Group-theoretic constructions of erasure-robust frames

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February 21, 2013

The Restricted Isometry Property (RIP)

Definition: Fix $K \leq M \leq N$ and let $\Phi = [\varphi_1 \ \cdots \ \varphi_N] \in \mathbb{R}^{M \times N}$.

We say Φ has the (K, δ)-**Restricted Isometry Property (RIP)** if for every K-element subset $\mathcal{K}$ of $\{1, \ldots, N\}$, we have

$$(1 - \delta) \sum_{n \in \mathcal{K}} |y(n)|^2 \leq \left\| \sum_{n \in \mathcal{K}} y(n) \varphi_n \right\|^2 \leq (1 + \delta) \sum_{n \in \mathcal{K}} |y(n)|^2,$$

for all $y \in \mathbb{R}^N$.

Fact: For any K-element subset $\mathcal{K}$ of $\{1, \ldots, N\}$, consider the $M \times K$ submatrix $\Phi_{\mathcal{K}}$ of Φ with columns $\{\varphi_n\}_{n \in \mathcal{K}}$.

Then Φ is (K, δ)-RIP if and only if the eigenvalues of $\Phi_{\mathcal{K}}^{\mathrm{T}} \Phi_{\mathcal{K}}$ lie in $[1 - \delta, 1 + \delta]$ for all $\mathcal{K}$.

RIP: Il buono, il brutto, il cattivo

The Good: Candès and Tao showed that L1-minimization can be used to quickly and stably find a unique K-sparse solution y to an *underdetermined* linear system $\Phi y = z$, provided the matrix Φ is $(2K, \delta)$-RIP and a sparse solution exists.

Moreover, they showed that with overwhelming probability, certain random matrices will be (K, δ)-RIP for $K = \mathcal{O}(M/\log(N))$.

The Bad: All known deterministic constructions of RIP matrices are only guaranteed to be (K, δ)-RIP for $K = \mathcal{O}(M^{\frac{1}{2}+\varepsilon})$. This is known as the **square root bottleneck**.

The Ugly: Directly checking whether or not a given Φ has the RIP involves estimating the singular values of $\binom{N}{K}$ possible submatrices of Φ; see "Certifying the Restricted Isometry Property is Hard" by Bandeira, Dobriban, Mixon and Sawin (2013).

Numerically Erasure-Robust Frames (NERFs)

Definition: Fix $M \leq K \leq N$ and let $\Phi = [\varphi_1 \ \cdots \ \varphi_N] \in \mathbb{R}^{M \times N}$.

We say $\{\varphi_n\}_{n=1}^N$ is a (K, α, β)-**NERF** for $\mathbb{R}^M$ if there exists $0 < \alpha \leq \beta < \infty$ such that for every K-element subset $\mathcal{K}$ of $\{1, \ldots, N\}$ we have $\{\varphi_n\}_{n \in \mathcal{K}}$ is a **frame** for $\mathbb{R}^M$ with frame bounds α and β:

$$\alpha \|x\|^2 \leq \sum_{n \in \mathcal{K}} |\langle x, \varphi_n \rangle|^2 \leq \beta \|x\|^2, \quad \forall x \in \mathbb{R}^M.$$

Here we want the eigenvalues of $\Phi_{\mathcal{K}} \Phi_{\mathcal{K}}^{\mathrm{T}}$ to lie in $[\alpha, \beta]$ for all $\mathcal{K}$.

Recall: For $K \leq M \leq N$, Φ has the (K, δ)-RIP if the eigenvalues of $\Phi_{\mathcal{K}}^{\mathrm{T}} \Phi_{\mathcal{K}}$ lie in $[1 - \delta, 1 + \delta]$ for all $\mathcal{K} \subseteq \{1, \ldots, N\}$, $|\mathcal{K}| = K$.

