Time-Frequency Analysis and the Dark Side of Representation Theory

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February 21, 2014

We consider time-frequency translations on $L^2(\mathbb{R})$:

$$T_x f(t) = f(t+x), \qquad M_y f(t) = e^{2\pi i y t} f(t)$$

We have $T_x M_y = e^{2\pi i x y} M_y T_x$, so the collection of operators

$$\left\{ e^{2\pi i z} M_y T_x : x, y, z \in \mathbb{R} \right\}$$

forms a group, essentially the (real) Heisenberg group. More precisely, the *real Heisenberg group* $H_{\mathbb{R}}$ is $\mathbb{R}^3$ equipped with the group law

$$(x, y, z)(x', y', z') = (x + x',\, y + y',\, z + z' + x y').$$

Given $\tau, \omega > 0$, consider the subgroup generated by the $T_{j\tau}$ and $M_{k\omega}$ with $j, k \in \mathbb{Z}$, namely,

$$\left\{ e^{2\pi i \tau \omega l} M_{k\omega} T_{j\tau} : j, k, l \in \mathbb{Z} \right\}.$$

There is a large literature on the use of families $\{M_{k\omega} T_{j\tau} \phi\}$ as building blocks to synthesize more general functions.

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By rescaling, we can and shall take $\tau = 1$. This is a unitary representation of the *discrete Heisenberg group* H, whose underlying set is $\mathbb{Z}^3$ and whose group law is

$$(j, k, l)(j', k', l') = (j + j',\, k + k',\, l + l' + jk').$$

That is, the representation in question is defined by

$$\rho_\omega(j, k, l) f(t) = e^{2\pi i \omega l} e^{2\pi i \omega k t} f(t + j) \qquad (f \in L^2(\mathbb{R})).$$

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How does this representation decompose into irreducible representations?

Some Background

- A *(unitary) representation* of a locally compact group G is a continuous homomorphism $\rho : G \to U(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space.
- ρ is *irreducible* if there are no nontrivial closed subspaces of $\mathcal{H}$ that are invariant under the operators $\rho(g)$, $g \in G$.
- $\rho : G \to U(\mathcal{H})$ and $\rho' : G \to U(\mathcal{H}')$ are *(unitarily) equivalent* if there is a unitary map $V : \mathcal{H} \to \mathcal{H}'$ such that $V\rho(g) = \rho'(g)V$ for all $g \in G$.
- The set of equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$.

If G is compact, every unitary representation of G is a direct sum of irreducible representations. The equivalence classes (elements of $\widehat{G}$) occurring in it and the multiplicities with which they occur are uniquely determined.
If G is noncompact, there are "continuous families" of irreducible representations, and in general one must employ direct integrals instead.

Direct Integrals

Suppose we have a family $\{\pi_\alpha : \alpha \in A\}$ of representations of G parametrized by a measure space (A, μ), where π_α acts on $\mathcal{H}_\alpha$. The *direct integral* of the Hilbert spaces $\mathcal{H}_\alpha$ is the Hilbert space

$$\begin{aligned}\mathcal{H} &= \int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha) \\ &= \left\{ f : A \to \bigcup \mathcal{H}_\alpha : f(\alpha) \in \mathcal{H}_\alpha \ \forall \alpha, \ \int \|f(\alpha)\|^2_{\mathcal{H}_\alpha} \, d\mu(\alpha) < \infty \right\}.\end{aligned}$$

(Some issues of measurability are being swept under the rug, but note that if the $\mathcal{H}_\alpha$ are all the same, say $\mathcal{H}_\alpha = \mathcal{K}$ for all α, then $\mathcal{H}$ is just $L^2(A, \mathcal{K})$.) The *direct integral* of the representations π_α is the representation

$$\pi = \int^{\oplus} \pi_\alpha \, d\mu(\alpha) \text{ on } \mathcal{H} \text{ defined by } [\pi(g)f](\alpha) = \pi_\alpha(g)[f(\alpha)].$$

