Series of Chromatic Differences

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Outline of talk

Taylor's Series—what's wrong?

History of chromatic derivatives and series

What's wrong with them?

Extension to Slowly growing BL signals

Chromatic Differences and Series

Problem with Taylor's series

(i) $f(t)=\sum^{\infty}_{n=0} f^{(n)}(0)t^n/n!$ converges only locally.

(ii) Representation of bandlimited functions not bandlimited

Ignjatovic (1990) used other derivatives

(*chromatic derivatives*) $p_n(-iD)f(0)$, not $f^{(n)}(0)$

$\{p_n(x)\}$ orthogonal polynomials wrt weight w(x)

Chromatic series

Taylor series replaced by series

$$f(t) = \sum_{n=0}^{\infty} 2\pi \{p_n(-iD)f\}(0)\, \varphi_n(t),$$

$\varphi_n(t)$ inverse Fourier transform

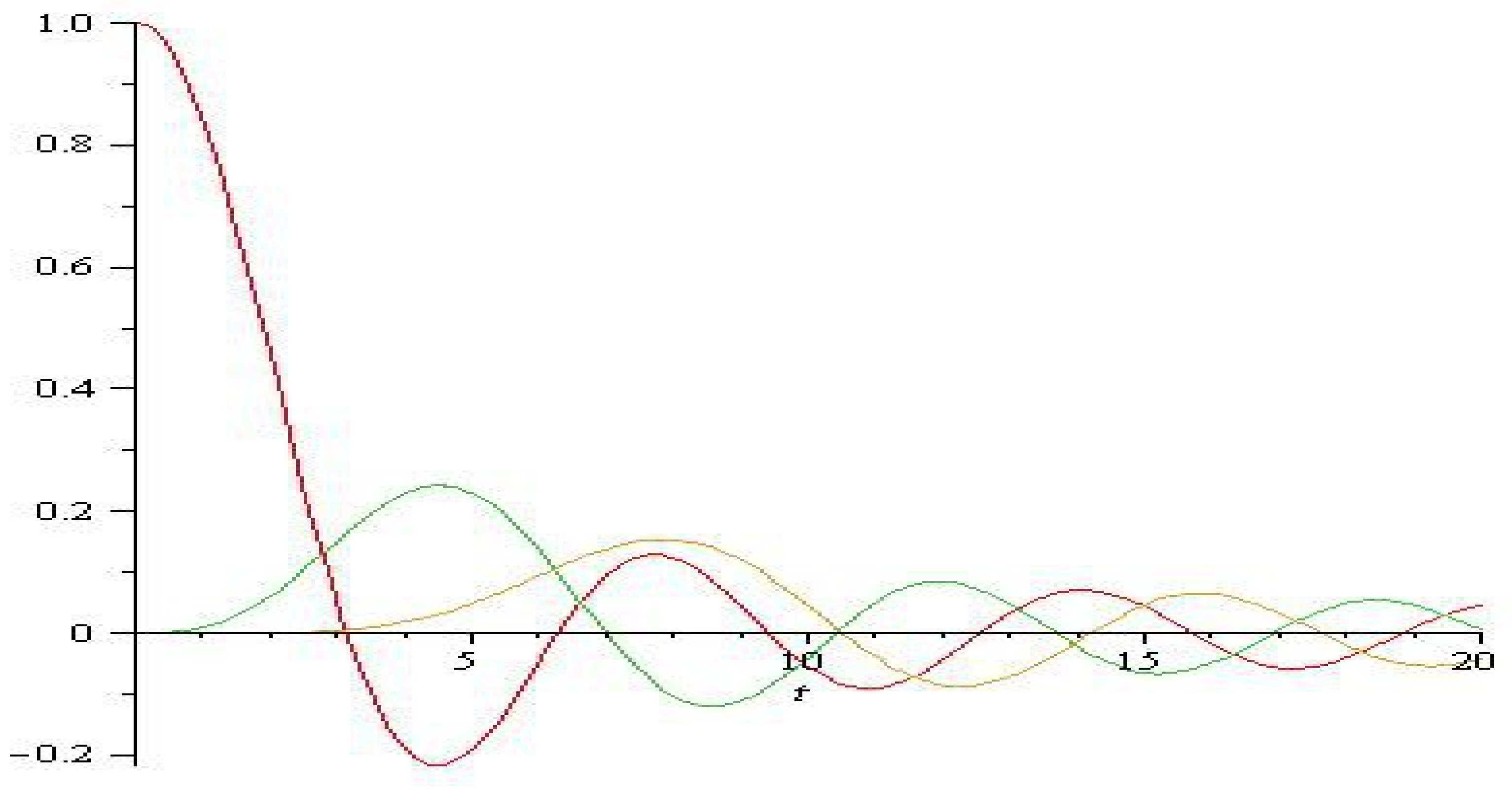
$$\varphi_n(t) := (1/(2\pi)) \int_{-\infty}^{\infty} e^{i\omega t} p_n(\omega) w(\omega)\, d\omega$$

convergence uniform on all of **R**.

