Global Estimates for Kernels of Neumann Series and Green's Functions

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February Fourier Talks, Norbert Wiener Institute

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Background

Joint work with Fedor Nazarov and Igor Verbitsky
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$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j.$$

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When is

$$\sum_{j=1}^\infty K_j(x, y) \le CK(x, y)?$$

Simple Result

Trivial theorem: Suppose there exists ϵ with $0 < \epsilon < 1$ such that $K_2 \leq \epsilon K_1$. Then

$$\sum_{j=1}^{\infty} K_j(x, y) \leq \frac{1}{1-\epsilon} K(x, y).$$

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Proof: $$K_3(x, y) = \int K_2(x, z) K(z, y)\, d\omega(z)$$

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$$\sum_{j=1}^{\infty} K_j \leq (1 + \epsilon + \epsilon^2 + \dots)K.$$

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there exists $C_1 > 0$ such that $K_2 \leq C_1 K$.

Main Result

Theorem: Let K be a quasi-metric kernel on Ω.

A.) (Lower bound) There exists $c = c(\kappa)$ such that

$$\sum_{j=1}^{\infty} K_j(x,y) \geq K(x,y) e^{c\,K_2(x,y)/K(x,y)}.$$

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The proof of the upper bound is quite involved, as reflected in our value for C:

$$C = \frac{3 \cdot 2^{11} \kappa^6 (6 + \log_2 \kappa)^2 \Gamma(6 + \log_2 \kappa)}{(1 - \|T\|)^{6+\log_2 \kappa}}.$$

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Then the same theorem holds for K. Apply previous theorem with kernel H and measure $d\nu = m^2 d\omega$. Note that the operator $Sf(x) = \int H(x, y)f(y)\, d\nu(y)$ satisfies

$$\|S\|_{L^2(\nu)\to L^2(\nu)} = \|T\|_{L^2(\omega)\to L^2(\omega)}.$$

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So
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implies
$$\sum_{j=1}^{\infty} \frac{K_j(x,y)}{m(x)m(y)} \approx \frac{K(x,y)}{m(x)m(y)}e^{CK_2(x,y)/K(x,y)}.$$

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Kalton and Verbitsky (TAMS, 1999) studied the existence of solutions $u \geq 0$ to

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Eventually we generalized to $(-\triangle)^{\alpha/2} u - qu = \varphi$, with $0 < \alpha \leq 2$ ($\alpha \neq 2$ in dimension 2), where $(-\triangle)^{\alpha/2}$ is defined probabilistically on a domain.

Green's functions

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$$\text{Let} \qquad \mathcal{G}(x, y) = \sum_{j=1}^{\infty} G_j(x, y).$$

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So $\mathcal{G}$ is the kernel of the solution operator for (*).

Green's Function Estimates for Schrödinger operators

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Theorem: Let $\Omega = \mathbb{R}^n$, or any bounded domain satisfying the boundary Harnack principle (e.g., any bounded Lipschitz domain).

A.) (Lower bound) Then there exists $c = c(\Omega, \alpha) > 0$ such that

$$\mathcal{G}(x, y) \geq G(x, y) e^{cG_2(x,y)/G(x,y)}.$$

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Then it is not difficult to see that $G(x, y)/(\delta(x)\delta(y))$ is a quasi-metric kernel.

About the proof, cont'd

More generally, it is known (Hansen) that for bounded domains satisfying the boundary Harnack principle, $G(x,y)/(m(x)m(y))$ is a quasi-metric kernel for

$$m(x) = \min(1, G(x, x_0)),$$

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Hence the results follow from the remarks earlier about modifiable kernels.

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For $\alpha = 2$, there is a probabilistic formula

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Using (1) and Jensen's inequality, the lower bound

$$\mathcal{G}(x,y) \geq G(x,y) e^{cG_2(x,y)/G(x,y)}$$

with sharp constant $(c = 1?)$ follows.

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Question: Is there a probabilistic proof of the upper bound? With a sharper constant?

