The abc-problems for Gabor systems and for sampling

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February Fourier Talks 2013

February 22, 2013

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Thank

the organizing committee

for inviting me to give a talk on February Fourier Talks 2013.

Abstract

In this presentation, I will talk about

- full classification of ideal windows such that the corresponding Gabor system is a Gabor frame (the abc-problem for Gabor systems)
- full classification of box impulse response such that signals in the corresponding shift-invariant space are recoverable from their uniform samples (the abc-problem for sampling)

Supported partially by the National Science Foundation

This presentation is based on a joint paper with Xinrong Dai (Sun Yat-sen University, China)

Gabor system

$$T_a : f(t) \longmapsto f(t-a)$$

(translation in time domain) and

$$M_\omega : f(t) \longmapsto e^{2\pi i\omega t} f(t)$$

(Modulation=translation in Fourier domain)

Gabor system associated with window function ϕ and time-frequency shift lattice $a\mathbb{Z} \times \mathbb{Z}/b$:

$$\mathcal{G}(\phi, a\mathbb{Z} \times \mathbb{Z}/b) = \{\phi_{m,n} := M_{n/b} T_{ma}\phi := \exp(-2\pi int/b)\phi(t-ma) : (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$$

A fundamental problem in time-frequency analysis

Problem: Find triples (ϕ, a, b) such that

$$\mathcal{G}(\phi, a\mathbb{Z}\times\mathbb{Z}/b) = \{\phi_{m,n} := \exp(-2\pi int/b)\phi(t-ma) : \ (m,n) \in \mathbb{Z}\times\mathbb{Z}\}$$

is a frame for L^2, that is,

$$A\|f\|_2^2 \le \sum_{m,n\in\mathbb{Z}} |\langle f, \phi_{m,n}\rangle|^2 \le B\|f\|_2^2, \quad f \in L^2.$$

Full classification of time-frequency shift lattices $a\mathbb{Z} \times \mathbb{Z}/b$ only known for very few window functions ϕ in last twenty years.

For the Gabor system $\mathcal{G} := \{\phi(\cdot - ma)e^{2\pi int/b}\}$ associated with Gaussian window $\phi(t) = \exp(-t^2/2)$

- Von Neumann claim (1932): span of $\{\phi(t-m)e^{2\pi int} : m, n \in \mathbb{Z}\}$ is dense in L^2 (Confirmed 1971)
- Gabor conjecture (1946): $f = \sum_{mn} c_{mn}(f)\phi(t-m)e^{2\pi int}$ (confirmed 1981, convergence in distributional sense)
- Daubechies and Grossman conjecture: $\{\phi(t-ma)e^{2\pi int/b}\}_{m,n}$ is a frame if and only if $a/b < 1$. (Confirmed by Lyubarskii (1992), Seip and Wallsten (1992/93))

- Daubechies, Grossman, Meyer (1986): there exists ϕ such that $\phi(t-ma)e^{2\pi int/b}$ is a frame if $a/b<1$
- Janssen and Strohmer (2002): Hyperbolic secant, two-sided exponential $\phi(t)=e^{-|t|}$, one-side exponential
- Gröchenig and Stöckler (2011): positive definitive function of finite type
- Janssen (2001, 2003), Gu and Han (2008): Ideal window (Janssen Tie)
- Feichtinger and Kabilinger(2004): Openness when ϕ has certain regularity in time-frequency domain.

An example

The ideal window function $\phi(x) = \chi_{[0,a)}$ on the interval $[0,a)$; lattice $a\mathbb{Z} \times \mathbb{Z}/a$ (i.e. $a = b$). In this case, $\{\phi_{m,n}\}$ is an orthonormal basis.

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} |\langle f, \phi_{m,n}\rangle|^2 &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \Big| \int_{ma}^{(m+1)a} f(t)e^{-2\pi itn/a} dt\Big|^2 \\
&= a^2 \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \Big| \int_0^1 f(ma+sa)e^{-2\pi itn} ds\Big|^2 \\
&= a^2 \sum_{m\in\mathbb{Z}} \int_0^1 |f(ma+at)|^2 dt = a\|f\|_2^2.
\end{aligned}$$

Gabor system with ideal window

Natural Question: When does the Gabor system $\{M_{n/b}T_{ma}\chi_I\}$ associated with the ideal window function χ_I on the interval I form a frame for L^2?

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Natural Question: When does the Gabor system $\{M_{n/b}T_{ma}\chi_I\}$ associated with the ideal window function χ_I on the interval I form a frame for L^2?

Jassen's tie suggests that it could be "arbitrarily complicated"!

In this talk, I will present a complete answer to the above question.

Reduction: By shift-invariance, $\mathcal{G}(\chi_I, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame if and only if $\mathcal{G}(\chi_{I+d}, a\mathbb{Z} \times \mathbb{Z}/b)$ for all d. Thus we may assume that $I = [0, c)$ (c=length of the interval I).

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Recall our question: When does the Gabor system $\{M_{n/b}T_{ma}\chi_I\}$ associated with the ideal window function χ_I on the interval I form a frame for L^2? It suffices to solve

The abc problem for Gabor systems: **Classifying all triples (a,b,c) of positive numbers (time shift parameter a, frequency shift parameter b, interval length c) such that $\mathcal{G}:=\{\chi_{[0,c)}(\cdot-ma)e^{2\pi int/b}\}$ is a frame for L^2.**

Before we provide an answer to the above abc problem for Gabor system, let us take a look another problem in sampling.

Shannon Sampling

Sampling theorem for bandlimited signals[1]: If a function $x(t)$ contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced $1/(2B)$ seconds apart.

