Analyzing fluid flows via the ergodicity defect

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FFT 2013, Norbert Wiener Center
February 22, 2013

Support:
ONR-MURI Ocean 3D+1(N00014-11-1-0087)

Outline

Background & Motivation

General Idea: Ergodicity Defect (ED)

With Jones, Redd, Mezić & Kuznetsov

ED & other metrics & some results

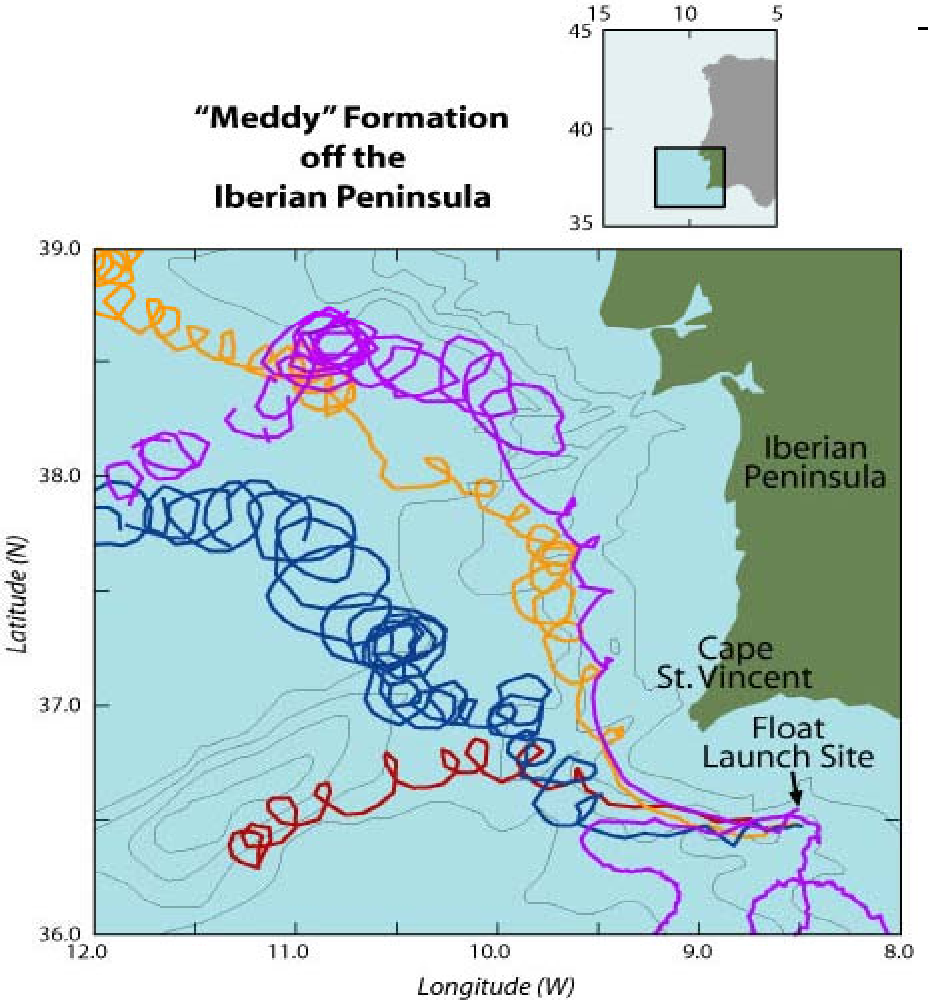
With Rypina, Pratt & Brown

ED & Other fluid flow aspects

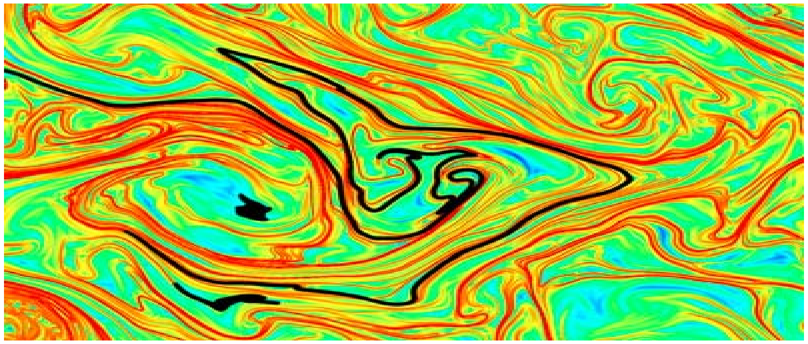
Future & preliminary work

Lagrangian Data in the Ocean

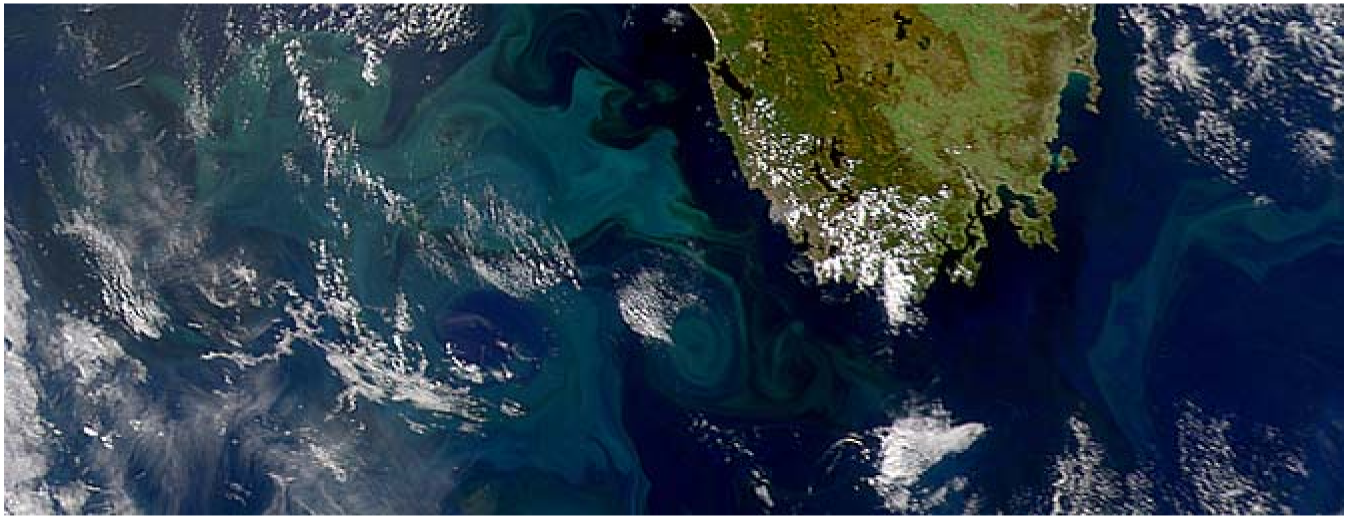
ALACE float
http://www.seabird.com/products/spec_sheets/41data.htm



Analyze Key Structures - Lagrangian Coherent Structures (LCS)



LCS:
"Organized patterns of Trajectories"



(NOAA website)

Why LCS?

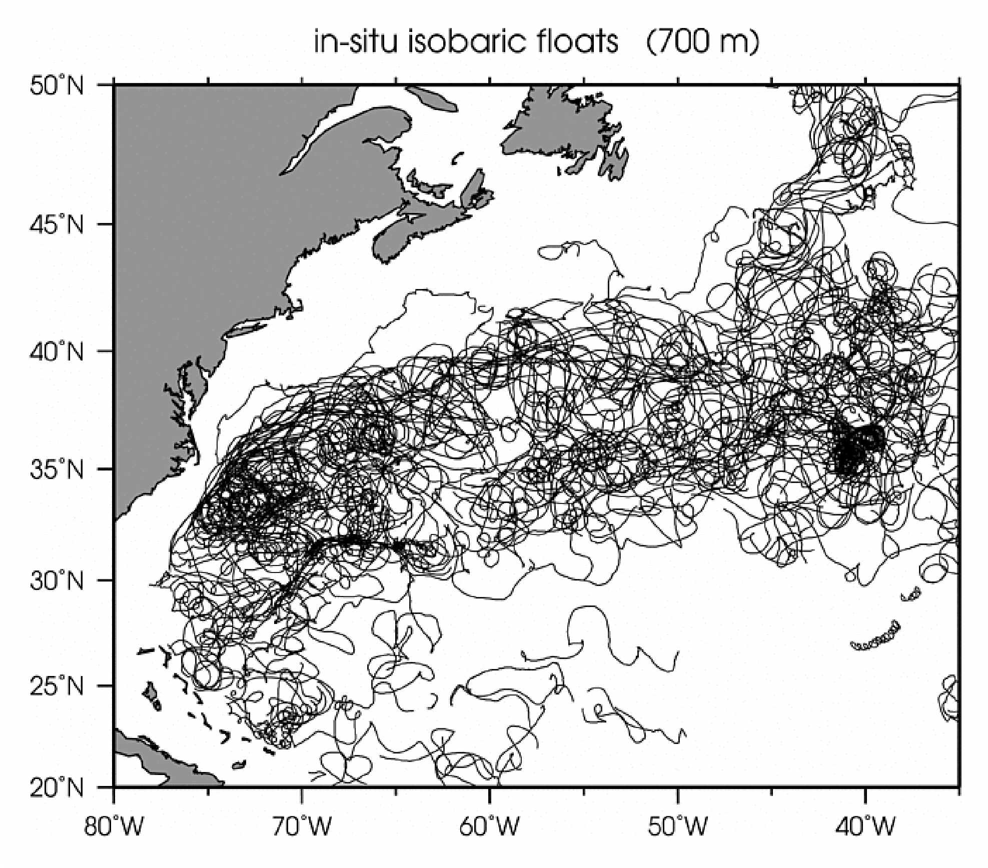
Understand transport of
materials/flow properties

transport barriers?

where/how transport happens?

