure: Sam and Josh Bagatell