Let $PW_\pi = \{\sum_k c(k)\mathrm{sinc}(t-k) : \sum_k |c(k)|^2 < \infty\}$ where $\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t}$. Then for all $0 < a \leq 1$ and $t \in \mathbb{R}$,

$$\|f\|_2^2 \approx \sum_k |f(t+ak)|^2 \quad f \in PW_\pi.$$

($a = 1$ for exact sampling, and $a < 1$ for oversampling. Observation: Oversampling leads to a more stable recovery)

[1]C. E. Shannon, "Communication in the presence of noise", Proc. Institute of Radio Engineers, vol. 37, no. 1, pp. 1021, Jan. 1949.

Spline space:

$$V_n = \{\sum_k c(k)B_n(t-k) : \sum_k |c(k)|^2 < \infty\}$$

($B_0 = \chi_{[0,1)}$, $B_1(t) = (1-|t|)_+$ hat function). Then for $0 < a < 1$, and $t \in \mathbb{R}$,

$$\|f\|_2^2 \approx \sum_k |f(t+ak)|^2 \quad f \in PW_\pi.$$

($a < 1$ for oversampling. Observation: oversampling leads to a local reconstruction) (Aldroubi and Gröchenig 2000; Sun 2010)

Does oversampling always help?

Giving an impulse response function ϕ, define the shift-invariant space

$$V(\phi, b) = \{\sum_k c(k)\phi(t - kb) : \sum_k |c(k)|^2 < \infty\}.$$

Does oversampling $a < b$ help for recovering signals in a shift-invariant space? For $0 < a < b$, and $t \in \mathbb{R}$,

$$\|f\|_2^2 \approx \sum_k |f(t + ak)|^2 \quad f \in V(\phi, b).$$

(Any time signal in $V(\phi, b)$ can be recovered from its uniform samples spaced every a seconds apart).

The abc-problem for sampling

Let us take a look at the example generated by the characteristic function χ_I with interval length $c = |I|$. Define

$$V(\chi_I, b) = \{\sum_k c(k)\chi_I(t - kb) : \sum_k |c(k)|^2 < \infty\}.$$

As $f \in V(\chi_I, b)$ if and only if $f(t + d) \in V(\chi_{I+d}, b)$, it suffices to consider the following abc **problem for sampling**:

Find all triples (a, b, c) such that any time signal in $V_{[0,c)}(\phi, b)$ can be stably recovered from its uniform samples spaced every a seconds apart.

The "almost" equivalence between the abc-problem for Gabor systems and for sampling

Theorem 1. *Let $a < b < c$ and $c/b \notin \mathbb{Z}$. Then the following are equivalent:*

- *all time signal in $V_{[0,c)}(\phi, b)$ can be recovered from its uniform samples spaced every a seconds apart*
- *$\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a frame for L^2.*

Conclusions: Oversampling does not always help for more stable recovery. Oversampling problem could be "arbitrary complicated".

The abc-problem for Gabor system

Due to the above equivalence between the abc-problem for Gabor systems and the abc-problem for sampling, from now on, we just work on the abc problem for Gabor system:

Classifying all triples (a, b, c) of positive numbers such that $\mathcal{G} := \{\chi_{[0,c)}(\cdot - ma)e^{2\pi int/b}\}$ is a frame for L^2

(time shift parameter a, frequency shift parameter b, interval length c).

The abc-problem I: Easy cases

Theorem 2. *Let (a, b, c) be a triple of positive numbers, and let $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ be the Gabor system generated by the characteristic function on the interval $[0, c)$.*

(I) If $a > c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame.

(II) If $a = c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame if and only if $a \le b$. (Previous example: $a = b = c = 1$)

(III) If $a < c$ and $b \le a$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame.

(IV) If $a < c$ and $b \ge c$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is a Gabor frame.

It remains to consider $a < b < c$.

The abc-problem II: Infinite matrices

For considering the case $a < b < c$, we introduce infinite matrices:

$$\mathbf{M}_{a,b,c}(t) := \big(\chi_{[0,c)}(t - \mu + \lambda)\big)_{\mu \in a\mathbb{Z}, \lambda \in b\mathbb{Z}}, \ \ t \in \mathbb{R}. \tag{1}$$

$$\mathbf{M}_{a,b,c}(0) = \left(\begin{array}{ccccccccccccc}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & & \\
 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & & \\
 & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & & \\
 & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\
 & & & & & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right) \tag{2}$$

for the triple $(a, b, c) = (\pi/4, 1, 23 - 11\pi/2)$ We define

$$\mathcal{D}_{a,b,c} = \{t : \mathbf{M}_{a,b,c}(t)\mathbf{v} = (\cdots, 2, 2, 2, \cdots)^T \ for\ some\ \mathbf{v} \in \mathcal{B}_b^0\}$$

where $\mathcal{B}_b^0 = \{(\mathbf{v}(\mu)) : \ \mathbf{v}(\mu) \in \{0, 1\}, \mathbf{v}(0) = 1\}$ is the set of all binary vectors. We show that if $a < b < c$ then

$$\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b) \text{ is a frame } \Leftrightarrow \mathcal{D}_{a,b,c} = \emptyset.$$

Applying the above equivalence, we could take one step forward in the way to solve the abc-problem for Gabor systems.

Theorem 3. *(Dai and S.) Set* $c_0 = c - \lfloor c/b \rfloor b$.

