# Dynamical Sampling

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# Outline

# Motivation and inspiration sources

Typical sampling and reconstruction problem:
Find a function $f$ from its samples $f(X) = \{f(x_j)\}_{x_j \in X}$.

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The source of our inspiration is the work of Lu, Vetterli, *et al*, who study spatiotemporal sampling and reconstruction problem coming from the (sparse) inverse initial problem for a one-dimensional diffusive field modeled by the heat equation:
*"... our analysis indicates that it is possible to compensate linearly for insufficient spatial sampling densities by oversampling in time."*

# Dynamical Sampling Problem and its potential applications

The field estimation problem deals with evolution processes such as diffusion. The goal is to recover as much information about the field as possible from its samples taken at various locations at various times. Potential applications include monitoring pollution processes, parameter estimation, perimeter breach classification, etc. In general the problem is very hard since the locations may not be known. Our initial goal was to understand when and why the assertion about the spatiotemporal trade-off is correct.

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**Dynamical Sampling**: Let $X_m = m\mathbb{Z}$, $m \in I\!N$. Study conditions under which $f = f_0$ can be recovered from samples $\{f(X_m), (A_1 f)(X_m), \dots, (A_T f)(X_m)\} \cup \{f(\Omega)\}$ in a stable way, where $\Omega \subset Z\!\!\!Z$ is a small extra sampling set.

# Dynamical Sampling Problem – illustration

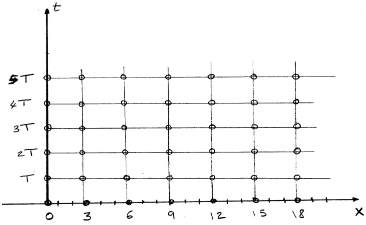


Figure: $m = 3$, $f \in \ell^2(\mathbb{Z}_d)$. $f_{kT} = A_{kT} f$ is sampled on $X_3 = 3\mathbb{Z}_d$ to obtain $\{f(X_3), A_T f(X_3), \ldots, A_{NT} f(X_3)\}$.

What is the minimum $N$? Do we need an extra sampling set $\Omega$, to recover $f = f_0$, and what would such a set look like?

# Dynamical Sampling Problem – connections

The theory we develop has connections to distributed sampling, frame theory, multiresolution analysis, filter banks, superresolution and interpolation theory. We have already used various ideas of discrete Fourier analysis (downsampling formulas), numerical linear algebra (Vandermonde matrices, Tikhonov/Wiener regularization), sparsity techniques (Prony's method, compressed sensing).

# General problem in finite dimensions

- The signal: $x \in \ell^2(\mathbb{Z}_d) \simeq \mathbb{C}^d$
- Evolution process governed by a $d \times d$ (invertible) matrix $A$
- Problem: recover $x$ from undersampled vectors $x$, $Ax$, $A^2x$,

Some examples:

- Sampling the diagonal
- Sampling at one node
- Downsampling (regular undersampling)

# Specific problem

**Invariant evolution systems**: Let $f = f_0 \in \ell^2(\mathbb{Z}_d) \simeq \mathbb{R}^d$. At time $t = nT$, $f_{nT} = a^n * f$, where $*$ is a spatial convolution $a^n = a * a \ldots. * a$.

Let $S_m$ be the sampling operator on $X_m = m\mathbb{Z}$, i.e., $S_m f(ml) = f(ml)$, and $S_m f(j) = 0$ if $j \neq ml$.

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**Problem:** Find $f$ from subsampled versions of the vectors $f$, $Af$, $A^2 f$, ...,i.e., find $f$ from

$$y_n = S_m A^{n-1} f, \quad n = 1, \dots, N.$$

where $A$ is a circular matrix corresponding to $a$, and possibly from the extra samples $f(\Omega)$ on a set $\Omega \subset \{1, \dots, d\}$.

# Specific problem – continued

Recall $f \in \ell(\mathbb{Z}_d) \simeq \mathbb{R}^d$, $Af = a * f$. Let $d = mJ$.