Motivation - Spaghetti plot

Complex fluid flow & wide range of trajectory behavior



*www.oceancurrents.rsmas.miami.edu/…/**analysis**/**Analysis**.htm*

Background:
The Idea

Understand flows/systems in terms of how trajectories sample/cover the space

Ergodicity?

Background: Definition of Ergodicity

Given a measure (μ) preserving flow T

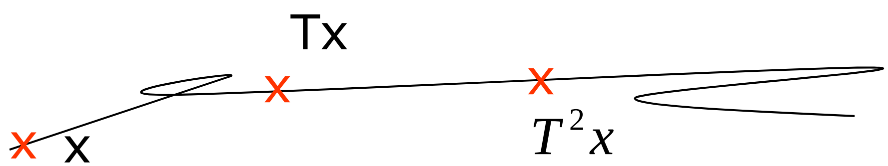
T is ergodic if the only T-invariant sets A are trivial, - i.e. are such that

$$\mu(A) = 0 \qquad \text{or} \qquad \mu(A) = 1$$

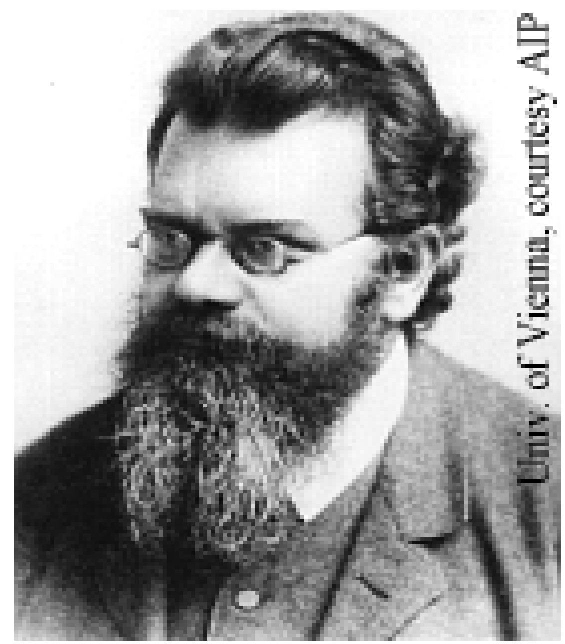
A is T- invariant if

$$A = T^{-1} A$$

Theory:
Boltzmann & The Ergodic Hypothesis



$$\underbrace{\left\{ f(x),\ f(Tx),\ f(T^2x),\ldots,\ f(T^nx) \right\}}_{f \text{ at first } n \text{ points of trajectory at } x}$$



$$\sum_{i=0}^{n} f(T^i x)\frac{1}{n} = \int f$$

Time avg = space avg? (~1860s)

Theory:
ED & Characterization of Ergodicity (George Birkhoff (1931))

T is ergodic if

for all $f \in L^1$ $\underbrace{\lim_{n\to\infty}(\frac{1}{n})\sum_{r=1}^{n} f(T^r(x))}_{time\ average,\ f^*(x,T)} = \frac{1}{\mu(X)} \underbrace{\int_X f d\mu}_{space\ average,\ \overline{f}}$ a.e. x

i.e, ergodic if for all integrable functions,

"time-average = space-average"

Ergodicity Defect (General Idea)

Think of ergodicity in terms of

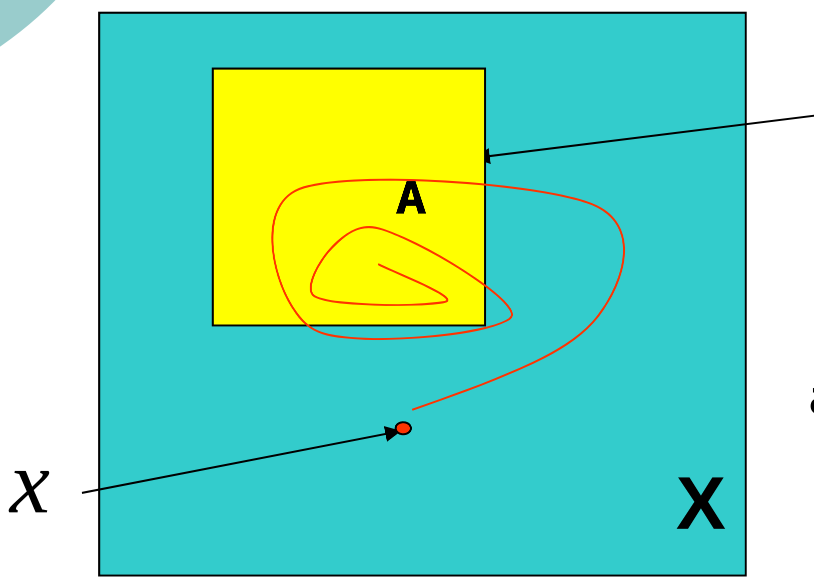
"time average of observables = space average of observables"

Ergodicity defect evaluates difference between time average and space average for a collection of observables (analyzing functions)

Characterization of Ergodicity (due to George Birkhoff (1931))

T is ergodic if for all $f \in L^1$

$$\underbrace{\lim_{n\to\infty} \left(\frac{1}{n}\right)\sum_{r=1}^{n} f(T^r(x))}_{time\ average\ ,\ f^*(x,T)} = \underbrace{\int_X f d\mu}_{space\ average\ ,\ \overline{f}}$$



$$f = \chi_A(x) = \begin{cases} 1, x \in A \\ 0, else \end{cases} \quad so \quad \int f = area(A)$$

average time spent in A = time average

Ergodicity Defect (ED) on unit square

Analyzing functions are 2 dimensional Haar father wavelets

$$\phi^{(s)}_{i_1 i_2}(x, y) = \phi^{(s)}_{i_1}(x)\phi^{(s)}_{i_2}(y) \quad i_1, i_2 = 1,\ldots 2^s$$

Corresponds to

Partition of unit square into 2^{2s} squares each of area $\frac{1}{2^{2s}}$ (where s is the spatial scale)

Deviation from Ergodicity with respect to Haar scaling functions

Haar ergodicity defect

The ergodicity defect of T with respect to the Haar partition at scale s is given by

$$d(s,T) = \frac{2^s}{2^s - 1} \sum_{j=1}^{2^s} \int_X \left(\phi_j^{(s),*}(x,T) - \frac{1}{2^s} \right)^2 dx$$

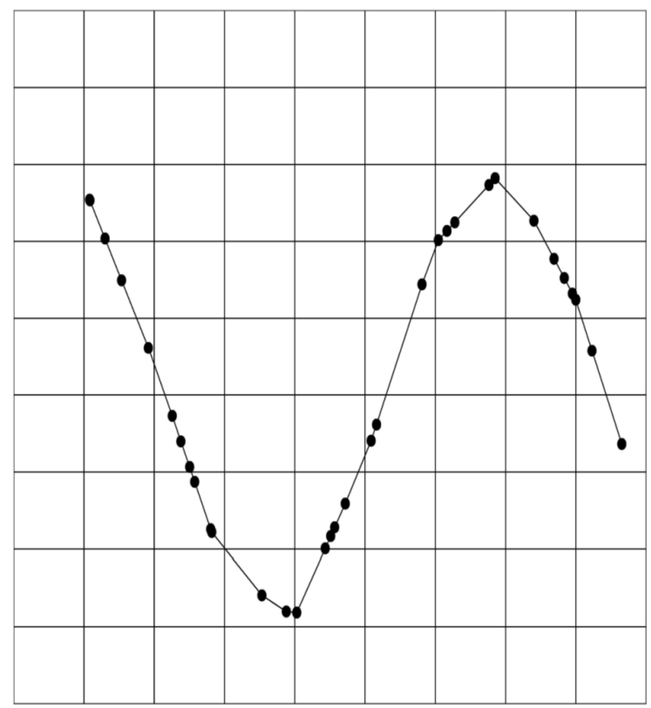
d(s,T) measures the degree of ergodicity