(V) *If* $c_0 \geq a$ *and* $c_0 \leq b - a$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame. (Janssen 03)*

(VI) *If* $c_0 \geq a$ *and* $c_0 > b - a$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is not a Gabor frame if and only if* $a/b = p/q$ *for some coprime integers, and either*

1) $c_0 > b - \gcd(\lfloor c/b \rfloor + 1, p)b/q$ *and* $\gcd(\lfloor c/b \rfloor + 1, p) \neq \lfloor c/b \rfloor + 1$, *or*
2) $c_0 > b - \gcd(\lfloor c/b \rfloor + 1, p)b/q + b/q$ *and* $\gcd(\lfloor c/b \rfloor + 1, p) = \lfloor c/b \rfloor + 1$.

(VII) *If* $c_0 < a$ *and* $c_0 \leq b - a$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is not a Gabor frame if and only if either*

3) $c_0 = 0$; *or*
4) $a/b = p/q$ *for some coprime integers* p *and* q, $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q$ *and* $\gcd(\lfloor c/b \rfloor, p) \neq \lfloor c/b \rfloor$; *or*
5) $a/b = p/q$ *for some coprime integers* p *and* q, $0 < c_0 < \gcd(\lfloor c/b \rfloor, p)b/q - b/q$ *and* $\gcd(\lfloor c/b \rfloor, p) = \lfloor c/b \rfloor$.

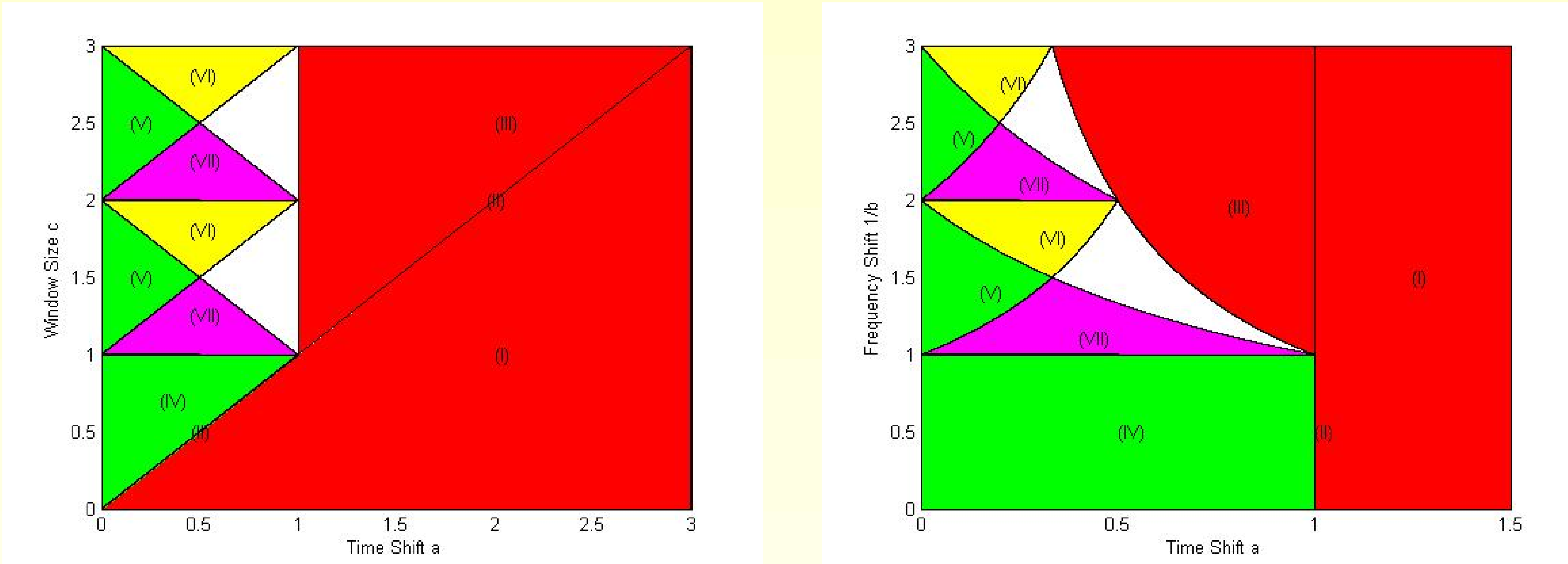

Figure 1: Needles growing from flooring and hanging from ceiling at rational positions, $b = 1$.

Next we consider $a < b < c$ and $b - a < c_0 < a$.

The abc-problem III: Maximal invariant Sets

Recall that: $\mathcal{D}_{a,b,c} = \{t : \mathbf{M}_{a,b,c}(t)\mathbf{v} = (\cdots, 2, 2, 2, \cdots)^T \; for\ some\ \mathbf{v} \in \mathcal{B}_b^0\}$. We introduced another set:

$$\mathcal{S}_{a,b,c} = \{t : \mathbf{M}_{a,b,c}(t)\mathbf{v} = (\cdots, 1, 1, 1, \cdots)^T \; for\ some\ \mathbf{v} \in \mathcal{B}_b^0\}$$

where $\mathcal{B}_b^0 = \{(\mathbf{v}(\mu)) : \ \mathbf{v}(\mu) \in \{0,1\}, \mathbf{v}(0) = 1\}$ is the set of all binary vectors.