Optimal NERF Bounds

Note: For any fixed K, the optimal NERF bounds α_K and β_K for $\{\varphi_n\}_{n=1}^N$ are the extreme eigenvalues of $\Phi_{\mathcal{K}}^{\mathrm{T}}\Phi_{\mathcal{K}}$:

$$\alpha_K := \min_{|\mathcal{K}|=K} \min_{\|x\|=1} \sum_{n\in\mathcal{K}} |\langle x, \varphi_n\rangle|^2, \ \beta_K := \max_{|\mathcal{K}|=K} \max_{\|x\|=1} \sum_{n\in\mathcal{K}} |\langle x, \varphi_n\rangle|^2.$$

Estimating α_K and β_K thus seems combinatorially difficult.

Idea: Rather than find the "worst x for any $\mathcal{K}$," let's instead find the "worst $\mathcal{K}$ for any x," namely interchange the optimizations:

$$\alpha_K := \min_{\|x\|=1} \min_{|\mathcal{K}|=K} \sum_{n\in\mathcal{K}} |\langle x, \varphi_n\rangle|^2, \ \beta_K := \max_{\|x\|=1} \max_{|\mathcal{K}|=K} \sum_{n\in\mathcal{K}} |\langle x, \varphi_n\rangle|^2.$$

For a fixed x, these worst $\mathcal{K}$'s are found by sorting $\left\{|\langle x, \varphi_n\rangle|^2\right\}_{n=1}^N$.

Problem: There are an infinite number of x's on the unit sphere.

ε-Nets

Definition: Given $\varepsilon > 0$, a sequence of unit norm vectors $\{\psi_p\}_{p=1}^P$ is an ε-**net** for $\mathbb{S}^{M-1}$ (with respect to chordal distance) if for all $x \in \mathbb{S}^{M-1}$ there exists ψ_p such that $|\langle x, \psi_p\rangle|^2 \geq 1 - \varepsilon^2$.

Idea: Given a ε-net $\{\psi_p\}_{p=1}^P$ for $\mathbb{S}^{M-1}$, estimate the optimal NERF bounds for $\{\varphi_n\}_{n=1}^N$ as the ε-**approximate NERF bounds**:

$$\alpha_{K,\varepsilon} := \min_{p=1,\dots,P} \sum_{n=1}^{K} |\langle \psi_p, \varphi_{\sigma(n)}\rangle|^2,$$

$$\beta_{K,\varepsilon} := \max_{p=1,\dots,P} \sum_{n=N-K+1}^{N} |\langle \psi_p, \varphi_{\sigma(n)}\rangle|^2,$$

where σ is a p-dependent permutation of $\{1,\dots,N\}$ chosen so that the values $\left\{|\langle \psi_p, \varphi_{\sigma(n)}\rangle|^2\right\}_{n=1}^N$ are nondecreasing.

Estimating NERF Bounds with ε-Nets

Theorem: [FJMP (2012)] Optimal NERF bounds α_K and β_K are estimated by ε-approximate bounds $\alpha_{K,\varepsilon}$ and $\beta_{K,\varepsilon}$ according to

$$\tfrac{1}{1-\varepsilon^2}\left(\alpha_{K,\varepsilon} - \tfrac{\varepsilon^2}{1-\varepsilon^2}\beta_{K,\varepsilon}\right) \leq \alpha_K \leq \alpha_{K,\varepsilon}, \quad \beta_{K,\varepsilon} \leq \beta_K \leq \tfrac{1}{1-\varepsilon^2}\beta_{K,\varepsilon}.$$

Moreover, if $\{\varphi_n\}_{n=1}^N$ is a **unit-norm tight frame (UNTF)** ($\Phi\Phi^{\mathrm{T}} = \frac{N}{M}\mathrm{I}$ and $\|\varphi_n\| = 1$ for all n) then we also have

$$\tfrac{1}{1-\varepsilon^2}(\alpha_{K,\varepsilon} - \varepsilon^2 \tfrac{N}{M}) \leq \alpha_K \leq \beta_K \leq \tfrac{N}{M}.$$

Note: For every fixed ψ_p, we compute $\left\{|\langle\psi_p, \varphi_n\rangle|^2\right\}_{n=1}^N$, and then sort these values so as to sum the K smallest and largest ones. Taking the minimum and maximum of these sums over all p yields $\alpha_{K,\varepsilon}$ and $\beta_{K,\varepsilon}$. This uses $\mathcal{O}((M + \log N)NP)$ operations overall.