(provided *w* has c.s.)

$\varphi_n(t)$ *takes the place of* $t^n/n!$ *in Taylor series*

φ_0, φ_3, φ_6 *look like this:*



Here's how CS works:

Let *g(t)* be function in B_π , *w* supported in *[-π,π]*

Take polynomial expansion of F.T. $\hat{g}(\omega)$ in form

$$\hat{g}=\sum_n \{\int_{-\pi}^{\pi} \hat{g}(\omega)\, p_n(\omega)\, d\omega\}\, p_n\, w$$

Take inverse Fourier transform

$$g(t)=\sum_n \int_{-\pi}^{\pi} \{\ \hat{g}(\omega)\, p_n(\omega)\, d\omega\}\ \ \varphi_n(t)$$

But $\int_{-\pi}^{\pi} \hat{g}(\omega)\, p_n(\omega)\, d\omega = 2\pi\{p_n(-iD)g\}(0)$

since

$$\int_{-\pi}^{\pi} \hat{g}(\omega)\, \omega^n\, d\omega = \int_{-\pi}^{\pi} e^{i\omega t}\, \hat{g}(\omega)\, \omega^n\, d\omega \Big|^{t=0} = 2\pi\{(-iD)^n\, g\}(0)$$

Chromatic series

are globally convergent (for *f* bandlimited)

are bandlimited (if *w* has compact support)

in contrast to Taylor series

Example: Legendre Polynomials

$w(\omega)=\chi_{[-1,1]}(\omega)$,

$P_0(\omega)=1$, $P_1(\omega)=\omega, \ldots,$

$(n+1)P_{n+1}(\omega)=(2n+1)\omega P_n(\omega)-nP_{n-1}(\omega)$,

$\varphi_n(t):=(1/(2\pi)) \int_{-1}^{1} e^{i\omega t} P_n(\omega) d\omega / \|P_n\|^2$

(Spherical Bessel Function)

What's Wrong?

Need input $\{p_n(-iD)g\}(0)$

Need to compute

$$\varphi_n(t) := (1/(2\pi)) \int_{-\infty}^{\infty} e^{i\omega t} p_n(\omega) w(\omega) d\omega$$

Paley - Wiener space B_π doesn't include all signals; e.g., periodic signals, polynomials.

Extending Paley-Wiener Space

Denote by B_{π}^{-m}, m integer ≥ 0,

$\{g \varepsilon C(\mathbf{R}) / \hat{g} \varepsilon S'$ of order m with support in $[-\pi, \pi]$ }.

B_{π}^{-m} includes periodic signals, polynomials, for $m>1$.

Example 1

Let $f(t) = t^j$ for some positive integer j.

Fourier transform of t^j is $2\pi i^j \delta^{(j)}$

and has support $\{0\} \subset [-\pi, \pi]$

Chromatic Derivatives in B_{π}^{-m}

Computations the same in B_{π}^{-m}

i.e., $\{p_n(-iD)g\}(0)$, $\varphi_n(t)$ still needed.

Convergence weaker;

in sense of *S' (tempered distributions).*

S' convergence

<u>Thm</u>. *Let f ε* B_{π}^{-m} *,m integer ≥ 0;*

then chromatic series of f converges

in sense of S' to f.