- if T is ergodic, d=0
- the normalization factor is chosen such that d(s,Id) = 1

We call this d(s,T) the Haar ergodicity defect

ED in 2 dimensions for a trajectory–

- Take mapped trajectory in unit square
- Partition the unit square into squares of length s and equal area s^2
- Space average = s^2
- Use number of trajectory points N_j inside jth square to estimate the average time spent in each square (time average)



ED in 2 dimensions – Numerical Algorithm

For a trajectory with initial conditions $\vec{x}_0, t_0$

$$d(s;\vec{x}_0,t_0) = \sum_{j=1}^{s^{-2}} \left(\frac{N_j(s)}{N} - s^2\right)^2$$

Time average for jth square

Space average

"Ergodic" (most complex) trajectory:

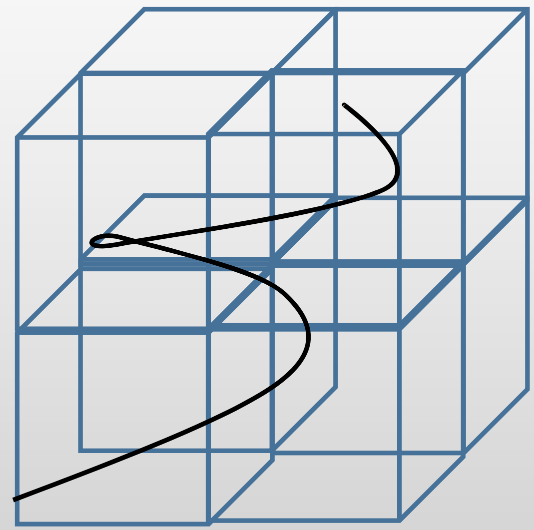
$$d = 0$$

Stationary (least complex) trajectory:

$$d = 1 - s^2 \to 1 \quad \text{as} \quad s \to 0$$

ED in 3 dimensions – Numerical Algorithm

- Take trajectory mapped into unit cube
- Partition the unit cube into smaller cubes of length s and equal volume s^3
- Space average = s^3
- Use number of trajectory points $N_j(s)$ inside jth cube to estimate the average time spent in each cube (time average)

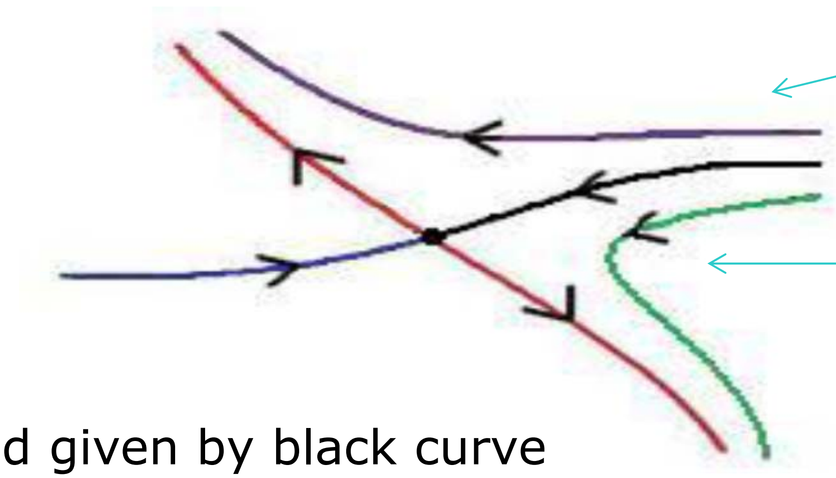


Partition of cube for s=1/2

For a trajectory with initial conditions $\vec{x}_0, t_0$

$$d(s;\vec{x}_0,t_0) = \sum_{j=1}^{s^{-3}} \left(\frac{N_j(s)}{N} - s^3\right)^2$$

ED & LCSs:
Conceptual illustration



Move into different regions

Stable manifold given by black curve

Complexities for trajectories along stable manifold are similar to each other (all similar to hyperbolic point) but

DIFFERENT FROM

Complexities of trajectories on opposite sides of stable manifold which also often differ in complexity

Manifolds correspond to level sets of ED values

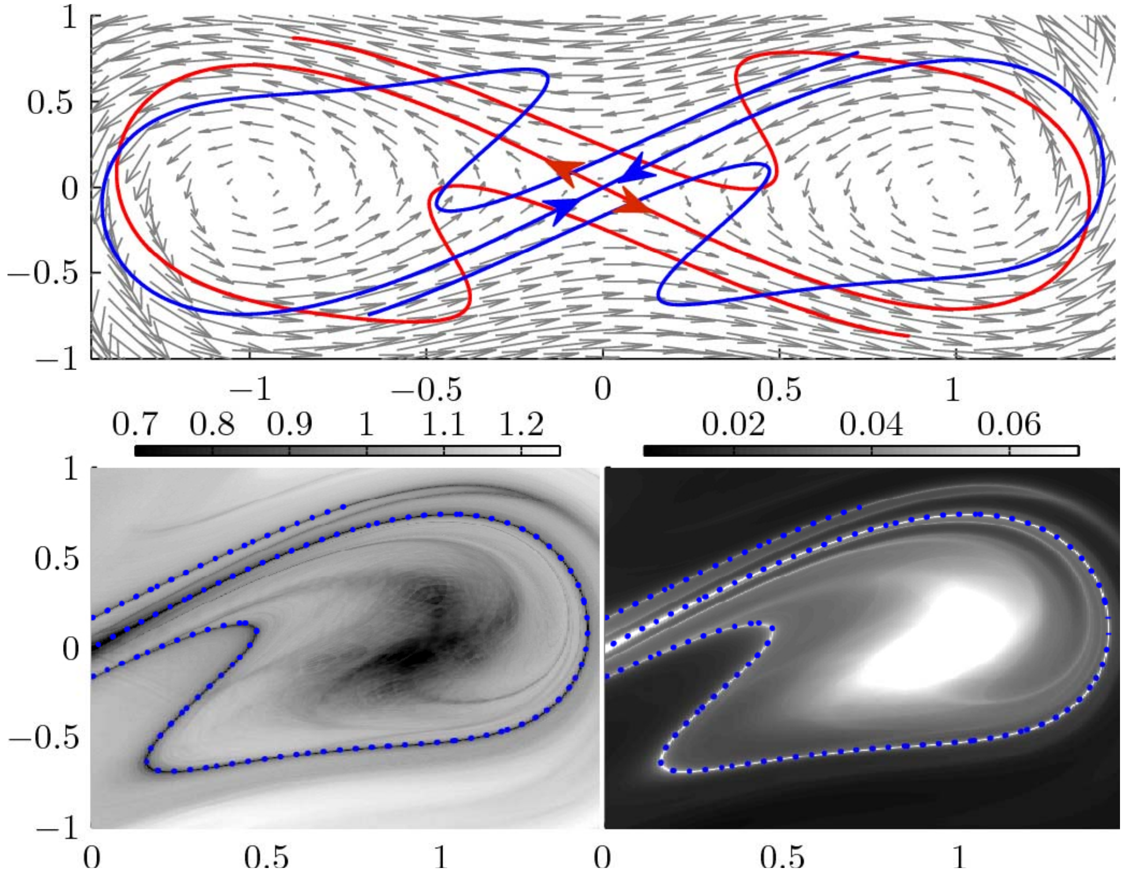
ED & Lagrangian Coherent Structures (LCSs)

Compute the ergodicity defect of individual fluid particle trajectories d_{mean}

Take the mean over scales of interest -

Distinguish each trajectory by the manner in which it samples the space (i.e., by its complexity)

ED & LCSs: Duffing Oscillator Example



Blue curve = stable manifold from a direct evolution method
Have minimizing ridges of C (left) maximizing ridges of d (right)