Problem: Good ε-nets are enormous, e.g. $P = (1 + \frac{2}{\varepsilon})^M$.

Group Frames

Definition: Let $\mathcal{U} = \{U_q\}_{q=1}^{Q}$ be a finite group of $M \times M$ orthogonal matrices. We say that $\{\varphi_n\}_{n=1}^{N}$ is $\mathcal{U}$-**invariant** if

$$\forall q \ \exists \text{ a permutation } \sigma \text{ of } \{1,\ldots,N\} \text{ s.t. } U_q\varphi_n = \pm\varphi_{\sigma(n)}, \forall n.$$

Note: We focus exclusively on the $2^M M!$-element group of signed permutation matrices that arises the symmetry group of the hypercube in $\mathbb{R}^M$. This group is **irreducible** meaning the orbit of any unit norm vector under its action is a UNTF.

Example: When $M = 4$, there are $2^4 4! = (16)(24) = 384$ distinct 4×4 signed permutation matrices. The following 4×12 UNTF is invariant under the action of this group:

$$\Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}.$$

Estimating NERF Bounds with Group-Generated ε-Nets

Idea: If $\{\varphi_n\}_{n=1}^N$ is $\mathcal{U}$-invariant, then $\forall q$, $\left\{|\langle U_q\psi_r, \varphi_n\rangle|^2\right\}_{n=1}^N$ has the same K smallest and largest values as $\left\{|\langle \psi_r, \varphi_n\rangle|^2\right\}_{n=1}^N$.

Theorem: [FJMP (2012)] Let $\mathcal{U}$ be a finite group of $M \times M$ orthogonal matrices and let $\{\varphi_n\}_{n=1}^N$ be $\mathcal{U}$-invariant. Choosing $\{\psi_r\}_{r=1}^R \subseteq \mathbb{S}^{M-1}$ such that $\{U_q\psi_r\}_{q=1,\ r=1}^{Q\quad R}$ is an ε-net for $\mathbb{S}^{M-1}$, the corresponding ε-approximate NERF bounds are:

$$\alpha_{K,\varepsilon} = \min_{r=1,\dots,R} \sum_{n=1}^{K} |\langle \psi_r, \varphi_{\sigma(n)}\rangle|^2,$$

$$\beta_{K,\varepsilon} = \max_{r=1,\dots,R} \sum_{n=N-K+1}^{N} |\langle \psi_r, \varphi_{\sigma(n)}\rangle|^2,$$

where σ is chosen so that $\left\{|\langle \psi_r, \varphi_{\sigma(n)}\rangle|^2\right\}_{n=1}^N$ is nondecreasing.

An ε-Net for Nonnegative, Nonincreasing Vectors

Note: For any $x \in \mathbb{S}^{M-1}$, there exists a signed permutation U_q such that $U_q x$ is nonnegative and nonincreasing, namely such that

$$U_q x \in \mathbb{S}^{M-1}_{\mathrm{nn}} := \{x \in \mathbb{S}^{M-1} : \; 0 \le x(1) \le \cdots \le x(M)\}.$$

Lemma: [FJMP (2012)] Let $\{\psi_r\}_{r=1}^R \subseteq \mathbb{S}^{M-1}_{\mathrm{nn}}$ and let $\{U_q\}_{q=1}^Q$ be the group all $M \times M$ signed permutations. Then $\{U_q \psi_r\}_{q=1,\, r=1}^{Q \quad R}$ is an ε-net for $\mathbb{S}^{M-1}$ if and only if $\{\psi_r\}_{r=1}^R$ is an ε-net for $\mathbb{S}^{M-1}_{\mathrm{nn}}$.

