Shift Invariant Spaces and BMO

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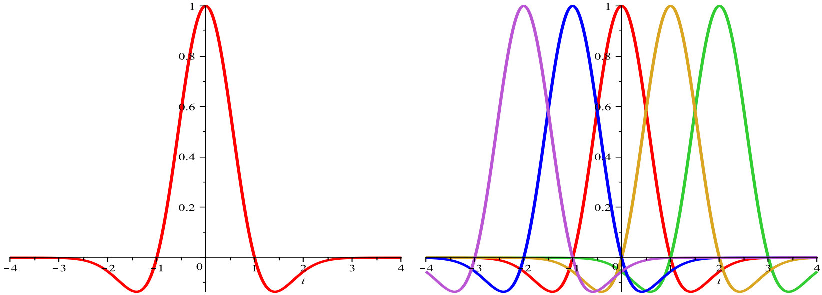
(Integer) Shifts of a Fixed Function

Given a function $\psi \in L_2(\mathbb{R}^d)$, one of the most basic operations we can consider is translation: $T_k\psi := \psi(\cdot - k)$, $k \in \mathbb{Z}^d$.



Translation is a fundamental operator in harmonic analysis since it is "simple" and behaves well under the Fourier transform:

$$\mathcal{F}(T_k\psi) = e^{-2\pi i k\cdot}\hat{\psi}, \qquad \text{where } \mathcal{F}(f)(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\cdot\xi}\,dx.$$

Shift Invariant Spaces

A finitely generated shift-invariant (FSI) subspaces of $L_2(\mathbb{R}^d)$ is a subspace $S \subset L_2(\mathbb{R}^d)$ for which there exists a finite family Ψ of $L_2(\mathbb{R}^d)$-functions such that

$$S = S(\Psi) := \overline{\text{span}\{\psi(\cdot - k) : \psi \in \Psi, k \in \mathbb{Z}^d\}}.$$

Remark

To keep the notation simple, we only consider the most basic case: $d = 1$ and $\#\Psi = 1$ [PSI space].

Applications

FSI/PSI subspaces are used in several applications.

- Wavelets and other multi-scale methods are based on PSI subspaces
- FSI/PSI subspaces play an important role in multivariate approximation theory such as spline approximation and approximation with radial basis functions.

Stable generating set

Given the structure of S, it is natural to consider a generating sets of integer translates. That is, a system with the following structure,

$$\{\varphi(\cdot - k) : k \in \mathbb{Z}\},$$

Often we take $\varphi = \psi$, but φ may be different from ψ. However, we always require that $S(\varphi) = S(\psi)$.

Basic Fourier Analysis of $S(\psi)$

It can easily be deduced from the identity,

$$f = \sum_k c_k \psi(\cdot - k) \Rightarrow \hat{f} = \sum_k c_k e^{-2\pi i k \cdot} \hat{\psi}$$

$$\Rightarrow \|\hat{f}\|_2^2 = \int_{\mathbb{T}} \left| \sum_k c_k e^{-2\pi i k \xi} \right|^2 \sum_j |\hat{\psi}(\xi + j)|^2 \, d\xi$$

that

$$J_\psi m := (m \cdot \hat{\psi})^\vee$$

is an isometry from $L_2(\mathbb{T}; p_\psi)$ onto $S(\psi)$, where p_ψ is the periodization of $|\hat{\psi}|^2$, given by

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2, \qquad \xi \in \mathbb{R}.$$

Observation

The system $\{e^{2\pi i k \xi}\}_k$ in $L_2(\mathbb{T}; p_\psi)$ is mapped by J_ψ to $\{\psi(\cdot - k)\}_k$.

Some well-known classical results

Orthonormal and Riesz bases

Let $\psi \in L_2(\mathbb{R})$ and consider

$$B := \{\psi(\cdot - k) : k \in \mathbb{Z}\}.$$

We let

$$p_\psi(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\psi}(\xi + k)|^2, \qquad \xi \in \mathbb{R}.$$

Then

- B forms an orthonormal basis for $S(\psi)$ provided $p_\psi \equiv 1$.
- B forms a Riesz basis for $S(\psi)$ provided that $p_\psi \asymp 1$.