ED & LCSs: Two Measures of Complexity

c Correlation dimension

measures area occupied by a trajectory

For F(s) = $\frac{1}{N^2}\sum_j (N_j(s))^2$

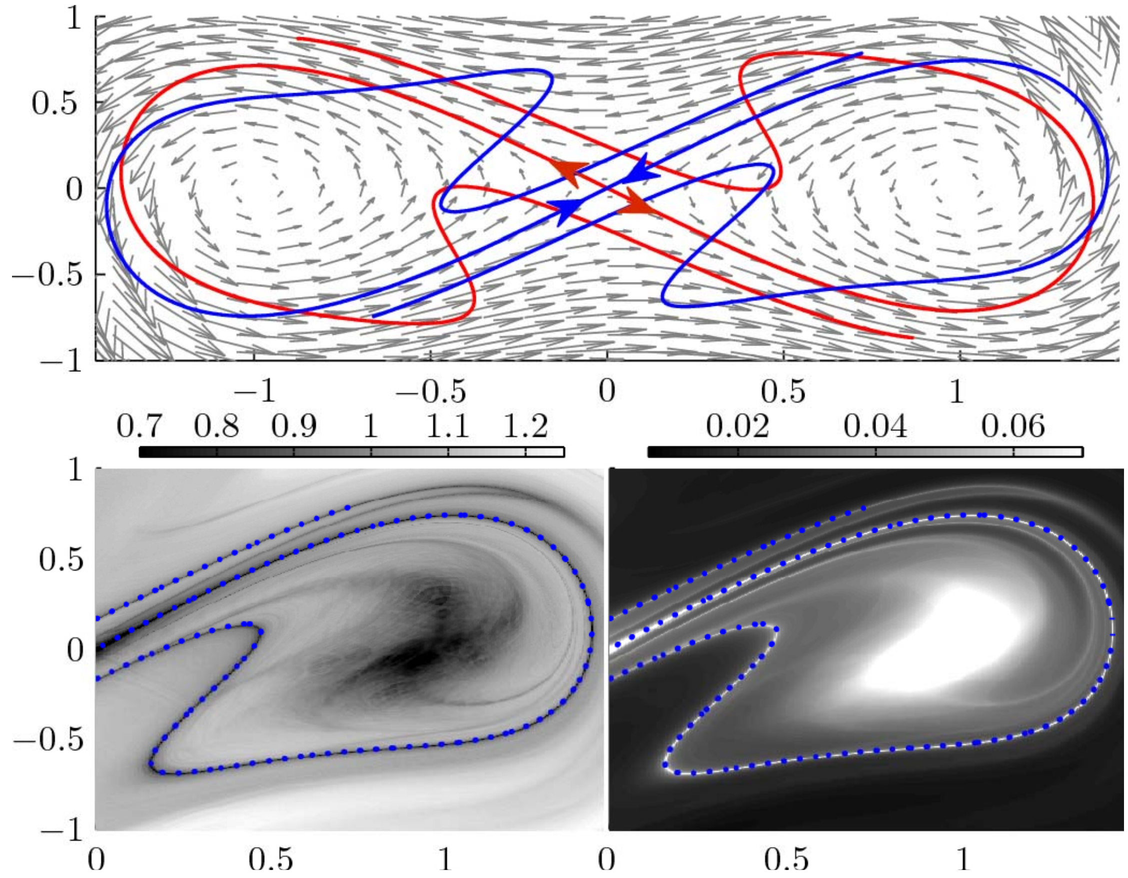
Use $F(s) \propto s^c$ to estimate

d Ergodicity defect

measures the manner in which the trajectory samples the space

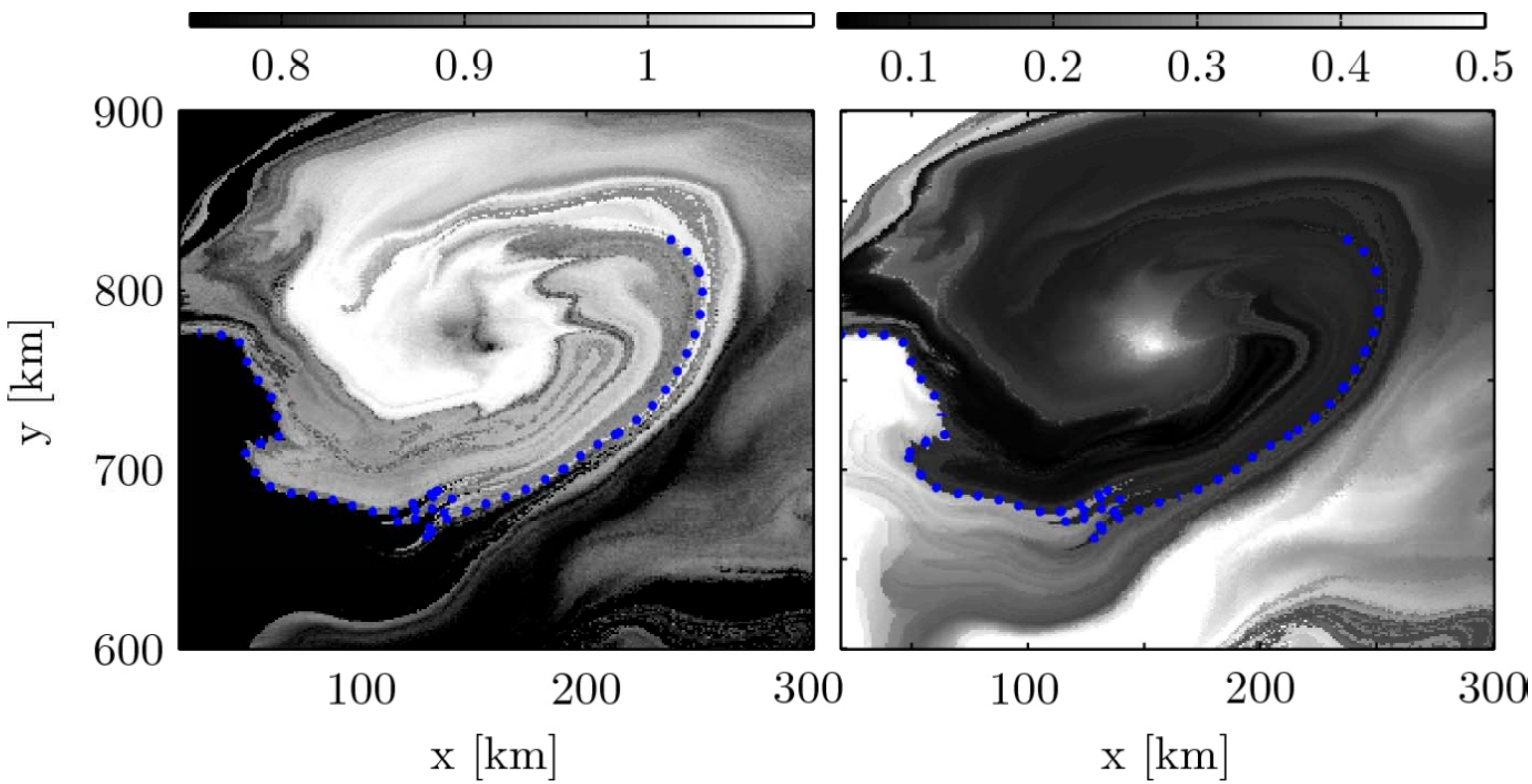
Small d $\Leftrightarrow$ Large c

ED & LCSs: Duffing Oscillator Example



Blue curve = stable manifold from a direct evolution method
Have minimizing ridges of C (left) maximizing ridges of d (right)

ED & LCSs: Numerically generated flow field from Regional Ocean Model System velocities



c (on left) d (on right)

Other Methods:

(1) Finite Time Lyapunov Exponent (FTLE)

- separation rates between trajecs

(George Haller)

(2) Correlation Dimension, c

- how trajecs fill/cover the space (Procaccia et al)

(3) M functions

- arclengths of trajecs

(A. Mancho)

