Bilinear pseudodifferential operators of Hörmander type

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Outline of the talk

- Linear ψDOs
- Some classical boundedness results
- Bilinear ψDOs
- Results and comparison to linear case

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Fourier analysis

For a function f, two complementary representations:

- The function $f(x)$ itself (spatial behavior)
- The Fourier transform $\widehat{f}(\xi)$ (frequency behavior)

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx$$

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

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Linear multipliers

The synthesis formula above is:

$$Id(f)(x) = \int_{\mathbb{R}^d} \underbrace{(2\pi)^{-d}}_{m} \widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

Translation invariant extension:

$$T_m(f)(x) = \int_{\mathbb{R}^d} m(\xi)\widehat{f}(\xi) e^{ix\cdot\xi}\, d\xi$$

Theorem (Mihlin, 1956)

If $|\partial^\beta m(\xi)| \lesssim (1+|\xi|)^{-|\beta|}$*, then* $T_\sigma : L^p \to L^p, 1 < p < \infty$.

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Linear pseudodifferential operators (ψDOs)

Non-translation invariant extension:

$$T_\sigma(f)(x) = \int_{\mathbb{R}^d} \sigma(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi$$

Theorem (Ching, 1972; a question of Nirenberg)

If $|\partial_\xi^\beta \sigma(x,\xi)| \lesssim (1+|\xi|)^{-|\beta|}$, *then* $T_\sigma : L^2 \not\to L^2$.

Boundeness requires also some a priori smoothness in x!

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(Linear) Hörmander classes of symbols

Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. A symbol $\sigma(x, \xi)$ belongs to the Hörmander class $S^m_{\rho,\delta}$ if

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim (1 + |\xi|)^{m + \delta|\alpha| - \rho|\beta|}$$

In particular: $\sigma \in S^0_{1,0} \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim (1 + |\xi|)^{-|\beta|}$.

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Connection to Calderón-Zygmund theory

Note that: $S^0_{1,0} \subset S^0_{1,\delta} \subset S^0_{1,1}$.

Theorem

The class $S^0_{1,1}$ is the largest one such that T_σ has a Calderón-Zygmund kernel.

That is,

$$T_\sigma(f)(x) = \int K(x,y) f(y)\, dy,$$

where $K(x,y)$ satisfies

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \lesssim |x-y|^{-n-|\alpha|-|\beta|}.$$

In particular, $T_\sigma : L^p \to L^p \Leftrightarrow T_\sigma : L^2 \to L^2$.

$$S^0_{1,\delta} : L^2 \to L^2, 0 \le \delta < 1 \text{ but } S^0_{1,1} : L^2 \not\to L^2$$

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Some examples

1. Let $a_k \in C^\infty$ and $|\partial_x^\alpha a_k(x)| \lesssim 1$. Define the PDO

$$T = \sum_{|k| \leq m} a_k(x) \partial_x^k.$$

Then: $T = T_\sigma$, where

$$\sigma(x, \xi) = \sum_{|k| \leq m} a_k(x)(i\xi)^k.$$

We have: $\sigma \in S^m_{1,0}$.

2. Let $|\partial_x^\alpha a_k(x)| \lesssim 2^{k|\alpha|}$ and $\psi(\xi)$ supported in $1/2 \leq |\xi| \leq 2$. Define

$$\sigma(x, \xi) = \sum_{k=1}^{\infty} a_k(x) \psi(2^{-k}\xi).$$

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3. The heat operator

$$L = \partial_t - \sum_{k=1}^{n} \partial^2_{x_k^2}$$

has an approximate inverse $T = T_\sigma$ $(LT \sim I)$ and

$$\sigma \in S^{-1}_{1/2,0}.$$

The classes $S^0_{\rho,\rho}$

Motivation

Kumano-go, Nagase-Shinkai ('70s): applications to parabolic and semi-elliptic operators

Theorem (Calderón-Vaillancourt, 1970)

If $\sigma \in S^0_{0,0}$, then $T_\sigma : L^2 \to L^2$ (but not on L^p, $p \neq 2$, in general).

Recall that

$$\sigma \in S^0_{0,0} \Leftrightarrow |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \lesssim 1.$$

Theorem (Cordes, 1975)

If $\sigma \in S^0_{\rho,\rho}, 0 \leq \rho < 1$, then $T_\sigma : L^2 \to L^2$.

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Theorem (Fefferman-Stein, 1972)

If $\sigma \in S^m_{\rho,0}, 0 < \rho < 1, -(1-\rho)n/2 < m \leq 0$, then $T_\sigma : L^2 \to L^2$.

Theorem (Fefferman, 1973)

If $\sigma \in S^{-(1-\rho)n/2}_{\rho,0}, 0 < \rho \leq 1$, then $T_\sigma : L^\infty \to BMO$.

Fefferman's result uses the fact (due to Hörmader, '70s) that

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$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}$$

Associated to such a symbol we have a bilinear ψDO :

$$T_\sigma(f, g)(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix\cdot(\xi+\eta)} d\xi d\eta.$$

Bilinear ψDOs generalize the product of two functions $f \cdot g$.

Question

Do the results for linear ψDOs go through in the bilinear case?

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1. Let $\xi, \eta \in \mathbb{R}$ and $\sigma(\xi, \eta) = \xi^k \eta^l (1 + |\xi|^2 + |\eta|^2)^{-1/2}$.
We have: $\sigma \in BS^{k+l}_{1,0}$.