Note: When combined with the previous result, this means that in order to estimate the NERF bounds of a $\mathcal{U}$-invariant frame, we only need to compute $\left\{|\langle \psi_r, \varphi_n \rangle|^2\right\}_{n=1}^N$ at every point ψ_r of an ε-net for $\mathbb{S}^{M-1}_{\mathrm{nn}}$ instead of at every point of an ε-net for $\mathbb{S}^{M-1}$.

The surface area of $\mathbb{S}^{M-1}_{\mathrm{nn}}$ is that of $\mathbb{S}^{M-1}$ divided by $2^M M!$.

Vector Quantization by "Rounding Up"

Lemma: [FJMP (2012)] For any positive integer M and $\varepsilon > 0$, let $\delta = [M(L-1)]^{-\frac{1}{2L}}$ and take any $L \geq 2$ such that

$$(L-1)(1-\varepsilon^2)^L \leq \tfrac{1}{M}\left(\tfrac{L-1}{L}\right)^L.$$

Then for any $x \in \mathbb{S}_{\mathrm{nn}}^{M-1}$, the step function $\psi_x = \hat{\psi}_x / \|\hat{\psi}_x\|$,

$$\hat{\psi}_x(m) := \begin{cases} \delta^l, & \delta^{l+1} < x(m) \leq \delta^l, \\ \delta^{L-1}, & 0 \leq x(m) \leq \delta^{L-1}, \end{cases}$$

satisfies $\langle x, \psi_x \rangle > \sqrt{1-\varepsilon^2}$.

Note: The set of all such ψ_x's forms an ε-net for $\mathbb{S}_{\mathrm{nn}}^{M-1}$. Since each ψ_x arises from a unique nonincreasing $\{1,\ldots,L\}$-valued function over $\{1,\ldots,M\}$, "stars and bars" reveals the number of elements in this ε-net to be at most $\binom{M+L-1}{L-1} \leq C_1 M^{C_2(\varepsilon)\log M}$.

Main Result

Theorem: [FJMP (2012)] Let $\{\varphi_n\}_{n=1}^N$ be a UNTF for $\mathbb{R}^M$ which is invariant under signed permutations. For any $\varepsilon > 0$, take δ and L as in the previous lemma and construct $\{\psi_r\}_{r=1}^R$ by normalizing all $\{\delta^l\}_{l=0}^{L-1}$-valued nondecreasing step functions.

Then for any $M \le K \le N$, the optimal NERF bounds α_K and β_K of $\{\varphi_n\}_{n=1}^N$ satisfy the estimates

$$\tfrac{1}{1-\varepsilon^2}\Big(\alpha_{K,\varepsilon} - \varepsilon^2 \min\big\{\tfrac{N}{M}, \tfrac{1}{1-\varepsilon^2}\beta_{K,\varepsilon}\big\}\Big) \le \alpha_K \le \alpha_{K,\varepsilon},$$
$$\beta_{K,\varepsilon} \le \beta_K \le \min\big\{\tfrac{N}{M}, \tfrac{1}{1-\varepsilon^2}\beta_{K,\varepsilon}\big\},$$

where $\alpha_{K,\varepsilon}$ and $\beta_{K,\varepsilon}$ are found by the following process:

For any $r = 1, \dots, R$, let $\alpha_{K,\varepsilon,r}$ and $\beta_{K,\varepsilon,r}$ be the sums of the K smallest and largest values of $\big\{|\langle\psi_r, \varphi_n\rangle|^2\big\}_{n=1}^N$, respectively.

Let $\alpha_{K,\varepsilon} = \min_r \alpha_{K,\varepsilon,r}$ and $\beta_{K,\varepsilon} = \max_r \beta_{K,\varepsilon,r}$.

