Fourier Bases on Fractals

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Coauthors

This is joint work with

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- Karen Shuman (Grinnell College).

Outline

Convolution measure

Let $\lambda \in (0,1)$.

A fractal Cantor subset of $\mathbb{R}$ is the unique set X_λ satisfying the invariance relation:

$$X_\lambda = \lambda(X_\lambda + 1) \cup \lambda(X_\lambda - 1). \quad (1)$$

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The Bernoulli convolution measure μ_λ is the unique probability measure satisfying:

$$\int f \, \mathrm{d}\mu_\lambda = \frac{1}{2} \int f(\lambda x + \lambda) + f(\lambda x - \lambda) \, \mathrm{d}\mu_\lambda(x). \tag{2}$$

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Historical note:

The Bernoulli measures date back to work of Erdös and others in the 1930s and 40s. μ_λ is the distribution of the random variable $\sum_k \pm\lambda^k$ where $+$ and $-$ have equal probability.

Some properties of Bernoulli measures

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- [Hutchinson, 1981] Existence of (X, μ), from an IFS perspective.
- [Solomyak, 1995] For almost every $\lambda \in (\frac{1}{2}, 1)$, μ_λ is absolutely continuous.

$\widehat{\mu}_\lambda$ as an infinite product

Consider the Hilbert space $L^2(\mu_\lambda)$. Is it possible for $L^2(\mu_\lambda)$ to have a Fourier basis, i.e. an orthonormal basis (ONB) of complex exponential functions?

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$$\int f \, \mathrm{d}\mu_\lambda = \frac{1}{2} \int f(\lambda x + \lambda) + f(\lambda x - \lambda) \, \mathrm{d}\mu_\lambda(x).$$

Then the Fourier transform of μ_λ is:

$$\begin{aligned}
\widehat{\mu}_\lambda(t) &= \int e^{2\pi i x t} \, \mathrm{d}\mu_\lambda(x) \\
&= \frac{1}{2} \int e^{2\pi i(\lambda x + \lambda)t} \, \mathrm{d}\mu_\lambda(x) + \frac{1}{2} \int e^{2\pi i(\lambda x - \lambda)t} \, \mathrm{d}\mu_\lambda(x) \\
&= \cos(2\pi\lambda t)\widehat{\mu}_\lambda(\lambda t) \\
&= \cos(2\pi\lambda t)\cos(2\pi\lambda^2 t)\widehat{\mu}_\lambda(\lambda^2 t) \\
&= \prod_{k=1}^{\infty} \cos(2\pi\lambda^k t)
\end{aligned}$$

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Lemma

The two exponentials $e_\gamma, e_{\tilde\gamma}$ are orthogonal if and only if one of the factors in the infinite product above is zero. This is equivalent to

$$\gamma-\tilde\gamma \in \left\{\frac{1}{4}\lambda^{-k}(2m+1)\ :\ k\in\mathbb{N}, m\in\mathbb{Z}\right\} =: \mathcal{Z}_\lambda.$$

Test for ONB

We use the zero set $\mathcal{Z}_\lambda$ to check orthogonality. Parseval's identity provides a test for an ONB.

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Let $\Gamma \subset \mathbb{R}$ be a set and let $E(\Gamma)$ be the set $\{e_\gamma : \gamma \in \Gamma\}$. If $E(\Gamma)$ is an ONB, then for every value of $t \in \mathbb{R}$, we have

$$\begin{aligned} 1 = \|e_t\|^2_{\mu_\lambda} &= \sum_{\gamma\in\Gamma} |\langle e_t, e_\gamma\rangle|^2 \\ &= \sum_{\gamma\in\Gamma} |\widehat{\mu}_\lambda(t-\gamma)|^2 \end{aligned}$$

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It other words,

$$c_\Gamma(t) = \sum_{\gamma\in\Gamma} [\widehat{\mu}_\lambda(t-\gamma)]^2 = \sum_{\gamma\in\Gamma}\prod_{k=1}^{\infty} \cos^2\left(2\pi\Big(\lambda\Big)^k(t-\gamma)\right) \equiv 1.$$

The function c_Γ is sometimes called a spectral function .

First results

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Example

$E(\Gamma_{\frac{1}{4}})$ is an ONB for $L^2(\mu_{\frac{1}{4}})$, where

$$\Gamma_{\frac{1}{4}} = \left\{ \sum_{j=0}^{p} a_j 4^j \ : \ a_j \in \{0,1\}, p \text{ finite} \right\} = \{0,1,4,5,16,17,20,21,64,\ldots\}.$$

Another surprise: $\lambda = \frac{1}{3}$

Theorem (Jorgensen, Pedersen 1998)

Not only is there no Fourier basis when $\lambda = \frac{1}{3}$, but orthogonal collections of exponential functions can have at most 2 elements.

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Theorem (Jorgensen, Pedersen 1998)

Given $\lambda = \frac{1}{n}$, if n is even, there is an ONB of exponentials for $L^2(\mu_{\frac{1}{2n}})$ but when n is odd, there can be only finitely many elements in any orthogonal collection of exponentials.

More recent progress

Theorem (Jorgensen, K, Shuman 2008; Hu, Lau 2008)

For $\lambda = \frac{a}{b}$, if b is odd, then any orthonormal collection of exponentials in $L^2(\mu_\lambda)$ must be finite. If b is even, then there exists countable collections of orthonormal exponentials in $L^2(\mu_\lambda)$.