The set $\mathcal{S}_{a,b,c}$ just introduced is a supset of $\mathcal{D}_{a,b,c}$ and

$$\begin{aligned} \mathcal{D}_{a,b,c} &= \Big(\mathcal{S}_{a,b,c} \cap ([0, c_0 + a - b) + a\mathbb{Z}) \cap (\mathcal{S}_{a,b,c} - \lfloor c/b \rfloor b)\Big) \\ &\quad \cup \Big(\mathcal{S}_{a,b,c} \cap (\cup_{\lambda \in [b, (\lfloor c/b \rfloor - 1)b] \cap b\mathbb{Z}} (\mathcal{S}_{a,b,c} - \lambda))\Big). \end{aligned}$$

We define piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$ on the real line:

$$R_{a,b,c}(t) := \begin{cases} t + \lfloor c/b \rfloor b + b & \text{if } t \in [0, c_0 + a - b) + a\mathbb{Z} \\ t & \text{if } t \in [c_0 + a - b, c_0) + a\mathbb{Z} \\ t + \lfloor c/b \rfloor b & \text{if } t \in [c_0, a) + a\mathbb{Z}, \end{cases} \tag{3}$$

and

$$\tilde{R}_{a,b,c}(t) := \begin{cases} t - \lfloor c/b \rfloor b & \text{if } t \in [c - a, c - c_0) + a\mathbb{Z} \\ t & \text{if } t \in [c - c_0, c + b - c_0 - a) + a\mathbb{Z} \\ t - \lfloor c/b \rfloor b - b & \text{if } t \in [c + b - c_0 - a, c) + a\mathbb{Z}, \end{cases} \tag{4}$$

Here we assume that $0 < c_0 + a - b < c_0 < a$ and $c_0 = c - \lfloor c/b \rfloor b$.

Both $\mathcal{D}_{a,b,c}$ and $\mathcal{S}_{a,b,c}$ are invariant under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$,

$$R_{a,b,c}E \subset E \text{ and } \tilde{R}_{a,b,c}E \subset E$$

and they have empty intersection with the black holes $[c_0+a-b, c_0)+a\mathbb{Z}$ and $[c-c_0, c-c_0+b-a)+a\mathbb{Z}$ of the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$,

$$E \cap ([c_0+a-b, c_0)+a\mathbb{Z}) = E \cap ([c-c_0, c-c_0+b-a)+a\mathbb{Z}) = \emptyset.$$

Our extremely important observation is that $\mathcal{S}_{a,b,c}$ is the **maximal** set that is invariant under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and that has empty intersection with their black holes $[c_0+a-b, c_0)+a\mathbb{Z}$ and $[c-c_0, c-c_0+b-a)+a\mathbb{Z}$.

Theorem 4. *(Dai and S.) Set* $c_1 = \lfloor c/b \rfloor b - \lfloor (\lfloor c/b \rfloor b/a) \rfloor a$.

(VIII) If $\lfloor c/b \rfloor = 1$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame. (Janssen03, Han and Gu 08)*

(IX) If $\lfloor c/b \rfloor \geq 2$ *and* $c_1 > 2a - b$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame.*

(X) If $\lfloor c/b \rfloor \geq 2$ *and* $c_1 = 2a - b$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame if and only if* $a/b = p/q$ *for some coprime integers* p *and* q, $c_0 \leq b - a + b/q$ *and* $\lfloor c/b \rfloor + 1 = p$.

(XI) If $\lfloor c/b \rfloor \geq 2$ *and* $c_1 = 0$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame if and only if* $a/b = p/q$ *for some coprime integers* p *and* q, $c_0 \geq a - b/q$ *and* $\lfloor c/b \rfloor = p$.

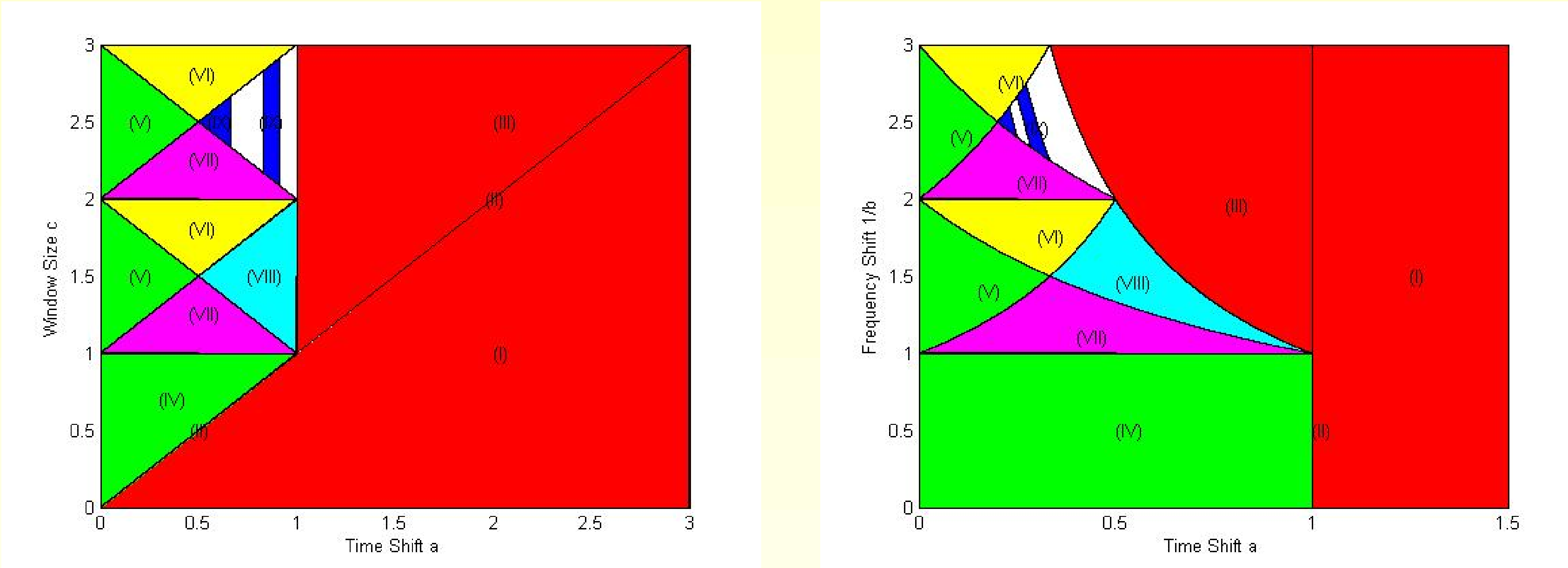

Figure 2: Needles grows sometimes on the boundary. $b = 1$

Next consider: $a < b < c, b - a < c_0 < a, \lfloor c/b \rfloor \geq 2$ and $0 < c_1 < 2a - b$.