Numerical Example: $M = 4$, $N = 12$

Size of ε-net:

ε^2	L	$\binom{M+L-1}{L-1}$	R_{improved}
2^{-1}	6	126	45
2^{-2}	19	7315	1107
2^{-3}	47	230300	15916
2^{-4}	110	6438740	202628
2^{-5}	249	164059875	2366922

ε-approximate lower NERF bounds $\alpha_{K,\varepsilon}$:

$\varepsilon^2 \backslash K$	1	2	3	4	5	6	7	8	9	10	11	12
2^{-1}	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.72	1.00	1.58	2.10	3.00
2^{-2}	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.52	2.03	3.00
2^{-3}	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.50	2.01	3.00
2^{-4}	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.50	2.00	3.00
2^{-5}	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.50	2.00	3.00
α_K	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.50	2.00	3.00

$\frac{1}{1-\varepsilon^2}\left(\alpha_{K,\varepsilon} - \varepsilon^2 \min\left\{\frac{N}{M}, \frac{1}{1-\varepsilon^2}\beta_{K,\varepsilon}\right\}\right)$:

$\varepsilon^2 \backslash K$	1	2	3	4	5	6	7	8	9	10	11	12
2^{-1}	-3.00	-3.00	-3.00	-3.00	-3.00	-3.00	-2.23	-1.54	-0.99	0.16	1.21	3.00
2^{-2}	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00	-0.49	-0.04	0.33	1.02	1.71	3.00
2^{-3}	-0.42	-0.42	-0.42	-0.42	-0.42	-0.42	0.00	0.39	0.71	1.29	1.87	3.00
2^{-4}	-0.20	-0.20	-0.20	-0.20	-0.20	-0.20	0.20	0.56	0.86	1.40	1.93	3.00
2^{-5}	-0.09	-0.09	-0.09	-0.09	-0.09	-0.09	0.29	0.64	0.93	1.45	1.96	3.00
α_K	0.00	0.00	0.00	0.00	0.00	0.00	0.38	0.71	1.00	1.50	2.00	3.00

More Numerical Examples

Example: Let $M = 6$ and let $\{\varphi_n\}_{n=1}^N$ be the 80 signed permutations of $\varphi = [1\ 1\ 1\ 0\ 0\ 0]^{\mathrm{T}}$ which are distinct modulo negation. Taking $\varepsilon = \frac{1}{2}$, our MATLAB code took around 8.84 seconds to show that any 61 of these 80 frame elements span $\mathbb{R}^6$. Obtaining this same fact directly involves forming each of the $\binom{80}{61} \approx 1.16 \times 10^{18}$ such submatrices.

Example: Let $M = 8$ and let $\{\varphi\}_{n=1}^N$ be the 560 distinct signed permutations of $\varphi = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]^{\mathrm{T}}$. Taking $\varepsilon = \frac{1}{2}$, our MATLAB code took around three minutes to show that any 399 of these 560 frame elements span $\mathbb{R}^9$. Note $\binom{560}{399} \approx 2.94 \times 10^{144}$.

Example: Let $M = 10$ and let $\{\varphi\}_{n=1}^N$ be the 4032 distinct signed permutations of $\varphi = [1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0]^{\mathrm{T}}$. Taking $\varepsilon = \frac{1}{2}$, our MATLAB code took around 77 minutes to show that any 2883 of these 4032 frame elements span $\mathbb{R}^{10}$. Note $\binom{4032}{2883} \approx 3.65 \times 10^{1044}$.

Conclusions and Future Work

Conclusions:

- Using explicit ε-nets reduces the problem of numerically estimating the optimal NERF bounds of any given finite frame from being exponential in N to being exponential in M.
- Exploiting symmetry can further reduce this problem to being subexponential in M provided the frame is group-invariant.
- Even after these speedups, estimating NERF bounds still involves a large amount of computation.

Future Work:

- Methods for constructing smaller ε-nets for $\mathbb{S}_{\mathrm{nn}}^{M-1}$?
- Apply these techniques to the deterministic RIP problem?