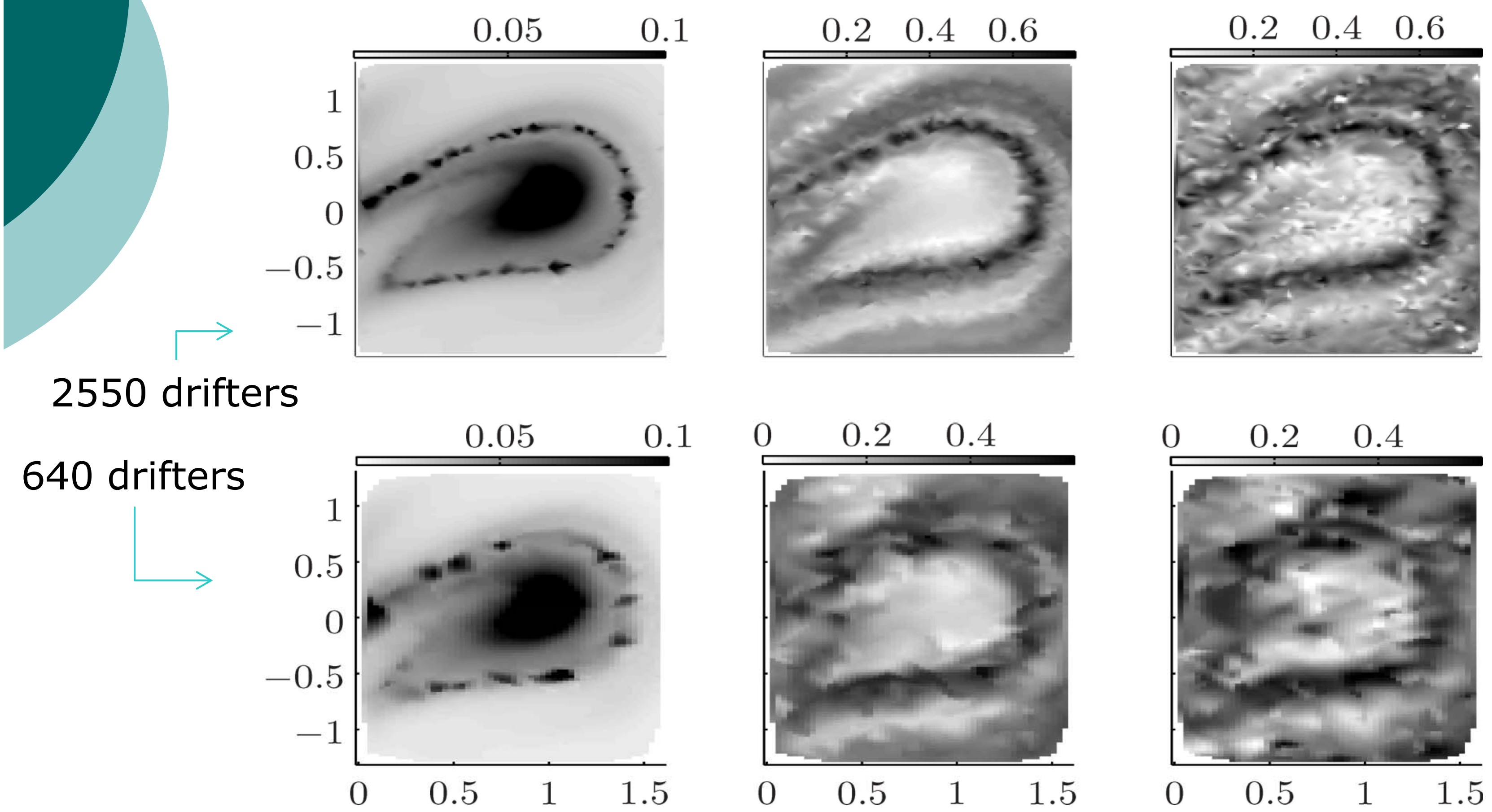
(4) Ergodic quotient (Mezic et al)

Identifying LCSs: Addressing a Challenge

Often data is not amenable to traditional analysis methods such as FTLE

if drifter trajectories are sparse and non-uniformly spaced then individual trajectory methods have an advantage

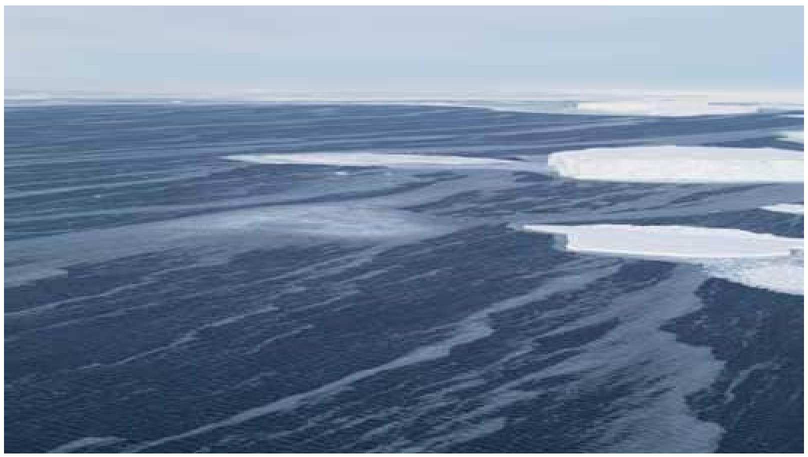
ED & LCSs:
Advantages with sparse & non-uniform data



(left) d (middle) FTLE using Lekien and Ross (2010) method
(right) conventional FTL (darkest color =stable manifold)

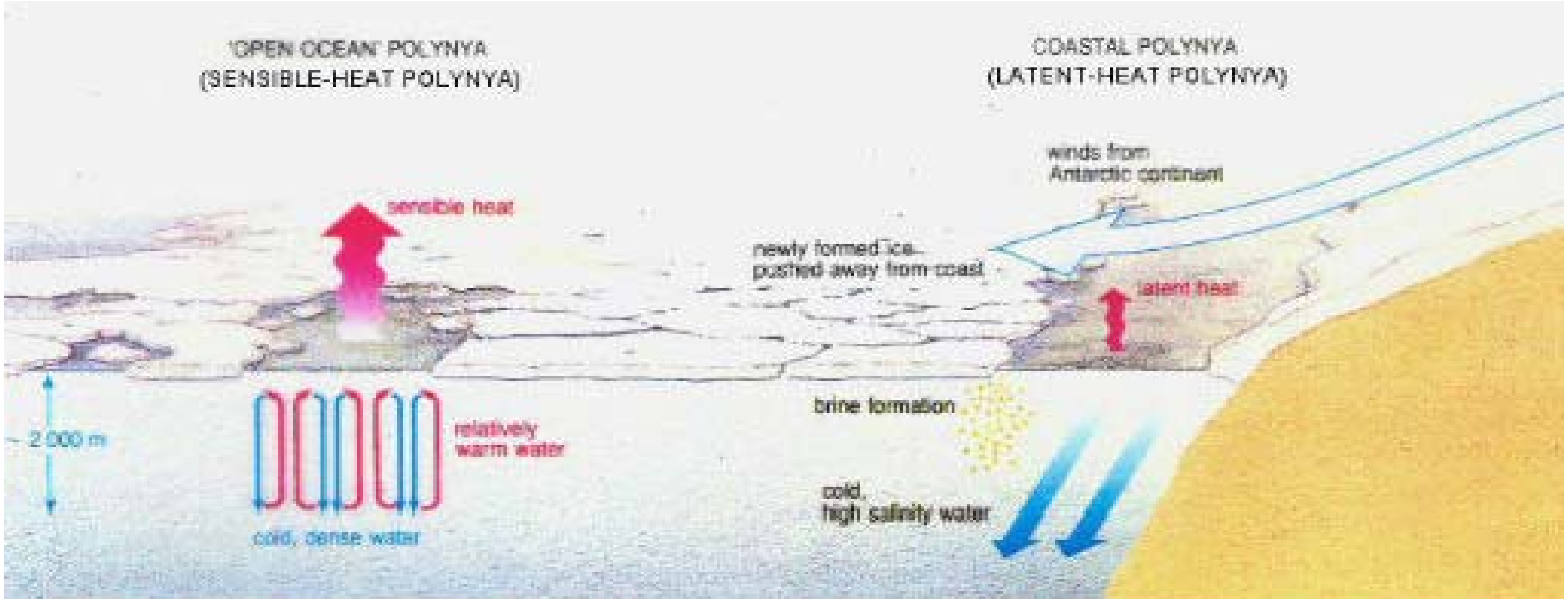
Ergodicity Defect & Polynyas (3D + time dependence)

persistent open water where we would expect to find sea ice



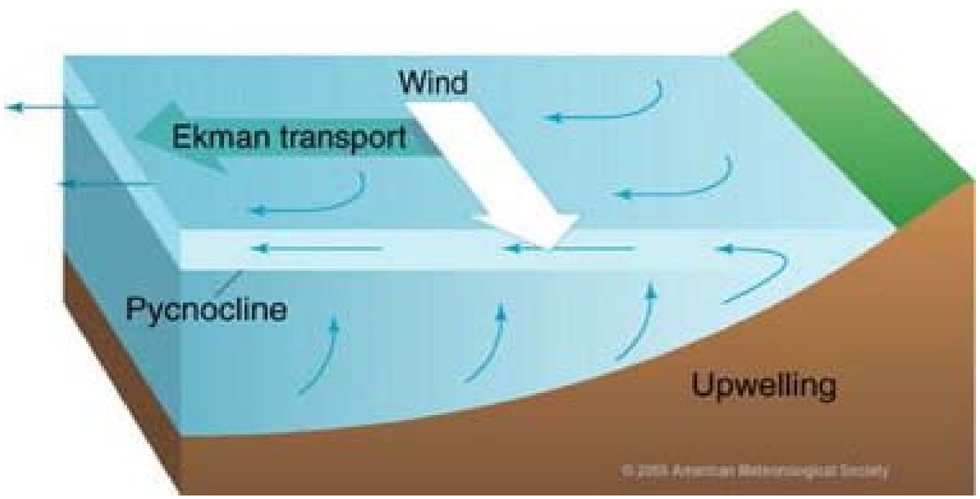
Note: 3D data primarily from floats/drifters/gliders etc i.e., from trajectories