The abc-problem III: holes-removal surgery

Recall that $\mathcal{S}_{a,b,c}$ is the **maximal** set that is invariant under the piecewise linear transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$, and that has empty intersection with their black holes $[c_0+a-b, c_0)+a\mathbb{Z}$ and $[c-c_0, c-c_0+b-a)+a\mathbb{Z}$.

We related the set $\mathcal{S}_{a,b,c}$ to the dynamic system associated with the transformations $R_{a,b,c}$ and $\tilde{R}_{a,b,c}$: The black hole of the piecewise linear transformation $R_{a,b,c}$ attracts the black hole of the piecewise linear transformation $\tilde{R}_{a,b,c}$ when applying the piecewise linear transformation $R_{a,b,c}$ finitely many times

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If $\mathcal{S}_{a,b,c}\neq\emptyset$, then there exists $N\geq 0$ such that

$$\mathbb{R}\backslash\mathcal{S}_{a,b,c}=\cup_{n=0}^{N}(R_{a,b,c})^n([c-c_0, c+b-c_0-a)+a\mathbb{Z})$$

and

$$(R_{a,b,c})^N([c-c_0, c+b-c_0-a)+a\mathbb{Z}) = [0, c_0+a-b)+a\mathbb{Z}.$$

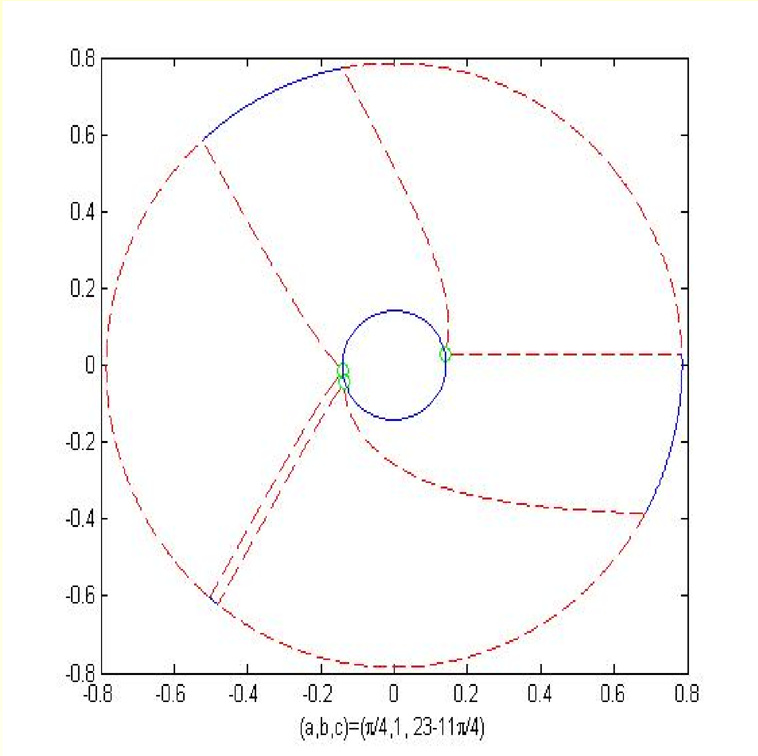

Figure 3: The maximal invariant set $\mathcal{S}_{a,b,c} = \left[18 - \frac{23\pi}{4}, 11 - \frac{7\pi}{2}\right) \cup \left[12 - \frac{15\pi}{4}, 5 - \frac{3\pi}{2}\right) \cup \left[6 - \frac{7\pi}{4}, 17 - \frac{21\pi}{4}\right) + \frac{\pi}{4}\mathbb{Z}$ for the triple $(a, b, c) = (\pi/4, 1, 23 - 11\pi/2)$ consists of intervals of different lengths on one period,

Recall: The maximal invariant set $\mathcal{S}_{a,b,c}$ is the complement of finitely many mutually disjoint periodic holes.

Hole-removal surgery:

- We squeeze out those holes from the line and then reconnect their endpoints.
- After performing the above holes-removal surgery, the maximal invariant set $\mathcal{S}_{a,b,c}$ becomes the real line with marks (image of the holes).