Extension to FSI spaces

The above result can be extended to FSI spaces using the Grammian for the generating set.

Question

Is stability of B possible even if $p_\psi \not\asymp 1$?

Weaker notion of stability: Schauder bases

Definition

A family $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ of vectors in a Hilbert space $\mathbb{H}$ is a *Schauder basis* for $\mathbb{H}$ if there exists a unique dual sequence $\{y_n : n \in \mathbb{N}\} \subset \mathbb{H}$ such that for every $x \in \mathbb{H}$,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \langle x, y_n \rangle x_n = x \qquad \text{(norm convergence).}$$

Ordering of the system

The Schauder basis convergence may not be unconditional so the ordering of the system becomes important.

Schauder bases of translates and Muckenhoupt weights

The Muckenhoupt A_2-class

A measurable, 1-periodic function $w : \mathbb{R} \to (0, \infty)$ is an $A_2(\mathbb{T})$-weight provided that

$$[w]_{A_2} := \sup_{I \in \mathcal{I}} \left(\frac{1}{|I|} \int_I w(\xi)\, d\xi \right) \left(\frac{1}{|I|} \int_I w(\xi)^{-1}\, d\xi \right) < \infty,$$

where $\mathcal{I}$ is the collection of intervals (arcs) on $\mathbb{T}$.

Proposition [Sikic and N., ACHA (2008)]

Let $\psi \in L_2(\mathbb{R})\backslash\{0\}$. The system $B := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$ forms a Schauder basis for $S(\psi)$, with $\mathbb{Z}$ ordered the natural way as $0, 1, -1, 2, -2, \ldots$, if and only if the periodization function p_ψ satisfies the $A_2(\mathbb{T})$ condition.

Remark

- The result is based on the well-known Hunt-Muckenhoupt-Wheeden Theorem.
- Similar results for Gabor systems were obtained by Heil and Powell [J. Math. Phys. (2006)].
- The PSI result can be generalized to multivariate FSI spaces using a theory of product A_2-matrix weights [N., JFAA (2010)].

Examples

- Define $\psi \in L_2(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \sqrt{\ln\big(\ln(2+|\xi|^{-1})\big)} \cdot \chi_{[0,1)}(\xi).$$

  It follows that $p_\psi(\xi) = \ln\big(\ln(2+|\xi|^{-1})\big)$, $\xi \in [-1/2, 1/2)$. A direct calculation shows that $p_\psi \in A_2(\mathbb{T})$, so $B := \{\psi(\cdot - k) : k \in \mathbb{Z}\}$ forms a Schauder basis for $S(\psi)$. However, p_ψ is not bounded and consequently B fails to be an unconditional Riesz basis for $S(\psi)$.

- Another example is provided by $\psi \in L_2(\mathbb{R})$ defined by

$$\hat{\psi}(\xi) = |\xi|^\alpha \cdot \chi_{[0,1)}(\xi),$$

  with $\alpha \in (-1/2, 1/2)$

Integer translates, A_2, and BMO

The A_2 class is closely related to the functions of bounded mean oscillation.

Definition

Let $f \in L_{1,\text{loc}}(\mathbb{R})$ be 1-periodic, and let $\mathcal{I}$ be the collection of intervals (arcs) on $\mathbb{T}$. We say that $f \in BMO(\mathbb{T})$ provided that

$$\|f\|_{BMO(\mathbb{T})} := \sup_{I \in \mathcal{I}} \frac{1}{|I|} \int_I |f(x) - f_I|\, dx < \infty,$$

where $f_I := \frac{1}{|I|} \int_I f(x)\, dx$.

- One can verify that $\log(A_2(\mathbb{T})) \subset BMO(\mathbb{T})$.
- Conversely, for $f \in BMO(\mathbb{T})$ there is some $\alpha > 0$ such that $e^{\alpha f} \in A_2(\mathbb{T})$ [by the John-Nirenberg inequality].
- It is also easy to check that $L_\infty(\mathbb{T}) \hookrightarrow BMO(\mathbb{T})$.

