Absolute continuity and singularity of measures without measure theory

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F. & M. Riesz Theorem

Every non-zero analytic measure ν *on* $\mathbb{T}$ *is mutually absolutely continuous with respect to Lebesgue measure* λ*.*

Corollary (Szegö's Theorem)

Let σ *be a Borel probability measure on* $\mathbb{T}$ *that annihilates some set of positive Lebesgue measure. Then the powers* z^n*,* $n \in \mathbb{N}$*, span* $L^2(\sigma)$*.*

Proof of the F. & M. Riesz Theorem

Let $\mu := |\nu|$ and f the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. That is,

$$d\nu = f d\mu \quad \text{and} \quad |f| = 1 \quad \mu - \text{a.e.}$$

Then the analyticity hypothesis on ν can be written

$$\int_{\mathbb{T}} z^n f(z) d\mu(z) = 0 \quad \forall n \in \mathbb{N}. \tag{1}$$

Let $\langle , \rangle$ and $\| \cdot \|$ denote inner product and norm in $L^2(\mu)$ and U the unitary operator of multiplication by z. By "span" will be meant "closed linear span in $L^2(\mu)$."

According to (1) the constant function 1 is orthogonal to every $U^n f$ $(n \in \mathbb{N})$, so the set

$$M := \text{(closed) span } \{U^n f : n \in \mathbb{N}\} \text{ in } L^2(\mu) \tag{2}$$

is a *proper* subspace of $L^2(\mu)$, evidently U-invariant. In fact,

$$UM \subsetneq M. \tag{3}$$

Now the linear span of $\{z^n : n \in \mathbb{Z}\}$ is dense in $C(\mathbb{T})$ (why?), hence also in $L^2(\mu)$. So, given $g \in L^2(\mu)$, since $|f| = 1$ μ-a.e. $\bar{f}g \in L^2(\mu)$ and accordingly some P_n in this linear span satisfy $\|P_n - \bar{f}g\| \to 0$. That is, $\|P_n f - g\| \to 0$, showing that

$$\text{span}\,\{z^n f : n \in \mathbb{Z}\} = L^2(\mu). \tag{4}$$

If we suppose, contrary to (3), that $UM = M$, then

$$U^* M = U^* UM = M,$$

so M contains, along with $z\,f$,

$$(U^*)^m z\,f = (\bar{z})^m z\,f = z^{-m+1} f \quad \forall m \in \mathbb{N},$$

and consequently

$$z^n f \in M \quad \forall n \in \mathbb{Z},$$

which with (4) contradicts the proper inclusion $M \subsetneq L^2(\mu)$. This contradiction confirms (3).

Form the orthocomplement

$$M \ominus UM \neq \{0\}$$

and note that the closed subspaces $U^n(M \ominus UM)$ are orthogonal, which is pretty clear when they are written as

$$U^n M \ominus U^{n+1} M.$$

As a special case

$$\{U^n h\}_{n\in\mathbb{Z}} \text{ is an orthonormal sequence in } L^2(\mu) \tag{5}$$

for every unit vector $h \in M \ominus UM$.

Note that

$$\bigcap_{k\geq 0} U^k M \text{ is orthogonal to } U^n(M \ominus UM) \quad \forall n \in \mathbb{Z}.$$

For if $m_1 \in M \ominus UM$ and m_0 lies in this intersection, then $m_0 = U^{|n|+1}m_2$ for some $m_2 \in M$, and so

$$\langle m_0, U^n m_1\rangle = \langle U^{|n|+1}m_2, U^n m_1\rangle = \langle U^{|n|-n+1}m_2, m_1\rangle = 0,$$

since $U^{|n|-n+1}m_2 \in UM$. The same argument shows that

$$\bigcap_{k\geq 0} U^k M \text{ is orthogonal to } U^n 1 \quad \forall n \in \mathbb{Z}. \tag{6}$$

The *Wold decomposition* says that M is the orthogonal sum

$$M = \bigcap_{k \geq 0} U^k M \oplus \bigoplus_{n \geq 0} U^n (M \ominus UM). \tag{7}$$

Again, this is pretty transparent when the right side is written out as

$$\bigcap_{k \geq 0} U^k M \oplus (M \ominus UM) \oplus (UM \ominus U^2 M) \oplus (U^2 M \ominus U^3 M) \oplus \cdots$$

As previously noted, vectors $U^n 1 = z^n$ ($n \in \mathbb{Z}$) span a dense subspace of $L^2(\mu)$. From (6), then $\cap_{k \geq 0} U^k M$ must be $\{0\}$ and (7) reads

$$M = \bigoplus_{n \geq 0} U^n (M \ominus UM). \tag{8}$$

Next we aim to show the non-zero space

$$M \ominus UM \text{ is 1-dimensional.} \tag{9}$$

If not, $\exists$ orthogonal unit vectors $g, h \in M \ominus UM$. By familiar maneuvers,

$$U^m h \perp U^k g \quad \forall m, k \in \mathbb{N}_0,$$

so

$$0 = \langle U^m h, U^k g\rangle = \langle U^{m-k} h, g\rangle \quad \forall m, k \in \mathbb{N}_0,$$

whence

$$0 = \langle U^n h, g\rangle = \int_{\mathbb{T}} z^n h\bar{g}\, d\mu \quad \forall n \in \mathbb{Z}.$$

Again, due to denseness of the powers z^n, this entails $h\bar{g} = 0$ μ-a.e. That is,

$$|h||g| = 0 \quad \mu\text{-a.e.} \tag{10}$$

As noted in (5)

$$\int_{\mathbb{T}} z^n |h|^2 \, d\mu(z) = \langle U^n h, h \rangle = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

that is, the measure $|h|^2 d\mu$ has exactly the same Fourier coefficients as λ, so

$$|h|^2 d\mu = d\lambda.$$

And the same is true for g. Thus,

$$|h|^2 d\mu = d\lambda = |g|^2 d\mu, \qquad (11)$$

whence, by (10),

$$|h|^3 d\mu = |h||g|^2 d\mu = 0,$$

contrary to $\|h\|^2 = \int |h|^2 d\mu = 1$. Thus $g = 0$, and (9) is confirmed.