Polynyas
(3D + time)



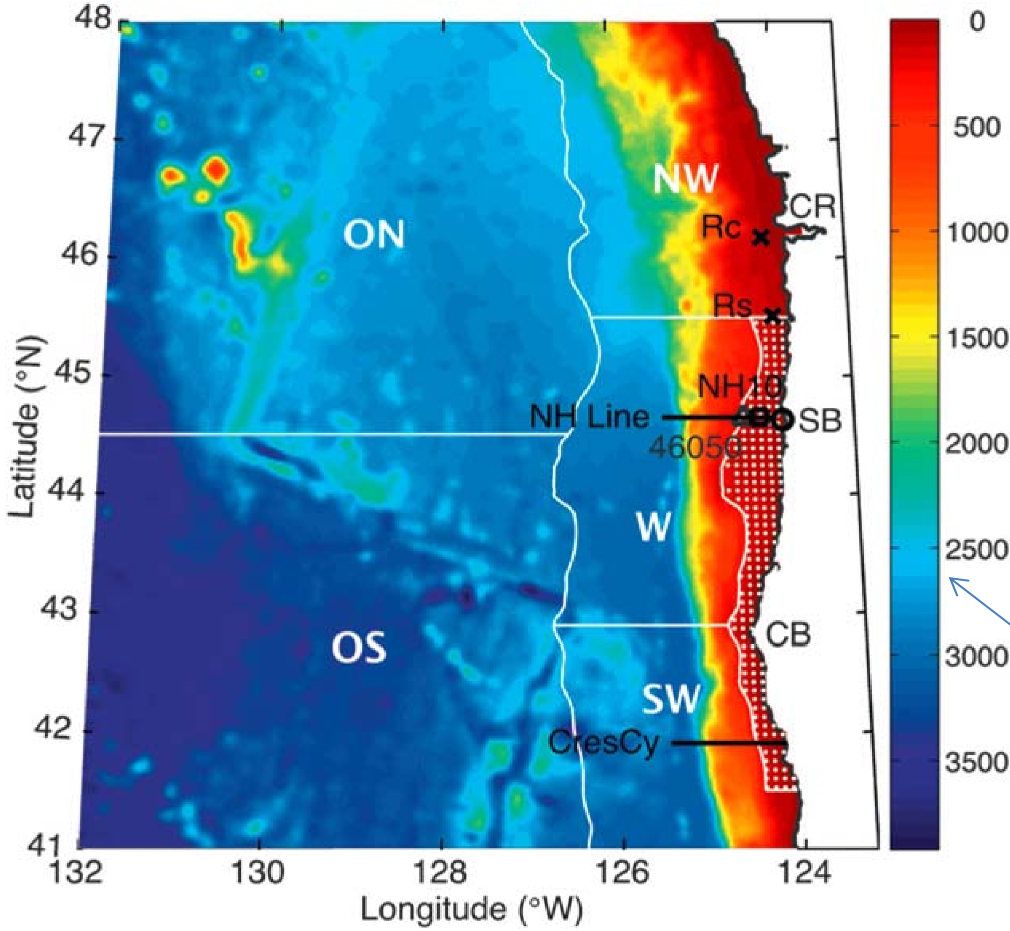
Not polynya but upwelling flow (3D + time)

- Coastal upwelling



ED & an Upwelling flow (Rivas & Samelson) (3D + time example)

Strong Vertical Velocity in Ocean?



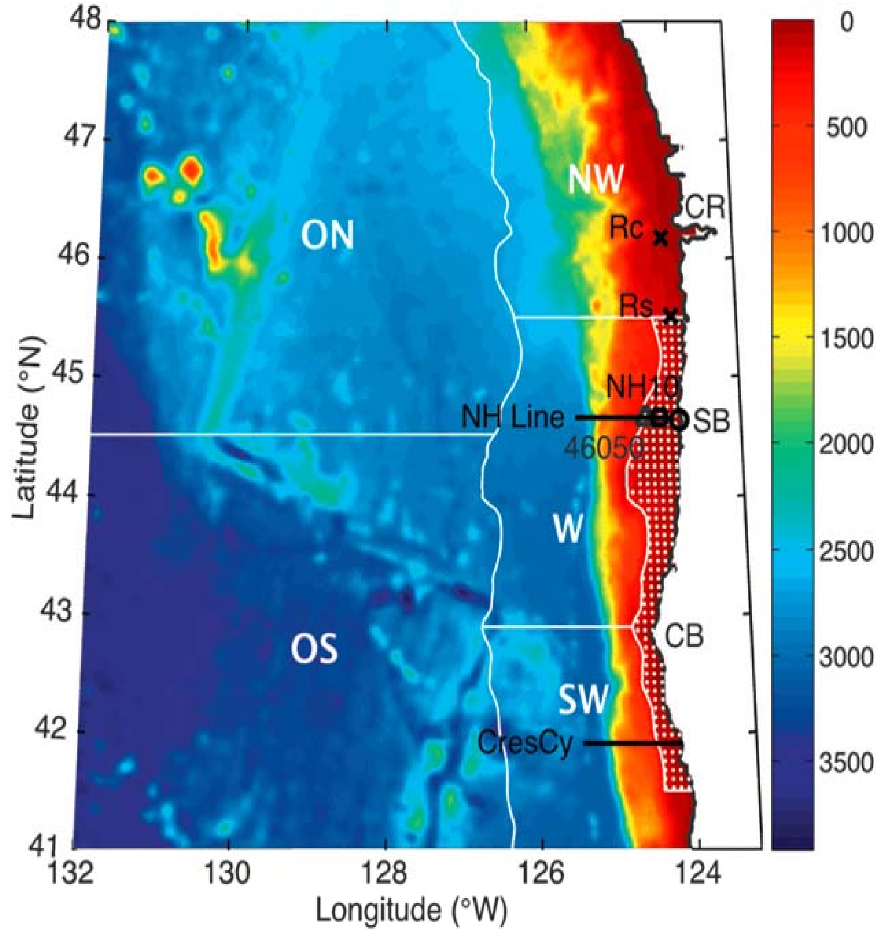
Numerical model off Oregon coast in 2005

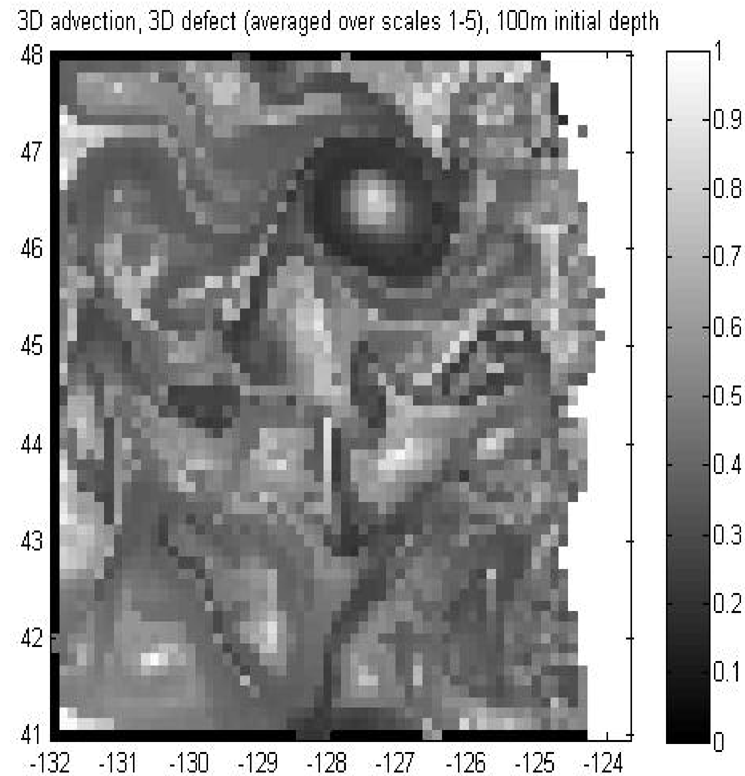
Use Ergodicity Defect to Identify Vertical LCS?

Does 3D Defect (sampling in x,y, & z) give more/different info than just 2D?