All of this is related to stability of integer translates by the fact that

$$\{\psi(\cdot - k) : k \in \mathbb{Z}\} \text{ forms a Riesz basis } \iff \log(p_\psi) \in L_\infty$$

Question

Can we use the distance to L_∞ of $\log(p_\psi) \in BMO(\mathbb{T})$ to quantify the "quality" of a conditional Schauder basis?

Distance to L_∞

For $f \in BMO(\mathbb{T})$ we let

$$\text{dist}(f, L_\infty(\mathbb{T})) := \inf_{g \in L_\infty(\mathbb{T})} \|f - g\|_{BMO(\mathbb{T})}.$$

The distance to L_∞ in BMO

One additional observation

It is known that L_∞ is not a closed subset of BMO. In fact,

$$\{f \in BMO(\mathbb{T}) : \operatorname{dist}(f, L_\infty) = 0\} = \left\{ f \in BMO : e^{mf} \in A_2, m \in \mathbb{Z} \right\}.$$

This follows from the celebrated result by Garnett and Jones that asserts that $\operatorname{dist}(f, L_\infty)$ and

$$\varepsilon(f) := \inf\{\lambda > 0 : [e^{f/\lambda}]_{A_2(\mathbb{T})} < \infty\}$$

are in fact equivalent independent of $f \in BMO(\mathbb{T})$.

Theorem (Garnett and Jones)

There exist positive constants C_1 and C_2 such that for $f \in BMO(\mathbb{T})$,

$$C_1\varepsilon(f) \le \operatorname{dist}(f, L_\infty(\mathbb{T})) \le C_2\varepsilon(f).$$

The BMO subset $\text{dist}(f, L_\infty(\mathbb{T})) = 0$

Example

An example of an unbounded BMO function in $\{f : \text{dist}(f, L_\infty) = 0\}$ is given by

$$f(x) = \ln\left(\ln(2 + |x|^{-1})\right), \qquad x \in \mathbb{T}.$$

This is a consequence of the fact that $\ln^N(2 + |x|^{-1}) \in A_2(\mathbb{T})$ for any $N \in \mathbb{N}$, which follows by direct calculation.

Improved stability: The coefficient space

Let $B = \{x_n\}_{n\in\mathbb{N}}$ be a Schauder basis for $\mathbb{H}$ with dual system $\{y_n\}_{n\in\mathbb{N}}$. The coefficient space associated with B is the sequence space given by

$$\mathcal{C}(B) := \left\{\{\langle x, y_n\rangle\}_{n\in\mathbb{N}} : x \in \mathbb{H}\right\}.$$

Controlling $\mathcal{C}(\mathcal{B})$

For a Riesz basis B, we have

$$\mathcal{C}(B) = \ell_2.$$

For a normalized conditional Schauder basis $\mathcal{B}$ in $\mathbb{H}$ one can find $2 \leq p < \infty$ (possibly very large) such that

$$\mathcal{C}(B) \hookrightarrow \ell_p.$$

[Gurariĭ and Gurariĭ, 1971]

Improved conditioning based on $\text{dist}(\ln(p_\psi), L_\infty(\mathbb{T}))$

Theorem [Šikić and N. JFA (2014)]

Let $\psi \in L_2(\mathbb{R})$ and suppose that $p_\psi \in A_2(\mathbb{T})$. We let $\mathcal{C}(\mathcal{E})$ denote the coefficient space for the Schauder basis $\mathcal{E} = \{\psi(\cdot - k)\}_k$ for $S(\psi)$. Define $\varepsilon = \varepsilon(\ln p_\psi) := \inf\{\lambda > 0 : [p_\psi^{1/\lambda}]_{A_2} < \infty\}$. Then the following inclusion holds

$$\mathcal{C}(\mathcal{E}) \subset \bigcap_{p_0 < p < \infty} \ell_p(\mathbb{Z}), \qquad p_0 := \frac{2}{1-\varepsilon}.$$

In particular, if $\text{dist}(\ln(p_\psi), L_\infty(\mathbb{T})) = 0$ then

$$\mathcal{C}(\mathcal{E}) \subset \bigcap_{2 < p < \infty} \ell_p(\mathbb{Z}).$$