That is, for any $h \in M \ominus UM$ of norm 1

$$M \ominus UM = \mathbb{C}h,$$

so (8) says

$$\text{span } \{U^n h : n \in \mathbb{N}_0\} = M.$$

In particular, since $Uf = z\,f \in M$, we see that $z\,f$ lies in the span of $z^n h$. It follows that

$$\text{span } \{z^n f : n \in \mathbb{Z}\} \subset \text{ span } \{z^n h : n \in \mathbb{Z}\}.$$

Combined with (4) this says

$$\text{span}\ \{z^n f : n \in \mathbb{Z}\} = \text{span}\ \{z^n h : n \in \mathbb{Z}\} = L^2(\mu). \tag{12}$$

A little thought shows the equality of these two spans entails

$$f\, d\mu \ll h\, d\mu \ll f\, d\mu,$$

and thanks to (11)

$$d\lambda \ll |h|\, d\mu \ll d\lambda.$$

Thus $fd\mu = d\nu$ is mutually absolutely continuous with respect to $d\lambda$. □

Proof of Szegö

Denote by M the (closed) span in $L^2(\sigma)$ of the z^n, $n \in \mathbb{N}$, and assume $M \neq L^2(\sigma)$. There is then a non-zero function $g \in L^2(\sigma)$ orthogonal to M:

$$0 = \langle z^n, g \rangle_{L^2(\sigma)} = \int_{\mathbb{T}} z^n \bar{g}\, d\sigma \quad \forall n \in \mathbb{N}.$$

This says that $\bar{g}\, d\sigma$ is a (non-zero) analytic measure. Hence $d\lambda \ll \bar{g}\, d\sigma$. So for every Borel B,

$$\sigma(B) = 0 \Rightarrow \int_B \bar{g}\, d\sigma = 0 \Rightarrow \lambda(B) = 0.$$

Contrary to the hypothesis on σ, which has σ annihilating a B with $\lambda(B) > 0$. □

Holland [1974]

σ is a Borel probability measure on $\mathbb{T}$ which is singular with respect to λ.

$$F(z) := \int_{\mathbb{T}} \frac{u+z}{u-z}\, d\sigma(u),$$

a holomorphic self-map of $\mathbb{D}$.

$$A_k := k^{th} \text{ Taylor coefficient of } \frac{F(z)-1}{F(z)+1}.$$

Then

$$\sum_{k=1}^{\infty} |A_k|^2 = 1 \qquad \text{(i)}$$

and the polynomials $P_n(z) := \sum_{k=1}^{n} A_k z^k$, $n \in \mathbb{N}$, satisfy

$$\int_{\mathbb{T}} |1-P_n|^2\, d\sigma = 1 - \sum_{k=1}^{n} |A_k|^2 \quad \forall n \in \mathbb{N}. \qquad \text{(ii)}$$

(i) and (ii) show (very constructively!) that

$$1 \in \overline{\text{span}}\,\{z^n : n \in \mathbb{N}\},\ i.e.,$$

$$\overline{\text{span}}\,\{z^n : n \in \mathbb{N}_0\} = \overline{\text{span}}\,\{z^n : n \in \mathbb{N}\}.$$

By induction it follows

$$\overline{\text{span}}\,\{z^n : n \in \mathbb{Z}\} = \overline{\text{span}}\,\{z^n : n \in \mathbb{N}\},\ i.e.,$$

$$L^2(\sigma) = \overline{\text{span}}\,\{z^n : n \in \mathbb{N}\}$$

(Note the stronger hypothesis than in Szegö.)

Øksendal [1971]

A $\mathbb{C}$-valued Borel measure ν on $\mathbb{T}$ satisfying (A) is given and what has to be shown is that $\nu(K) = 0$ for every λ-null Borel K.
Because Borel measures are inner regular, it suffices to consider only compact K.
Clearly it further suffices to do this for the modified measure

$$\nu_0 := \nu - \nu(\mathbb{T})\lambda.$$

The measure ν_0 is also analytic but in addition annihilates 1. That is,

$$\hat{\nu_0}(-n) = \int_{\mathbb{T}} z^n \, d\nu_0(z) = 0 \quad \forall n \in \mathbb{N}_0. \qquad \text{(A*)}$$

For each $n \in \mathbb{N}$, an $N \in \mathbb{N}$, $z_j \in K$ and $\rho_j > 0$ are chosen appropriately and the rational functions

$$g_n(z) := 1 - \prod_{j=1}^{N} \frac{z - z_j}{z - (1 + \rho_j) z_j}$$

are introduced.

They are bounded by 2 on $\mathbb{T}$ and converge there to the indicator function of K. Since g_n is holomorphic in a neighborhood of $\overline{\mathbb{D}}$, the partial sums of its Taylor series at 0 approximate it uniformly on $\mathbb{T}$, and each sum has ν_0-integral 0, thanks to (A*). Consequently,

$$\int_{\mathbb{T}} g_n \, d\nu_0 = 0 \quad \forall n \in \mathbb{N}.$$

It follows from the Lebesgue Dominated Convergence Theorem that

$$\nu_0(K) = \lim_{n\to\infty} \int_{\mathbb{T}} g_n \, d\nu_0 = 0,$$

as wanted.