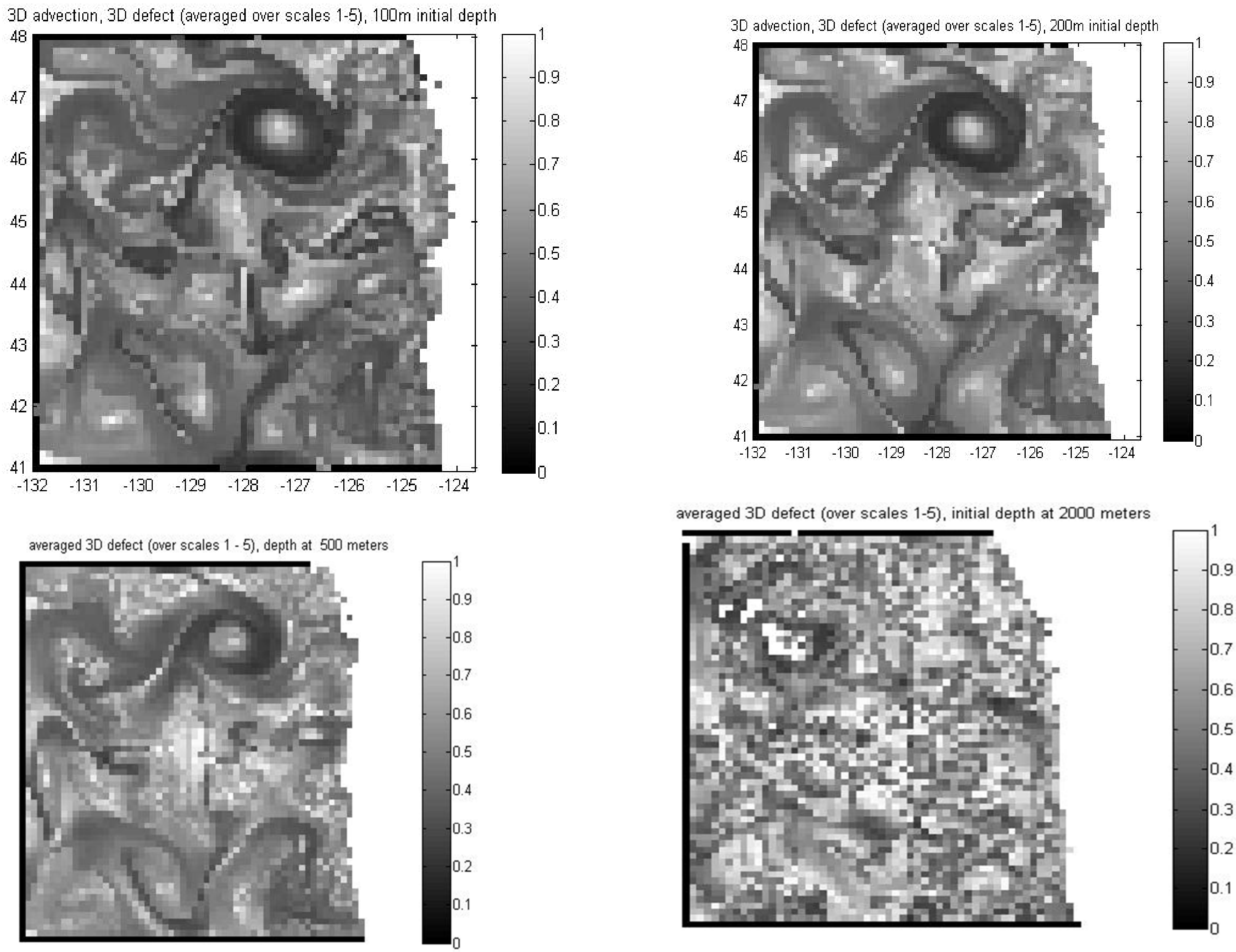
Color=bathymetry

ED & an Upwelling flow (Rivas & Samelson) (3D + time example)