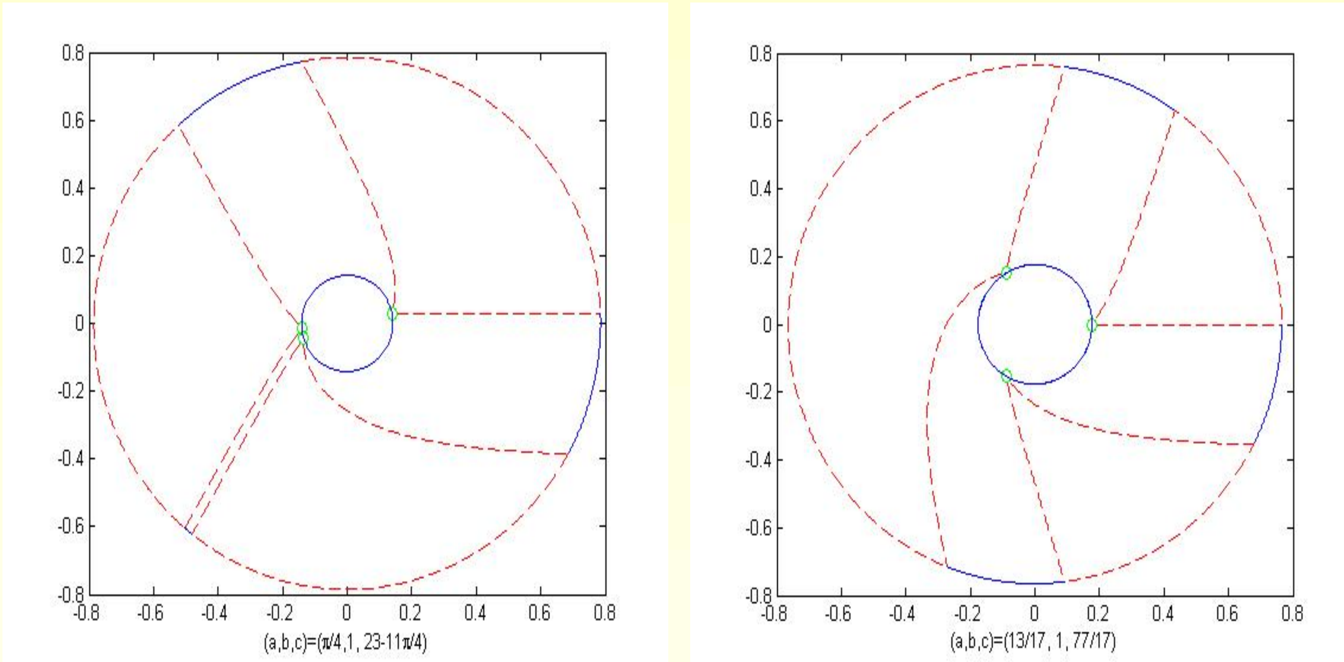

Figure 4: $a\mathbf{T} \ni a\exp(2\pi it/a) \longmapsto Y_{a,b,c}(a)\exp\left(-2\pi iY_{a,b,c}(t)/Y_{a,b,c}(a)\right) \in Y_{a,b,c}(a)\mathbb{T}$. The blue arcs in the big circle are contained in $a\exp(2\pi i\mathcal{S}_{a,b,c}/a)$, the red dashed arcs in the big circle belong to $a\exp(2\pi i(\mathbb{R}\backslash\mathcal{S}_{a,b,c})/a)$, and the circled marks in the small circle are $Y_{a,b,c}(a)\exp\left(-2\pi iK_{a,b,c}/Y_{a,b,c}(a)\right)$, where $K_{a,b,c}$ is the set of all marks on the line.

The above holes-removal surgery could be described by the one-to-one map $Y_{a,b,c}$ in the sense that the map $Y_{a,b,c}$ in

$$Y^d_{a,b,c}(t) = \begin{cases} |[0,t) \cap \mathcal{S}_{a,b,c}| & \text{if } t \geq 0, \\ -|[t,0) \cap \mathcal{S}_{a,b,c}| & \text{if } t \leq 0, \end{cases} \tag{5}$$

is an isomorphism from the set $\mathcal{S}_{a,b,c}$ to the line with marks.

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is an isomorphism from the set $\mathcal{S}_{a,b,c}$ to the line with marks.

Moreover, after performing holes-removal surgery,

- the restriction of piecewise linear transformation $R_{a,b,c}$ onto the maximal invariant set $\mathcal{S}_{a,b,c}$ becomes a linear isomorphism on the quotient group $\mathbb{R}/Y_{a,b,c}(a)\mathbb{Z}$;

i.e., the following diagram commutes,

$$\begin{array}{ccc} \mathcal{S}_{a,b,c} & \xrightarrow{R_{a,b,c}} & \mathcal{S}_{a,b,c} \\ {\scriptstyle Y_{a,b,c}}\downarrow & & \downarrow{\scriptstyle Y_{a,b,c}} \\ \mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z}) & \xrightarrow[S(\theta_{a,b,c})]{} & \mathbb{R}/(Y_{a,b,c}(a)\mathbb{Z}) \end{array}$$

- If $\mathcal{S}_{a,b,c} \neq \emptyset$, $a/b = p/q$ and $c \in b\mathbb{Z}/q$ for some coprime integers p and q, then marks on the line form a finite cyclic group generated by $Y_{a,b,c}(c_1 + b - a) + Y_{a,b,c}(a)\mathbb{Z}$,

$$K_{a,b,c} = Y_{a,b,c}(c_1 + b - a)\mathbb{Z} + Y_{a,b,c}(a)\mathbb{Z}, \tag{6}$$

where $K_{a,b,c}$ is the set of marks on the line.

The abc-problem III: Augmentation operation

- Holes-removal surgery: from the maximal invariant set $\mathcal{S}_{a,b,c}$ to line with marks

- Augmentation surgery: from line with marks to maximal invariant set $\mathcal{S}_{a,b,c}$
  - (irrational ratio a/b): the length, position of the first mark, the last mark and the number of marks. (the middle marks could be determined by the transformation $R_{a,b,c}$)
  - (rational ratio a/b): the length, position of the first mark, the number of marks, the lengths of big holes and small holes and left(right) side of marks for inserting holes.

We parameterize the augmentation surgery process and obtain the following final pieces to the full classification of triples (a, b, c) such that the Gabor system $\{\chi_{[0,c)}(\cdot - ma)e^{2\pi itn/b}$ is a frame. (Two integer parameters for $a/b \notin \mathbb{Q}$ and four integer parameters for $a/b \in \mathbb{Q}$.)