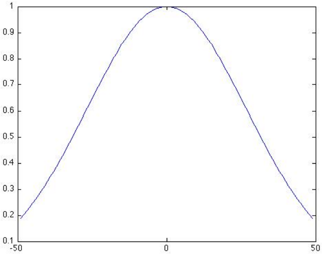
## Proposition (ADK, ACHA 2012)

*A vector $f \in \ell^2(\mathbb{Z}_d)$ can be recovered from the measurements $y_n = S_m A^{n-1} f$, $n = 1, \dots, m$, if and only if the $J = d/m$ matrices $\mathcal{A}_m(k) =$*

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}(k) & \hat{a}(k+J) & \dots & \hat{a}(k+(m-1)J) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(m-1)}(k) & \hat{a}^{(m-1)}(k+J) & \dots & \hat{a}^{(m-1)}(k+(m-1)J) \end{pmatrix},$$

*$k = 0, \dots, J-1$, are invertible.*

# Specific problem – Typical Filter

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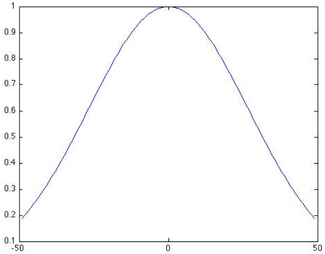


Figure: Fourier transform $\hat{a}$ of a typical convolution kernel $a$.

Such an operator never satisfies the conditions of the Proposition. Thus $f$ cannot be recovered.

# Specific problem – Extra sampling set

**Theorem (ADK, ACHA 2012)**

*Let the circular convolution matrix $A$ be generated by $a \in \ell^2(\mathbb{Z}_d)$ which is real symmetric and $\hat{a}$ is strictly monotonic on $\{0, \ldots, d/2\}$. Consider the dynamical sampling procedure $\mathbf{y} = \mathbb{A} f$, where $\mathbf{y} = (y_1, \ldots, y_m)$, $y_i \in \mathbb{R}^d$, and*

$$\mathbb{A} = \begin{pmatrix} S_m \\ S_m A \\ \vdots \\ S_m A^{m-1} \end{pmatrix}.$$

*Then we can recover $f$ by acquiring the extra samples $f(\Omega_0)$ on a set $\Omega_0 \subseteq \{1, \ldots, d\} - m\mathbb{Z}$ if and only if $\Omega_0$ contains a set of cardinality $\frac{m-1}{2}$ no two elements of which are m-congruent or have a sum divisible by m.*

# Sampling in invariant evolution systems: $\ell^2(\mathbb{Z})$

Let $f = f_0 \in \ell^2(\mathbb{Z})$, $a \in \ell^2(\mathbb{Z})$, $f_n = a^n * f$, $S_m$ be the sampling operator on $m\mathbb{Z}$, and $y_n = S_m A^{n-1} f$, $n = 1, \dots, m$.

## Theorem (ADK)

*Let* $\hat{a} \in L^\infty(\mathbb{T})$ *and define*

$$\mathcal{A}_m(\xi) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \dots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(m-1)}(\frac{\xi}{m}) & \hat{a}^{m-1}(\frac{\xi+1}{m}) & \dots & \hat{a}^{(m-1)}(\frac{\xi+m-1}{m}) \end{pmatrix},$$

$\xi \in \mathbb{T}$. *Then a vector* $f \in \ell^2(\mathbb{Z})$ *can be recovered in a stable way from* $y_n$, $n = 1, \dots, m$, *if and only if there exists* $\alpha > 0$ *such that the set* $\{\xi : |\det \mathcal{A}_m(\xi)| < \alpha\}$ *has measure zero.*

# Sampling in invariant evolution systems: $\ell^2(\mathbb{Z})$

Let $T_c$ be the shift by an integer $c$.