Example

If $G = \mathbb{R}$, the irreducible representations are all one-dimensional and are parametrized by $\xi \in \mathbb{R}$:

$$\pi_\xi(x) = e^{2\pi i \xi x}.$$

The direct integral $\pi = \int_{\mathbb{R}}^{\oplus} \pi_\xi \, d\xi$ acts on $L^2(\mathbb{R})$ by

$$\pi(x) f(\xi) = e^{2\pi i \xi x} f(\xi).$$

Conjugation by the Fourier transform

$$\mathcal{F} f(\xi) = \int e^{-2\pi i t \xi} f(t) \, dt$$

turns this into the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$:

$$\mathcal{F}^{-1} \pi(x) \mathcal{F} f(t) = f(t + x), \quad \text{i.e.,} \quad \mathcal{F}^{-1} \pi(x) \mathcal{F} = T_x.$$

What Should Happen:

- $\widehat{G}$ is a geometrically "reasonable" object, equipped with a natural σ-algebra of measurable sets, and we can choose a representative π_α from each equivalence class α in $\widehat{G}$ in a "reasonable" way.
- Given a representation ρ, there is a measure μ on $\widehat{G}$ and disjoint measurable sets $E_1, E_2, \ldots, E_\infty$ (some of which may be empty) such that

$$\rho \sim \int_{E_1}^{\oplus} \pi_\alpha \, d\mu(\alpha) \oplus 2 \int_{E_2}^{\oplus} \pi_\alpha \, d\mu(\alpha) \oplus \cdots \oplus \infty \int_{E_\infty}^{\oplus} \pi_\alpha \, d\mu(\alpha).$$

(The coefficients in front of the integrals denote multiplicities.) μ is determined up to equivalence (mutual absolute continuity), and the E_j are determined up to sets of μ-measure zero.

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- Some type I groups: Abelian groups; compact groups; connected Lie groups that are nilpotent, semisimple, or algebraic; discrete groups with an Abelian normal subgroup of finite index.
- Some non-type I groups: some solvable Lie groups, all other discrete groups.

Now back to the discrete Heisenberg group H with group law

$$(j,k,l)(j',k',l') = (j+j',\, k+k',\, l+l'+jk'),$$

and our representation ρ_ω of H,

$$\rho_\omega(j,k,l)f(t) = e^{2\pi i\omega l}e^{2\pi i\omega kt}f(t+j) \qquad (f \in L^2(\mathbb{R})).$$

Note that the center of H (also its commutator subgroup) is

$$Z = \big\{(0,0,l) : l \in \mathbb{Z}\big\},$$

and it acts by scalars:

$$\rho_\omega(0,0,l) = e^{2\pi i\omega l}I.$$

The representation $l \mapsto e^{2\pi i\omega l}$ of Z is called the *central character* of ρ_ω. Only those irreducible representations having the same central character will occur in ρ_ω.

Case 1: ω is rational, say $\omega = p/q$ ($p, q \in \mathbb{Z}_+$, $\gcd(p, q) = 1$). Here the central character is trivial on multiples of $(0, 0, q)$, so ρ_ω factors through the group

$$H_q = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_q \qquad (\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}),$$

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Subcase 1a: $\omega \in \mathbb{Z}$, i.e., $q = 1$. Here $H_1 = \mathbb{Z}^2$ with the standard Abelian group structure. Its irreducible representations are one-dimensional; they are the characters

$$\chi_{u,v}(j,k) = e^{2\pi i(ju+kv)}, \qquad u, v \in \mathbb{R}/\mathbb{Z}.$$

Claim:

If $\omega = p \in \mathbb{Z}$, then $\rho_\omega \sim p \int_{(\mathbb{R}/\mathbb{Z})^2}^{\oplus} \chi_{u,v}\, du\, dv$.

The intertwining operator that gives this equivalence is the *Zak transform*. This is a map from (reasonable) functions on $\mathbb{R}$ to functions on $\mathbb{R}^2$ defined by

$$\mathcal{Z}f(u, v) = \sum_{n \in \mathbb{Z}} e^{2\pi i n u} f(v + n).$$

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Note that

$$\mathcal{Z}f(u+m,v) = \mathcal{Z}f(u,v), \qquad \mathcal{Z}f(u,v+m) = e^{-2\pi i m u}\mathcal{Z}f(u,v),$$

so $\mathcal{Z}f$ is determined by its values on $[0,1)\times[0,1)$. Moreover, by the Parseval identity,

$$\int_0^1\int_0^1 |\mathcal{Z}f(u,v)|^2\,du\,dv = \sum_n \int_0^1 |f(v+n)|^2\,dv = \int_{\mathbb{R}} |f(t)|^2\,dt,$$

so $\mathcal{Z}$ is an isometry from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$ which is easily seen to be surjective, hence unitary.