Theorem 5. *(Dai and S.)*

(XII) *If* $a/b \notin \mathbb{Q}$, *then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is not a Gabor frame if and only if there exist nonnegative integers* d_1 *and* d_2 *such that (a)* $a \neq c - (d_1 + 1)(\lfloor c/b \rfloor + 1)(b - a) - (d_2 + 1)\lfloor c/b \rfloor (b - a) \in a\mathbb{Z}$; *(b)* $\lfloor c/b \rfloor b + (d_1 + 1)(b - a) < c < \lfloor c/b \rfloor b + b - (d_2 + 1)(b - a)$; *and (c)* $\# E_{a,b,c} = d_1$, *where* $m = ((d_1 + d_2 + 1)c_1 - c_0 + (d_1 + 1)(b - a))/a$ *and*

$$E_{a,b,c} = \Big\{ n \in [1, d_1 + d_2 + 1] \Big| \; n(c_1 - m(b - a)) \in [0, c_0 - (d_1 + 1)(b - a)) + (a - (d_1 + d_2 + 1)(b - a))\mathbb{Z} \Big\}.$$

Theorem 6 *(XIII) If $a/b = p/q$ for some coprime integers p and q, and $c \in b\mathbb{Z}/q$, then $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ is not a Gabor frame if and only if the triple (a, b, c) satisfies one of the following three conditions:*

6) $c_0 < \gcd(a, c_1)$ *and* $\lfloor c/b \rfloor(\gcd(a, c_1) - c_0) \neq \gcd(a, c_1)$.

7) $b - c_0 < \gcd(a, c_1 + b)$ *and* $(\lfloor c/b \rfloor + 1)(\gcd(a, c_1 + b) + c_0 - b) \neq \gcd(a, c_1 + b)$.

8) There exist nonnegative integers d_1, d_2, d_3, d_4 *such that (a)* $0 < a - (d_1 + d_2 + 1)(b - a) \in Nb\mathbb{Z}/q$; *(b)* $Nc_1 + (d_1 + d_3 + 1)(b - a) \in a\mathbb{Z}$; *(c)* $(d_1 + d_2 + 1)(Nc_1 + (d_1 + d_3 + 1)(b - a)) - (d_1 + d_3 + 1)a \in Na\mathbb{Z}$; *(d)* $\gcd(Nc_1 + (d_1 + d_3 + 1)(b - a), Na) = a$; *(e)* $\#E^d_{a,b,c} = d_1$; *(f)* $c_0 = (d_1+1)(b-a)+(d_1+d_3+1)(a-(d_1+d_2+1)(b-a))/N+\delta$ *for some* $-\min(a-c_0, (a-(d_1+d_2+1)(b-a))/N) < \delta < \min(c_0+a-b, (a-(d_1+d_2+1)(b-a))/N)$; *and (g)* $|\delta| + a/(N\lfloor c/b \rfloor + (d_1 + d_3 + 1)) \neq (a - (d_1 + d_2 + 1)(a - b))/N$, *where* $N := d_1 + d_2 + d_3 + d_4 + 2$ *and* $E^d_{a,b,c}$ *is defined by*

$$E^d_{a,b,c} = \Big\{n \in [1, d_1 + d_2 + 1] \Big|\ n(Nc_1 + (d_1 + d_3 + 1)(b - a)) \in (0, (d_1 + d_3 + 1)a) + Na\mathbb{Z}\Big\}. \quad (7)$$

(XIV) If $a < b < c, b - a < c_0 < a, \lfloor c/b \rfloor \geq 2$*,* $0 < c_1 < 2a - b$*,* $a/b = p/q$ *for some coprime integers* p *and* q*, and* $c \notin b\mathbb{Z}/q$*, then* $\mathcal{G}(\chi_{[0,c)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *is a Gabor frame if and only if both* $\mathcal{G}(\chi_{[0,\lfloor qc/b \rfloor b/q)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *and* $\mathcal{G}(\chi_{[0,\lfloor qc/b+1 \rfloor b/q)}, a\mathbb{Z} \times \mathbb{Z}/b)$ *are Gabor frames. (Janssen 03)*