Not very useful, better to get some

pointwise convergence

Uniform convergence

Thm. *Let $f(z)$ be given by a convergent power series for $|z| < r$;*

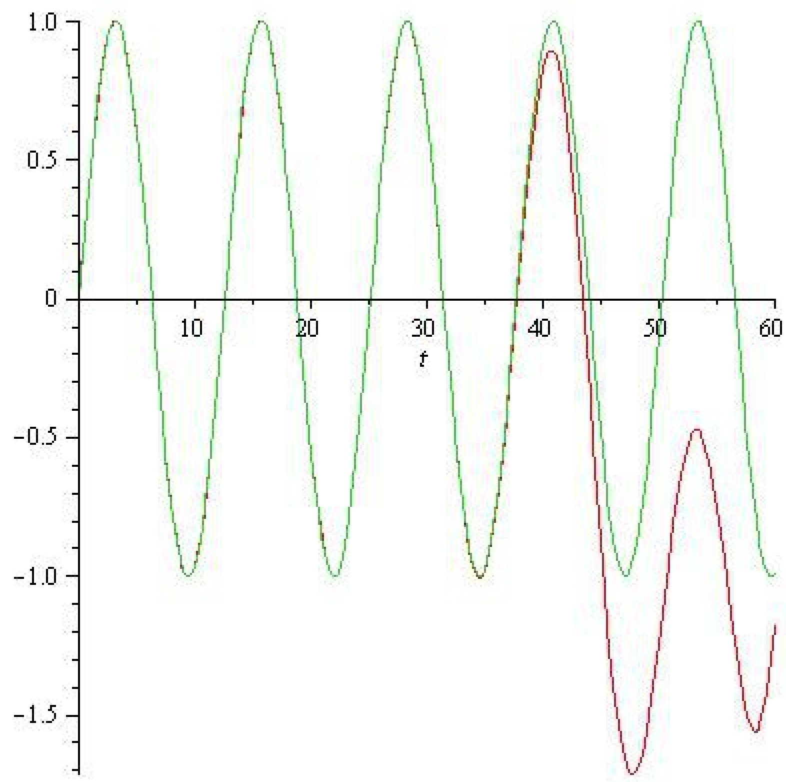
then $f(z)$ has chromatic series uniformly convergent to $f(z)$ on compact subsets of disk.

Examples

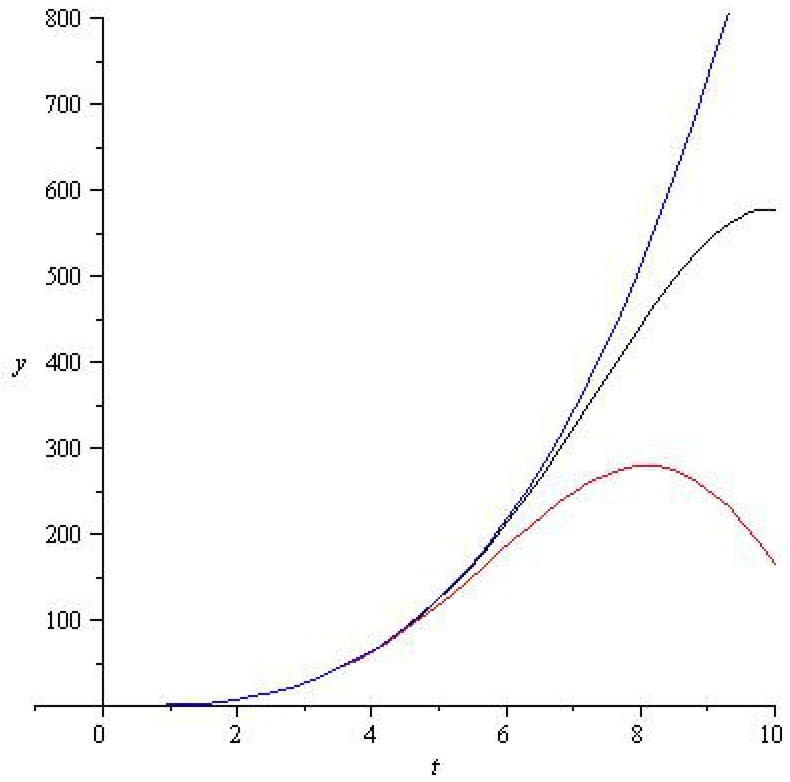
$f_2(t)=sin(t/2)$, then $f_2 \epsilon\ B^{-1}_{\pi}$,

$f_4(t)=t^3$, then $f_4 \epsilon\ B^{-4}_{\pi}$.

f_2 with 12 term partial sum of c.s.



$f_4(t)=t^3$, 3 and 4 term c.s.



Different approach: Chromatic Differences; polynomials orthogonal on circle

Start with $1, z, z^2, \ldots$ and orthogonalize on

$\{|z|=1\}$ with respect to weight function $v(z)/z$,

where $v(e^{i\theta})\chi_\pi(\theta) = w(\theta) \geq 0$ on $[-\pi, \pi]$

Denote by $\{p_n(z)\}$ resulting orthogonal system.