**Theorem (ADK)**

*Suppose $\hat{a}$ is real, symmetric, continuous, and strictly decreasing on $[0, \frac{1}{2}]$, $L \in \mathbb{N}$, and $\Omega = \{1, \dots, \frac{m-1}{2}\}$. Then the extra samples given by $\{S_{mL} T_c\}_{c \in \Omega}$ provide enough additional information to stably recover $f$, from $y_n = S_m A^{n-1} f, \quad n = 1, \dots, m$.*

# Stability in the presence of noise: $\ell^2(\mathbb{Z}_d)$

Let $\eta$ be an additive measurement noise: $\tilde{\mathbf{y}} = \mathbf{y} + \eta$,
$\mathbf{y} = (y_0, y_1, \dots, y_m)$ ($y_0 = S(\Omega_0)f$ is extra samples), and
$\eta = (\eta_0, \dots, \eta_m)^T$ and each $\eta_k$, $k = 0, \dots, m$, is iid $\mathcal{N}(0, \sigma^2)$,

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$$O(m) \leq \|\tilde{f} - f\| \leq O((1 + \sqrt{d/m - 1}) \max\left\{1, m^{\frac{3}{2}} \left(\frac{2}{\varepsilon}\right)^{m-1}\right\},$$

where $\varepsilon = \min_{p,q} |\hat{a}(p) - \hat{a}(q)|$

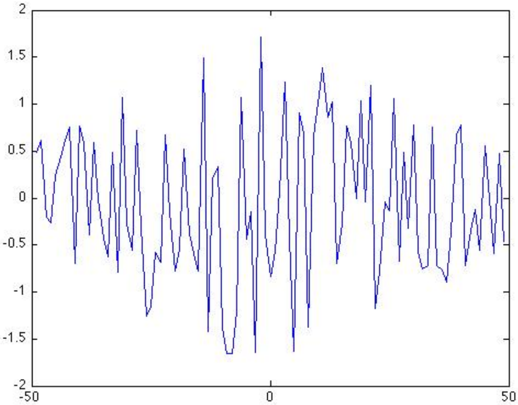
# Wiener Regularization

As the error estimates suggests, the recovery algorithms may perform poorly in the presence of noise. This is not surprising for two reasons.

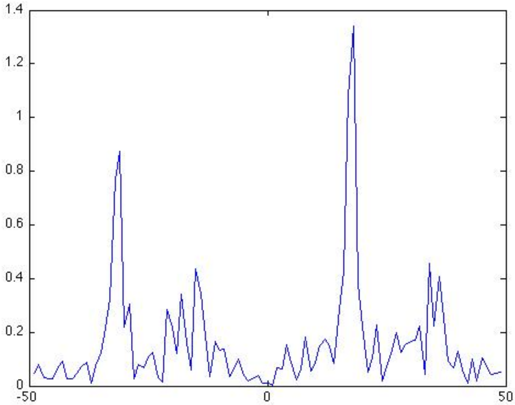
1. If all the samples were obtained after filtering, the recovery problem would require inverse filtering (deconvolution) which is known to be sensitive to noise.
2. The Vandermonde submatrices $\mathcal{A}_m$ introduce another source of sensitivity to noise.

A common approach for inverse filtering is the use of Wiener filters. Thus, it is natural to combine this kind of technique with dynamical sampling.

# Typical Signal



# Reconstruction error in Fourier domain (in %)



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**Theorem (AK, 2012)**

*Assume that $f$ is a realization of a random vector in $\ell^2(\mathbb{Z}_d)$. Assume also that the unknown evolution filter $a \in \ell^2(\mathbb{Z}_d)$ is such that $\hat{a}$ is non-vanishing, real, symmetric, and strictly decreasing on $\{0, \dots, \frac{m-1}{2}\}$. Then the convolution filter $a$ can be recovered almost surely from the measurements* $y_\ell = S_m A^\ell f,\ \ell = 0, \dots, 2m-1.$

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Proof is based on the method of Prony (1755-1839).

# Future and current directions

Many open problems. Even some seemingly simple problems that are easy to state end up being difficult.

- Questions related to the recovery of $A$ or $f$ under various assumptions on the sampling and the type of operators.
- Improving stability.
- Discretization models.
- Actual applications.
- ...

# THANK YOU