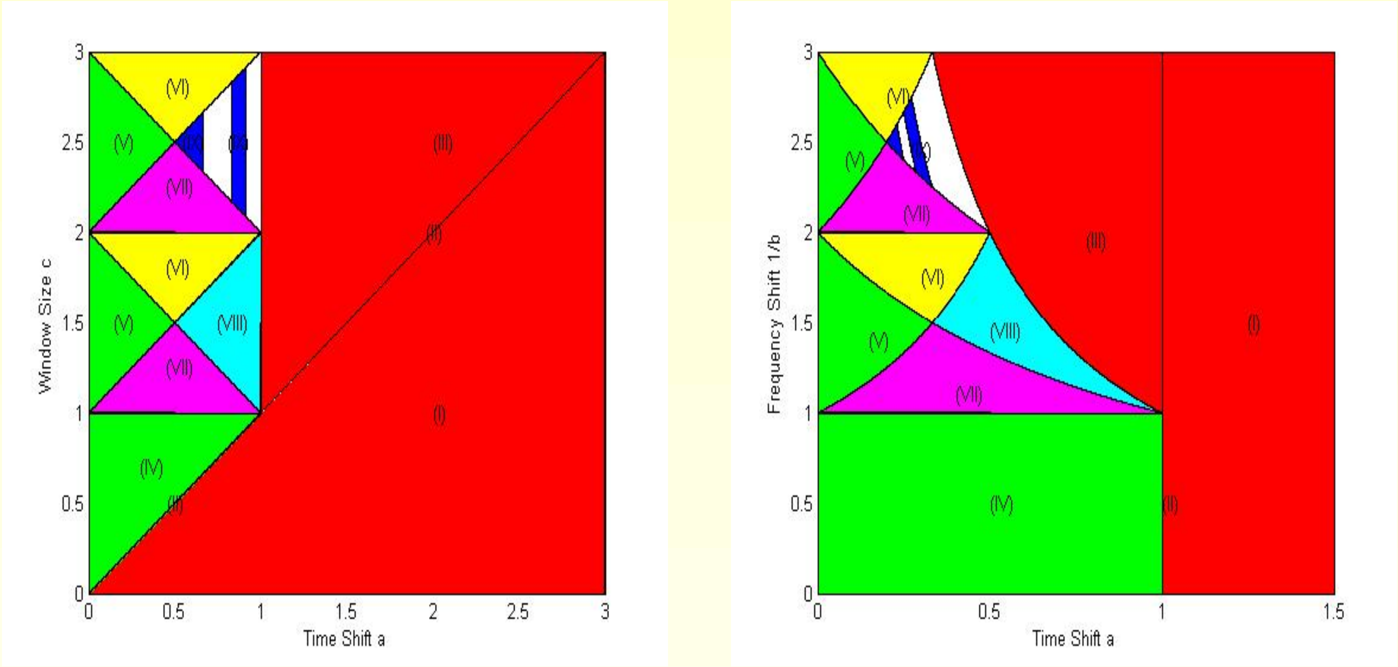
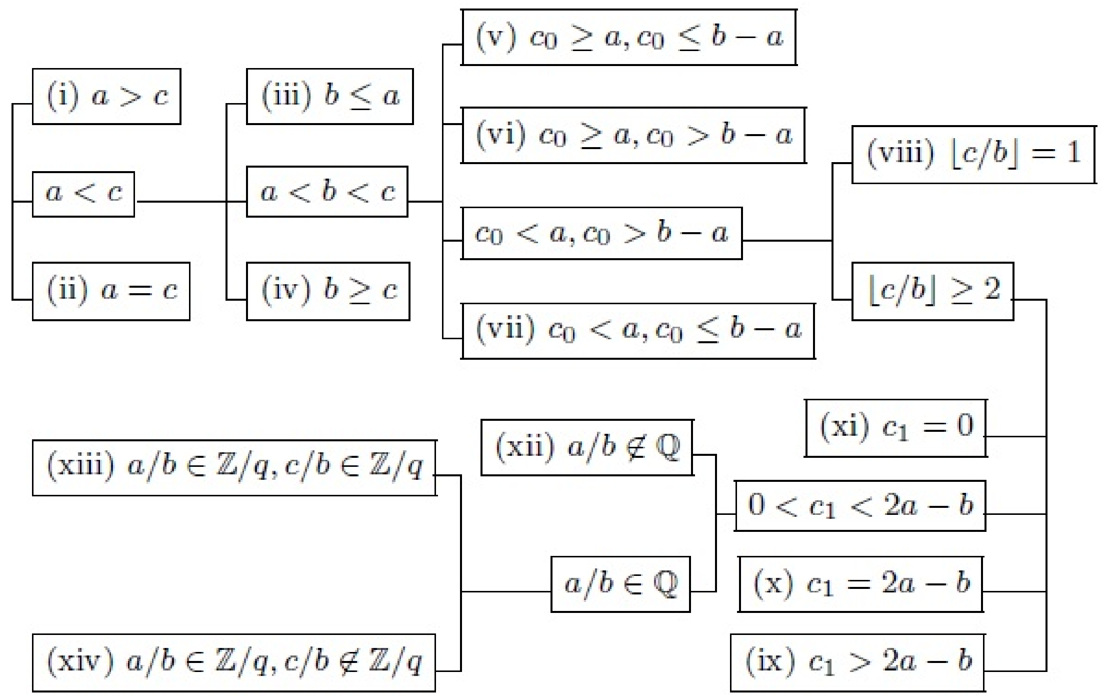


Figure 5: (Left $b = 1$): The red, green, blue, grey, purple, yellow regions in the conclusions (i) and (iii), (iv) and (v), (vi), (vii), (viii) and (ix) respectively. Needles growing from flooring and hanging from ceiling at rational positions. In the white region, some needles (line segments) on the vertical line growing from rational time shift locations and few needle holes (points) on the vertical line located at irrational time shifts. (Right $c = 1$)



Conclusion: The abc problem for Gabor systems and for sampling are completely solved.

Summary

- The abc problem for Gabor systems and for sampling are completely solved.

- The range of density parameters associated with ideal window on an interval is "arbitrarily complicated"! The range of density parameters a, b associated with ideal window on an interval $[0, c)$ is neither open nor connected, and it has very puzzling structure. Similarity and fractal structure?

- Novel techniques involving infinite matrices, dynamic system associated with piecewise linear transformation, maximal invariant sets, topological surgery and algebraic homomorphism. The linear transformation $R_{a,b,c}$ is not contractive but piecewise linear. The maximal invariant sets are important for the abc-problem for Gabor systems, and restriction of $R_{a,b,c}$ is measure-preserving.

- More challenging problem: How about abc-problem for wavelets (frame property for wavelet system associated with functions of Haar type)? Replacing the ideal window on an interval by on a measurable set? High dimensional problems? More general window, such as spline window? Similar problems for sampling and wavelets?

This finishes my presentation based on a joint work with Xinrong Dai (Sun Yat-sen University, China)

THANK YOU