Moreover, since $\rho_p(j,k,l)f(t) = e^{2\pi ipkt}f(t+j)$, we have

$$\begin{aligned}
\mathcal{Z}\rho_p(j,k,l)f(u,v) &= \sum_n e^{2\pi inu}e^{2\pi ipk(v+j)}f(v+j+n) \\
&= \sum_n e^{2\pi i(n-j)u}e^{2\pi ipkv}f(v+n) \\
&= e^{-2\pi iju}e^{2\pi ipkv}\mathcal{Z}f(u,v) \\
&= \chi_{-u,pv}(j,k)\mathcal{Z}f(u,v).
\end{aligned}$$

Thus $\mathcal{Z}$ intertwines ρ_p with

$$\int_{[0,1)^2}^{\oplus} \chi_{-u,pv}\,du\,dv \sim p\int_{(\mathbb{R}/\mathbb{Z})^2} \chi_{u,v}\,du\,dv.$$

Subcase 1b: $q > 1$. This is similar but a little more complicated. H_q is the semi-direct product of the Abelian subgroup $\{(j, 0, 0)\}$ with the normal Abelian subgroup $\{(0, k, l)\}$ which is "regular" in a certain sense, so a standard technique (the "Mackey machine") produces a complete list of inequivalent irreducible representations $\pi_{\alpha,\beta}$ of H_q with central character $l \mapsto e^{2\pi i(p/q)l}$, parametrized by $\alpha, \beta \in (\mathbb{R}/(1/q)\mathbb{Z})$. $\pi_{\alpha,\beta}$ acts on

$$\mathcal{H}_\alpha = \left\{f : \mathbb{Z} \to \mathbb{C} : f(m + kq) = e^{2\pi i\alpha kq} f(m)\right\} \quad \left(\cong \mathbb{C}^q\right)$$

by

$$\pi_{\alpha,\beta}(j, k, l)f(m) = e^{2\pi i\omega l} e^{2\pi ik(\beta+\omega m)} f(m + x).$$

A little Fourier analysis plus a rescaling of the Zak transform shows that

$$\rho_{p/q} \sim \int_{[0,p/q)\times[0,1/q)}^{\oplus} \pi_{\alpha,\beta}\, d\alpha\, d\beta \sim p \int_{[0,1/q)^2}^{\oplus} \pi_{\alpha,\beta}\, d\alpha\, d\beta.$$

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- Define $S : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $S(t) = t + \omega$.
- Given a Borel measure μ on $\mathbb{R}/\mathbb{Z}$, let $\mu_j(E) = \mu(S^j(E))$. μ is *quasi-invariant* (under S) if μ and μ_j are equivalent (mutually absolutely continuous) for all j.
- A Borel measure μ is *ergodic* (under S) if for any S-invariant set E, either E or its complement has μ-measure zero.

Given a σ-finite quasi-invariant ergodic measure μ on $\mathbb{R}/\mathbb{Z}$, define a representation ϕ_μ of H on $L^2(\mu)$ by

$$\phi_\mu(j,k,l)f(t) = e^{2\pi i\omega l}e^{2\pi ikt}\sqrt{(d\mu_j/d\mu)(t)}f(t+\omega j).$$

Then ϕ_μ is irreducible, and $\phi_\mu \sim \phi_\nu$ if and only if $\mu \sim \nu$.

What are the quasi-invariant, ergodic measures μ?

- Counting measure on any orbit of S.
- Lebesgue measure.
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Moreover, for each such μ there are *many other* inequivalent irreducible representations of H on $L^2(\mu)$ with the same central character, coming from nontrivial "cocycles." Again, it seems hopeless to classify them all.

In short, $\{[\pi] \in \widehat{G} : \pi(0, 0, l) = e^{2\pi i \omega l} I\}$ is *enormous* and *cannot be parametrized* in a geometrically nice way.