Let $p_n(z)=\Sigma^n_{k=0}\, c_k^{\,n}\, z^n$*; let* $h(t)$ *be* π *bandlimited*;

Then $a_n=\Sigma^n_{k=0}\, c_k^{\,n}\,(h*w^{(-1)})(k)$ *are the*

Chromatic Differences

Let $\psi_n(t) := 1/2\pi \int e^{i\omega t} p_n(e^{-i\omega t})\, w(\omega) d\omega.$

Then $\Sigma^{\infty}_{n=0} a_n \psi_n(t)$ *is*

Discrete chromatic series
of *h(t).*

Example

Take $w(\theta)=\chi_{\pi}(\theta)$

then $p_n(z)=z^n$, or $p_n(e^{i\theta})=e^{in\theta}$, $n=0,1,\ldots$

and $h(t)=\Sigma^{\infty}_{n=0} h(n)s(t-n)$,

where *s(t)* is sinc function.

Problem:

Discrete CS of *h(t)* converges in sense of Paley-Wiener space B_π ,

but doesn't always converge to *h(t).*

Decompostion of B_π

Note example includes only non-negative terms of exponential trig functions.

Define $B_\pi^{\ +} = \{f \,\varepsilon\, B_\pi \mid f^\wedge \varepsilon\, H^2[-\pi,\pi]\ \}$ *for* $w>0$ *on* $[-\pi,\pi]$.

For general w, define $B_w^{\ +} = \{f \,\varepsilon\, B_\pi \mid f^\wedge/w \,\varepsilon\, H^2[-\pi,\pi]\}$.

Discrete Chromatic Series
Convergence result

<u>Prop</u>. *Let* $h \varepsilon B_w^{+}$, $g=\hat{h}/w$, $p_n(z)=\Sigma^n_{k=0} c_k^n z^n$,

$\breve{g}$ *be inverse FT of* g, ψ_n *inv. FT of* $p_n(e^{-i\omega t})\, w$,

$a_n=\Sigma^n_{k=0} c_k^n \breve{g}(k)$;

then $\Sigma^{\infty}_{n=0} a_n \psi_n(t)$ *converges to* $h(t)$

uniformly on compact subsets of ***R.***

More examples:

1) $w(\theta)=((1+\cos\theta)/2)\chi_\pi(\theta)$, (Raised cosine)

Then inv. FT is $\psi(t)= (\sin \pi t)/2\pi t(1-t^2)$ *and*

$$\psi_n(t)=\Sigma^n_{k=0}\, c_k^{\,n}\, \psi(t-k)$$

2) $w(\theta)=(1-\cos^2\theta)^\lambda \chi_\pi(\theta)$, $\lambda>0$

(leads to Gegenbauer polynomial based $p_n(z)$)

Problem: Result only holds for $B_w^{\,+}$ not for all of B_π

Symmetric weight:

Then $\{p_n(z^{-1}) \mid n=1,2,..\}$
is also orthogonal system on circle.

Combine two systems by setting
$p_{-n}(z)=p_n(z^{-1}),\ n=1,2,...$
to get system $\{p_n(z) \mid n=0,\pm1,\pm2,..\}$

Then $\{p_n(e^{i\theta})\}_{n=-\infty}^{\infty}$ *is* <u>*Riesz basis*</u> *of* $L^2(w,[-\pi,\pi])$
and $\{\psi_n(t)\}$ *is Riesz basis of* B_π
(under certain conditions on w)

Other approach:

Orthogonalize $1, z^1, z^{-1}, z^2, \ldots$ on unit circle
with respect to weight $w(\theta)=\alpha(e^{i\theta})$
to get orthonormal system $\{\varphi_n\}$.

Prop. *Let $w(\theta)=w(-\theta) \geq 0$ on $[-\pi,\pi]$, then $\{\varphi_n\}$ is orthonormal basis of $L^2(w,[-\pi,\pi])$ and*

$$\varphi_n(e^{i\theta})=\Sigma_{k=-|n|}^{|n|} a_{k,n}e^{ik\theta}.$$

Discrete chromatic series on B_π

Thm. *Let* $h \in B_\pi$ *with* $g=\hat{h}/w \in L^2[-\pi,\pi]$*;*
then

$$h(t)= \Sigma^{\infty}_{n=-\infty}\Sigma^{|n|}_{k=-|n|}a_{k,n}\, g(k)\psi_k(t)$$

where

$$\psi_k(t):= 1/2\pi\int e^{i\omega t}\varphi_k(e^{-i\omega t})\, w(\omega)d\omega$$

and convergence is in $L^2(\mathbf{R})$ *and uniformly in* $\mathbf{R}$.

S' convergence

<u>Thm</u>. *Let f ε* $B_{\pi-\epsilon}^{-m}$ *,m integer ≥ 0, let w be trig polynomial*
э: $w(\theta)>0$ *on* $(-\pi,\pi)$ *&* $w^{(k)}(\pm\pi)=0, k\le m$*;*

Then discrete chromatic series of f converges in sense of S' to f and uniformly on compact sets.

Some references

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