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Remark: this is an equation of Ricatti type

Application: solvability of (**) and (***)

Let P be the Poisson kernel for a bounded C^2 domain Ω. Define the balyage operator P^* by

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Theorem: Suppose $\|T\| < 1$. Then there exists $C > 0$ such that if

$$\int_{\partial\Omega} e^{CP^*(\delta q)}\, d\sigma < \infty,$$

where $\delta(x) = dist(x, \partial\Omega)$, then $(**)$ and $(***)$ have solutions.

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$\|\delta q\|_C < \epsilon_1$, for ϵ_1 small enough, where $\|\delta q\|_C$ denotes the Carleson norm of the measure $\delta(x)q(x)\, dx$.

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Hence under assumption of theorem, get $u_0 \in L^1(d\omega)$, which implies $u_1 \in L^1(dx)$, so u_1 solves $(**)$. Then $v = \log u$ satsifies $(***)$.

Remark

Problems $(*)$ and $(**)$ have formal solutions. Recall that

$$u(x) = \int_\Omega \sum_{j=1}^{\infty} G_j(x, y) f(y)\, dy$$

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Taking $f = 1$, the formal solution of $(*)$ is

$$u_0(x) = \int_\Omega \sum_{j=1}^{\infty} G_j(x,y)\, dy.$$

Remark, cont'd

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So the only question is whether the formal solutions are finite a.e.

Sketch of proof

First, $(**)$ is related to $(***)$ as follows: if $u_1 > 0$ satisfies $(**)$, then $v = \log u$ satisfies $(***)$, and if v satisfies $(***)$ then $u_1 = e^v$ satisfies $(**)$.

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We will look for a condition that gives $u_0 \in L^1(d\omega)$. Then $u_1 \in L^1(dx)$, hence $u_1 < \infty$ a.e., so $(**)$ and $(***)$ are solvable.

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Enough to show: $\frac{u_0}{\delta} \leq C_1 \sum_{j=0}^{\infty} \frac{T^j \delta}{\delta} \leq C_1 e^{CT\delta/\delta}$.

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$$\sum_{j=0}^{\infty} \tilde{T}^j 1 \leq e^{C\tilde{T}1}.$$

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Imagine that we can add a point z to Ω such that $d(y, z) = 1/K(y, z) = 1$ for all $y \in \Omega$, with $\nu(\{z\}) = 0$. Then

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Works similarly if Ω is bounded and $d(y,z) = D >> diam(\Omega)$. So restrict to bounded subset, get estimates independent of $diam(\Omega)$, take limit.

Application: estimate of u_0 (cont'd)

Claim: there exists $c, C > 0$ such that

$$\int_{\partial\Omega} e^{CP^*(\delta q)(z)}\, d\sigma(z) \geq c|\partial\Omega| + c\int_{\Omega} u_0(y)\, d\omega(y).$$

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$$\geq c \lim_j \frac{\delta(x_j) + Tu_0(x_j)}{\delta(x_j)} \text{ since } u_0 - Tu_0 = G1 \approx \delta$$

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Application: estimate of u_0 (cont'd)

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$$\int_{\Omega} u^2 q \, dx \leq \beta \int_{\Omega} |(-\triangle)^{\alpha/4} u|^2 \, dx, \tag{2}$$

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Remark: Inequality (2) implies $\|T\| < 1$: Equivalently:

$$\int_\Omega |G^{(\alpha/2)} f|^2\, d\omega \le \beta \int_\Omega |f|^2\, dx,$$

or $\|G^{(\alpha/2)}\|_{L^2(\Omega,dx)\to L^2(\Omega,d\omega)} \le \sqrt{\beta}$, and
$T = G^{(\alpha/2)}(G^{(\alpha/2)})^*$.

Example

If $\Omega = \mathbb{R}^n$ and $q(x) = A/|x|^\alpha$ with

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Remark: There is a sharp result due to Maz'ya, Grigorian, and others with $c_2 = C_2 = \frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - A}$.