Let us examine the representations ϕ_μ described above when μ is counting measure on an orbit. Suppose $\beta \in \mathbb{R}/\mathbb{Z}$. If we identify the orbit of β, $\{\beta + m\omega : m \in \mathbb{Z}\}$, with $\mathbb{Z}$, by

$$\beta + m\omega \longleftrightarrow m,$$

ϕ_μ becomes a representation of H on $l^2 = L^2(\mathbb{Z})$ that we call π_β:

$$\pi_\beta(j,k,l)f(m) = e^{2\pi i\omega l}e^{2\pi ik(\beta+m\omega)}f(m+j).$$

The direct integral

$$\pi = \int_{[0,\omega)}^{\oplus} \pi_\beta \, d\beta$$

acts on $L^2([0,\omega)\times\mathbb{Z})$ by

$$\pi(j,k,l)f(\beta,m) = e^{2\pi i\omega l}e^{2\pi ik(\beta+m\omega)}f(\beta,m+j).$$

Define a unitary map $V : L^2(\mathbb{R}) \to L^2([0,\omega)\times\mathbb{Z})$ by

$$Vf(\beta,m) = \frac{1}{\sqrt{\omega}} f\left(\frac{\beta}{\omega} + m\right).$$

Then a simple calculation shows that V intertwines π with ρ_ω.

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And finally,

- *This irreducible decomposition of ρ_ω is far from unique.*

Nonuniqueness

Every $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) = Sp(1,\mathbb{R})$ acts as an automorphism of the real Heisenberg group $H_{\mathbb{R}}$:

$$\Phi_A(x,y,z) = \big(ax+by,\, cx+dy,\, z+\tfrac{1}{2}(acx^2+2bcxy+bdy^2)\big).$$

If $A \in SL(2,\mathbb{Z})$, the restriction of Φ_A to the discrete group H is an automorphism of H if ac and bd are even, and an isomorphism from H to a slightly different discrete subgroup otherwise. Our irreducible representations

$$\pi_\beta(j,k,l)f(m) = e^{2\pi i\omega l}e^{2\pi ik(\beta+m\omega)}f(m+j)$$

of H define irreducible representations of these modified groups too, so $\pi_\beta \circ \Phi_A$ is an irreducible representation of H for any $A \in SL(2,\mathbb{Z})$.

Our representation ρ_ω is the restriction to H of an *irreducible* representation of $H_{\mathbb{R}}$,

$$\rho_\omega(x,y,z)f(t) = e^{2\pi i\omega z}e^{2\pi i\beta y}f(t+x),$$

and $\rho_\omega \circ \Phi_A$ is another such representation with the same central character. By the Stone-von Neumann theorem, $\rho_\omega \sim \rho_\omega \circ \Phi_A$. (The intertwining operator comes from the metaplectic representation of $Sp(1,\mathbb{R})$.)
Hence, for any $A \in SL(2,\mathbb{Z})$,

$$\rho_\omega \sim \rho_\omega \circ \Phi_A \sim \int_{[0,\omega)}^{\oplus} \pi_\beta \circ \Phi_A \, d\beta.$$

But now let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$.

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Proof: $\pi_\beta \circ \Phi_A$ acts on $l^2 = L^2(\mathbb{Z})$ by

$$\pi_\beta \circ \Phi_A(j,k,l)f(m) = e^{2\pi i \omega l} e^{\pi i (acj^2 + 2bcjk + bdk^2)} e^{2\pi i k(\beta + \omega m)} f(m + aj + bk).$$

- If $aj + bk = 0$, $\pi_\beta \circ \Phi_A(j,k,l)$ has discrete spectrum: the canonical basis for l^2 is an eigenbasis.

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- If $aj + bk \neq 0$, $\pi_\beta \circ \Phi_A(j,k,l)$ is a weighted shift operator with weights of modulus 1, so it has *no* discrete spectrum.
- Since $A, A' \in SL(2, \mathbb{Z})$, we have $\gcd(a,b) = \gcd(a',b') = 1$. Hence, if $(a', b') \neq \pm(a, b)$, the equations $aj + bk = 0$ and $a'j + b'k = 0$ define *different* sets of (j,k)'s.

On the other hand, if $(a', b') = \pm(a, b)$, then $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $r \in \mathbb{Z}$, in which case the unitary map on l^2

$$f(m) \mapsto e^{\pi i \omega m^2} e^{\pm 2\pi i \beta r m} f(\pm m)$$

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Finally, given any integers a, b with $\mathrm{gcd(a, b)} = 1$, there exist integers c, d such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.
Hence we have an infinite family of completely inequivalent irreducible decompositions of ρ_ω, parametrized by $(a, b) \in \mathbb{Z}^2$.
This includes families described by Kawakami (1982).