2. Let $\sigma(\xi, \eta) = \varphi(\xi, \eta)(1 + |\xi|^2 + |\eta|)^{-1}$, where φ is a smooth function such that $\varphi = 1$ away from the set $\{(\xi, \eta) : \eta = 0\}$.
We have: $\sigma \in BS^{-1}_{\frac{1}{2},0}$.

3. Similarly, we have

$$\varphi(\xi, \eta)(1 + |\xi + \eta|^2 + |\xi|^2 + |\eta|)^{-1} \in BS^{-2}_{1,0}.$$

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Bilinear ψDOs : why?

1. Multilinear operators as intermediate tools to study specific linear and nonlinear operators (Coifman-Meyer, '70s)
2. Commutator estimates to study the regularity of solutions of nonlinear PDEs (Kato-Ponce, '88)
3. Proof of Calderón's conjecture on the boundedness of the bilinear Hilbert transform. This question was posed in connection with the Cauchy integral on Lipschitz curves and the so-called Calderón commutators (Lacey-Thiele, '97; Grafakos-Li, '01)
4. Bilinear pseudodifferential operators with non-smooth symbols (Gilbert-Nahmod, Muscalu-Tao-Thiele, '99)
5. Systematic study of multilinear singular integrals (Grafakos-Torres, '99)
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The bilinear Coifman-Meyer classes: $BS^0_{1,\delta}, 0 \leq \delta < 1$

Theorem (Coifman-Meyer '78; Grafakos-Torres '02; B.-Torres '03)

If $\sigma \in BS^0_{1,0}$, *then* $T_\sigma : L^p \times L^q \to L^r$, $1/p + 1/q = 1/r < 2$.

Theorem (B.-Oh, '10)

If $\sigma \in BS^0_{1,\delta}$, $0 \leq \delta < 1$, *then* $T_\sigma : L^p \times L^q \to L^r$, $1/p + 1/q = 1/r < 2$.

Tools: Littlewood-Paley theory; elementary symbols.

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Calderón-Zygmund theory and transposition calculus

Theorem (Grafakos-Torres, '02)

The class $BS^0_{1,1}$ is the largest one to produce bilinear Calderón-Zygmund kernels.

That is,

$$T_\sigma(f,g)(x) = \int\int K(x,y,z)f(y)g(z)\,dydz,$$

and $K(x,y,z)$ satisfies appropriate smoothness-decay estimates.

Both previous ψDO boundedness results on the Coifman-Meyer classes follow once we can establish a transposition symbolic calculus.

Theorem (B.-Maldonado-Naibo-Torres, '10)

If $\sigma \in BS^m_{\rho,\delta}, 0 \le \delta < \rho \le 1$, then $T^{*j}_\sigma = T_{\sigma^{*j}}$ with $\sigma^{*j} \in BS^m_{\rho,\delta}, j = 1,2$.

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The bilinear Calderón-Vaillancourt classes: $BS^0_{\rho,\rho}$

Theorem (B.-Torres, '04)

There exists a symbol in $BS^0_{0,0}$ such that $T : L^2 \times L^2 \not\to L^1$.

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)

If $\sigma \in BS^0_{\rho,\rho}$, $0 \leq \rho < 1$, then $BS^0_{\rho,\rho} : L^2 \times L^2 \not\to L^1$.

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Tool: almost orthogonality.

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The bilinear Calderón-Vaillancourt classes: $BS^0_{\rho,\rho}$

Theorem (B.-Torres, '04)

There exists a symbol in $BS^0_{0,0}$ such that $T : L^2 \times L^2 \not\to L^1$.

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)

If $\sigma \in BS^0_{\rho,\rho}$, $0 \leq \rho < 1$, then $BS^0_{\rho,\rho} : L^2 \times L^2 \not\to L^1$.

Theorem (B.-Torres, '04)

If $\sigma \in BS^0_{0,0}$ and $\partial^\alpha_\xi \sigma \in L^\infty_x L^1_\xi L^2_\eta$, $\partial^\alpha_\eta \sigma \in L^\infty_x L^1_\eta L^2_\xi$, then $T : L^2 \times L^2 \to L^1$.

Tool: almost orthogonality.

A link to modulation spaces

Theorem (B.-Gröchenig-Heil-Okoudjou, '05)

If $\sigma \in BS^0_{0,0}$, *then* $T : L^2 \times L^2 \to M^{1,\infty} \supseteq L^1$

An instructive statement (not completely correct):

$$f \in M^{p,q} \sim f \in L^p \text{ and } \hat{f} \in L^q$$

Fefferman's result in the bilinear case

Although the classes $BS^0_{\rho,\delta}$ fail to be bounded on products of Lebesgue spaces, we have surprisingly

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)

If $\sigma \in BS^{n(\rho-1)}_{\rho,0}$, $0 \le \rho < \frac{1}{2}$, *then* $T_\sigma : L^\infty \times L^\infty \to BMO$.

The crucial observation in the proof:

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)

If λ *is a symbol such that*

$$\sup_{\substack{|\beta| \le [\frac{n}{2}]+1 \\ |\alpha| \le 2(2n+1)}} \sup_{\xi, y \in \mathbb{R}^n} \|\partial_\xi^\alpha \partial_y^\beta \lambda(y, \xi - \cdot, \cdot)\|_{L^2} < \infty,$$

then $T_\lambda : L^2 \times L^2 \to L^2$.

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Thank you!