Sketch of proof

i. The A_2 condition implies that $L^2(\mathbb{T}, p_\psi) \hookrightarrow L^1(\mathbb{T})$

ii. Take $f = \lim_{N\to\infty} \sum_{|k|\le N} \langle f, \tilde{\psi}(\cdot - k)\rangle \psi(\cdot - k) \in S(\psi)$ and let $m_f = J_\psi^{-1}(f) \in L^2(\mathbb{T}, p_\psi)$.

iii. Using i., verify that $m_f = \sum_{k\in\mathbb{Z}} \langle m_f, e_k\rangle_{L^2(\mathbb{T})} e^{2\pi ikx}$.

iv. Now use the Reverse Hölder Inequality for p_ψ and the Hölder inequality to estimate

$$\|m_f\|_{L_r}$$

for $r \approx 2$.

v. Conclude using the Hausdorff-Young inequality.

An example

Example

Recall the previous example with $\psi \in L_2(\mathbb{R})$ defined by

$$\hat{\psi}(\xi) = \sqrt{\ln\left(\ln(2+|\xi|^{-1})\right)} \cdot \chi_{[0,1)}(\xi),$$

and $p_\psi(\xi) = \ln\left(\ln(2+|\xi|^{-1})\right)$, $\xi \in [-1/2, 1/2)$.
A direct calculation shows that $p_\psi^N \in A_2(\mathbb{T})$ for any $N \in \mathbb{N}$, so $\mathcal{E} = \{\psi(\cdot - k)\}_k$ forms a conditional Schauder basis for $S(\psi)$ with coefficient space for $\mathcal{E}$ controlled by

$$\mathcal{C}(\mathcal{E}) \subset \bigcap_{2<p<\infty} \ell_p(\mathbb{Z}).$$

Another point of view: Improved conditioning of Schauder bases

For a Schauder basis $\mathcal{B} = \{x_n : n \in \mathbb{N}\}$ in $\mathbb{H}$ with dual sequence $\{y_n : n \in \mathbb{N}\} \subset \mathbb{H}$, we consider the partial sum operators $S_N(x) = \sum_{n=1}^{N} \langle x, y_n \rangle x_n$. The basis constant for $\mathcal{B}$ is given by

$$\kappa(\mathcal{B}) := \sup_{N \in \mathbb{N}} \|S_N\|.$$

Theorem [Šikić and N. JFA (2014)]

Let $\psi \in L_2(\mathbb{R})$ with periodization function $p_\psi \in A_2(\mathbb{T})$. Suppose p_ψ satisfies $\mathrm{dist}(\ln p_\psi, L_\infty) = 0$. Let $\mathcal{E} = \{\psi(\cdot - k)\}_k$. Then

i. If $\ln p_\psi \in L_\infty(\mathbb{T})$ then $\mathcal{E}$ forms a Riesz basis for $S(\psi)$.

ii. If $\ln p_\psi \notin L_\infty(\mathbb{T})$ then for every $\eta > 0$ there exists $b \in L_\infty(\mathbb{T})$ such that $\tilde{\mathcal{E}} = \{\varphi(\cdot - k)\}_k$, with $\hat{\varphi} := \frac{\hat{\psi}}{e^b}$, forms a Schauder basis for $S(\psi)$ with Schauder basis constant at most $3 + \mathcal{O}(\eta)$. The Schauder bases $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are equivalent.

References

R. Hunt, B. Muckenhoupt, and R. Wheeden.
Weighted norm inequalities for the conjugate function and Hilbert transform.
Trans. Amer. Math. Soc., 176:227–251, 1973.

J. B. Garnett and P. W. Jones.
The distance in BMO to L^∞.
Ann. of Math. (2), 108(2):373–393, 1978.

M. Nielsen and H. Šikić.
Schauder bases of integer translates.
Appl. Comput. Harmon. Anal., 23(2):259–262, 2007.

M. Nielsen.
On stability of finitely generated shift-invariant systems.
J. Fourier Anal. Appl., 16(6):901–920, 2010.

M. Nielsen and H. Šikić.
On stability of Schauder bases of integer translates.
J. Funct. Anal., 266:2281-2293, 2014.