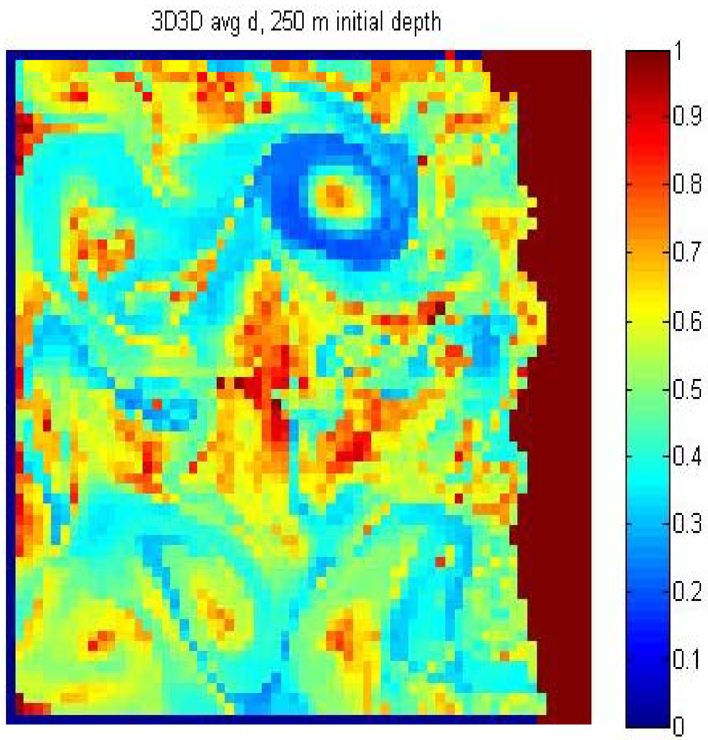


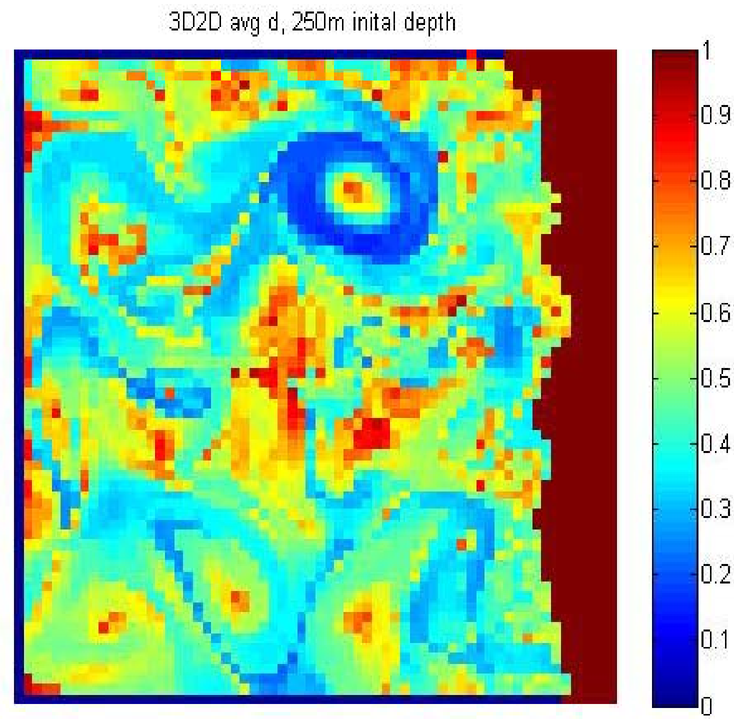


3D ED & Upwelling flow at different depths



ED & an Upwelling flow full domain, 3D advection



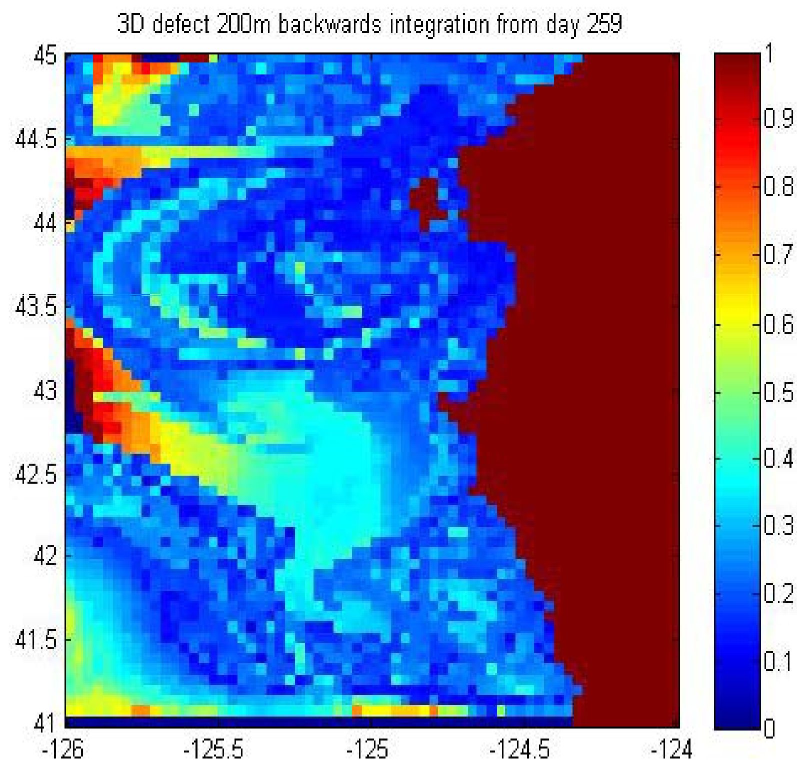


3D defect grayscale
x, y & z sampling

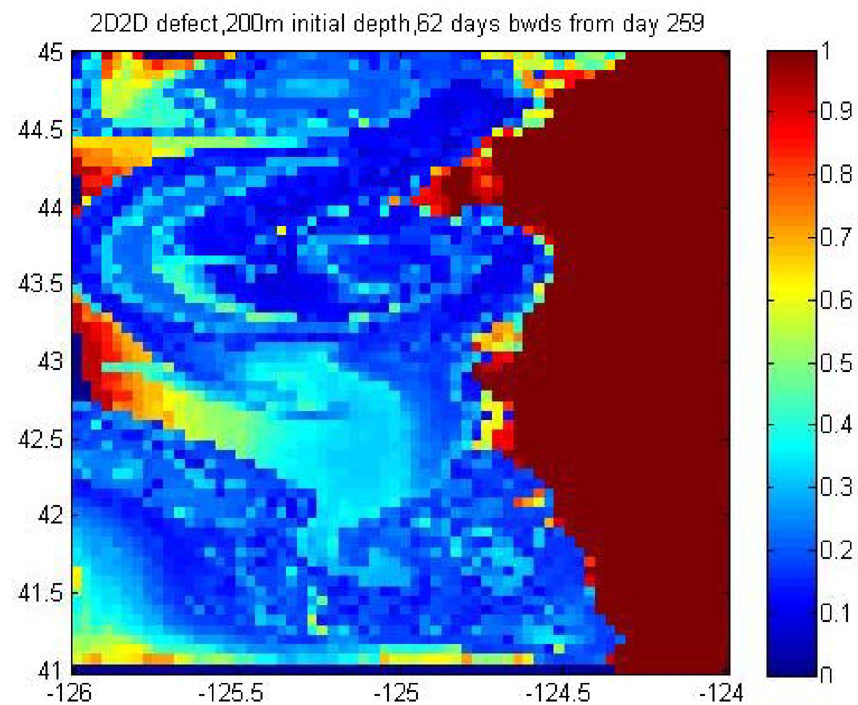
=

2D defect grayscale
x,y sampling

Upwelling flow
on smaller domain (closer to shore)



3D advection, 3D defect



2D advection, 2D defect

Still 3D defect grayscale pic similar 2D defect

Rerunning with better resolution

Other aspects of ED as a diagnostic

Ergodicity Defect (ED) distinguishes optimal trajectories/initial conditions

for assimilating data ?

for float/glider deployment strategy?

for estimating properties?

ED & other fluid flow aspects: Lagrangian Data Assimilation(LDA)

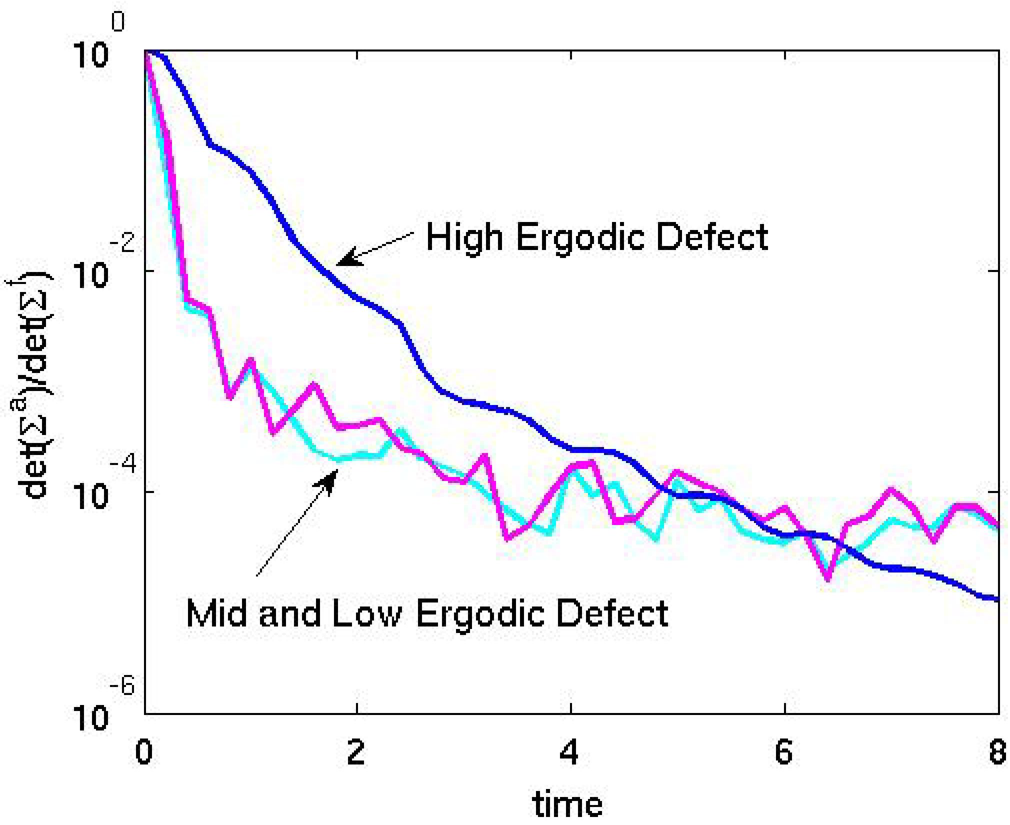
Want: estimate flow field

Have: positions of a drifter

Assimilate drifter positions into model
to estimate velocities

Ergodicity Defect & LaDA (Linearized Shallow Water & Particle Filter (E. Spiller))

Which trajectory? – Lower defect better?



How long?

Summary

Ergodicity Defect (ED) captures trajectory/flow complexity for identifying Lagrangian Coherent Structures

- Understanding barriers to transport
- Understanding/Determining transport of material/flow properties by coherent structures

Advantages of ED

- Distribution of trajectory can be non-uniform/sparse
- Works in both 2 and 3 dimensions
- Scaling analysis component/ other wavelet-like funcs

LDA Example - LSW

Linearized Shallow Water(LSW) Model

Have a flow field

u(x; y; t) = 2 sin(2∏x) cos(2∏y)uo + cos(2∏y)u1(t)

v(x; y; t) = 2 cos(2∏x) sin(2∏y)uo + cos(2∏y)v1(t)

h(x; y; t) = sin(2∏x) sin(2∏y)uo + sin(2∏y)h1(t)

Drifter trajectories given by:

$$dx/dt = u[x(t); y(t); t]$$

$$dy/dt = v[x(t); y(t); t]$$

ED & LCSs:
Two Measures of Complexity (CM)

(1) Correlation dimension c

Compute F(s) = $\frac{1}{N^2}\sum_j (N_j(s))^2$

Use $F(s) \propto s^c$ to estimate c

(2) Ergodicity defect d

adjust d to analyze individual trajectories and take the mean over scales of interest d_{mean}

Background: ED with respect to Haar mother wavelets

$$\psi(x) = \chi_{[0,1)}(x) = \begin{cases} 1, & x \in [0,1) \\ 0, & else \end{cases}$$

$$\psi_j^{(s)}(x) = \psi(2^s x - (j-1)), \qquad j = 1,\ldots, 2^s$$

Time averages

$$\boxed{d(s) = \frac{2^{s-1}-1}{2^s-1} d(s-1) + \frac{2^{s-1}}{2^s-1} \sum_{j=1}^{2^{s-1}} \| \psi_j^{(s-1),*} \|^2}$$

Better for scaling analysis

ED 3 dimensions + time – Numerical Algorithm

For different fixed initial depth (z) levels,

- Generate trajectory from (time) snapshots
- Take mapped trajectory in unit cube
- Partition the unit cube into smaller cubes with sides of length s
- Space average = s^3
- Use number of trajectory points $N_j(s)$ inside each cube to estimate the average time spent in each cube (time average)
- Combine info from all depth levels

ED & LCSs: General Setup

For 2d fluid flows, trajectories satisfy

$$d\vec{x}/dt = \vec{u}(\vec{x},t)$$

Trajectories exhibit a wide range of behavior

from stationary ⇨ densely covering

i.e., trajectories have different complexities