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Theorem (Dutkay, Han, Jorgensen 2009)

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Theorem (Xinrong Dai 2012)

The only spectral Bernoulli measures are for $\lambda = \frac{1}{2n}$.

"Canonical" ONBs

[Jorgensen, Pedersen 1998]

Definition

Let $\lambda = \frac{1}{2n}$ and consider the set from Jorgensen & Pedersen

$$\Gamma_{\frac{1}{2n}} = \left\{ \sum_{j=0}^{p} a_j (2n)^j \ : \ a_j \in \left\{0, \frac{n}{2}\right\}, p \text{ finite} \right\}.$$

We call $\Gamma_{\frac{1}{2n}}$ the canonical spectrum and $E(\Gamma_{\frac{1}{2n}})$ the canonical ONB for $L^2(\mu_{\frac{1}{2n}})$.

Note: We will justify the nomenclature by describing *alternate* bases for the same spaces.

Families of ONBs

Let $\lambda = \frac{1}{2n}$. We can construct alternate orthogonal families of exponentials from the canonical ONBs $E(\Gamma_{\frac{1}{2n}})$. We then determine whether these alternate sets are ONBs.

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Observe that if $\gamma - \tilde{\gamma} \in \mathcal{Z}_\lambda$ then $p\gamma - p\tilde{\gamma} \in \mathcal{Z}_\lambda$ as well, for any odd integer p. So scaling a canonical spectrum by p yields at least an orthogonal set, and sometimes an ONB.

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Note: Not every p yields an ONB, e.g. $p = 2n - 1$ for $\lambda = \frac{1}{2n}$. The set $E((2n-1)\Gamma_{\frac{1}{2n}})$ is not maximal, hence is not an ONB.

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Laba/Wang and Dutkay/Jorgensen have described many other values of p for which $p\Gamma_{\frac{1}{2n}}$ is a spectrum, particularly in the $\frac{1}{4}$ case.

Examples of $p\Gamma_{\frac{1}{2n}}$ ONBs

n	λ	p	Canonical Γ_λ
3	$\frac{1}{6}$	1, 3	$\{0, \frac{3}{2}, 9, \frac{21}{2}, \ldots\}$
4	$\frac{1}{8}$	1, 3	$\{0, 2, 16, 18, \ldots\}$
5	$\frac{1}{10}$	1, 3, 5	$\{0, \frac{5}{2}, 25, \frac{55}{2}, \ldots\}$
6	$\frac{1}{12}$	1, 3, 5, 7	$\{0, 3, 36, 39, \ldots\}$

Operators on L^2

Dutkay, Jorgensen, 2009:

When $\lambda = \frac{1}{4}$, both $\Gamma_{\frac{1}{4}}$ and $5\Gamma_{\frac{1}{4}}$ are spectra for $L^2(\mu_{\frac{1}{4}})$.

$$\Gamma_{\frac{1}{4}} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \ldots\}$$

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- U maps one ONB to another, so U is a unitary operator.

Structure of Γ

Example: Let $\lambda = \frac{1}{4}$ and denote $\mathcal{H} = L^2(\mu_{\frac{1}{4}})$.

$$\Gamma = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \ldots\}$$

$$\begin{aligned} \Gamma &= 4\Gamma \cup (1 + 4\Gamma) \\ &= 4^2\Gamma \cup 4(1 + 4\Gamma) \cup (1 + 4\Gamma) \\ &\vdots \\ &= 4^k\Gamma \cup 4^{k-1}(1 + 4\Gamma) \cdots 4(1 + 4\Gamma) \cup (1 + 4\Gamma) \end{aligned}$$

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- If we define $W_k = S_0^k(\mathcal{H}) \ominus S_0^{k+1}(\mathcal{H})$, then $\mathcal{H} = \overline{sp}(e_0) \oplus \bigoplus_{k=0}^{\infty} W_k$

The operator U

Recall $U : e_\gamma \mapsto e_{5\gamma}$. How does that scaling by $(\times 5)$ in the ONB frequencies interact with the inherent scaling $(\times 4)$ of the fractal measure space?

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- *With respect to the W_k ordering of Γ, the matrix of U has block diagonal form...and the infinite blocks are all the same!*
- *Even more, in the $(\times 4, \times 5)$ case U actually has a self-similar structure:*

$$U = (e_0 \otimes e_0) \oplus \bigoplus_{k=1}^{\infty} M_{e_1} U.$$

We call U an operator-fractal.

Matrix of U

	0	Γ_0	Γ_1	Γ_2	Γ_3	$\cdots$
0	1	0	0	0	0	$\cdots$
Γ_0	0	$M_{e_1}U$	0	0	0	$\cdots$
Γ_1	0	0	$M_{e_1}U$	0	0	$\cdots$
Γ_2	0	0	0	$M_{e_1}U$	0	$\cdots$
Γ_3	0	0	0	0	$M_{e_1}U$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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- *1 is the only eigenvalue of U. ($Ue_0 = e_0$)*
- *U is not spatially implemented; i.e. is not of the form $Uf = f \circ \tau$ for τ a point transformation on $[0, 1]$.*
- *U only fixes the constant functions; if $Uv = v$ then $v = ce_0$ for some $c \in \mathbb{C}$. In other words, U is an ergodic operator.*

Thank You