Bilinear pseudodifferential operators of Hörmander type

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Outline of the talk

- Linear $\psi$DOs
  - Some classical boundedness results
  - Bilinear $\psi$DOs
  - Results and comparison to linear case
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Fourier analysis

For a function $f$, two complementary representations:

- The function $f(x)$ itself (spatial behavior)
- The Fourier transform $\hat{f}(\xi)$ (frequency behavior)

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, dx
\]

\[
f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} \, d\xi
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f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i x \cdot \xi} \, d\xi
\]
The synthesis formula above is:

$$Id(f)(x) = \int_{\mathbb{R}^d} (2\pi)^{-d} \hat{f}(\xi) e^{ix \cdot \xi} d\xi$$

Translation invariant extension:

$$T_m(f)(x) = \int_{\mathbb{R}^d} m(\xi) \hat{f}(\xi) e^{ix \cdot \xi} d\xi$$

Theorem (Mihlin, 1956)

If $$|\partial^\beta m(\xi)| \lesssim (1 + |\xi|)^{-|\beta|}$$, then $$T_\sigma : L^p \to L^p, 1 < p < \infty.$$
Linear multipliers

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Theorem (Mihlin, 1956)

If $|\partial^\beta m(\xi)| \lesssim (1 + |\xi|)^{-|\beta|}$, then $T_\sigma : L^p \to L^p, 1 < p < \infty$. 
Non-translation invariant extension:

$$T_\sigma(f)(x) = \int_{\mathbb{R}^d} \sigma(x, \xi) \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi$$

Theorem (Ching, 1972; a question of Nirenberg)

If $$|\partial_\xi^\beta \sigma(x, \xi)| \lesssim (1 + |\xi|)^{-|\beta|}$$, then $$T_\sigma : L^2 \not\to L^2$$.

Boundeness requires also some a priori smoothness in $$x$$!
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(Linear) Hörmander classes of symbols

Let \( m \in \mathbb{R} \) and \( 0 \leq \rho, \delta \leq 1 \). A symbol \( \sigma(x, \xi) \) belongs to the Hörmander class \( S_{\rho,\delta}^m \) if

\[
| \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) | \lesssim (1 + |\xi|)^{m+\delta|\alpha|-\rho|\beta|}
\]

In particular: \( \sigma \in S_{1,0}^0 \iff | \partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi) | \lesssim (1 + |\xi|)^{-|\beta|} \).

**Theorem (Coifman-Meyer, ’70s)**

If \( \sigma \in S_{1,0}^0 \), then \( T_\sigma : L^p \to L^p, 1 < p < \infty \).
Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. A symbol $\sigma(x, \xi)$ belongs to the Hörmander class $S_{\rho,\delta}^m$ if

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**Theorem (Coifman-Meyer, ’70s)**

If $\sigma \in S_{1,0}^0$, then $T_\sigma : L^p \to L^p$, $1 < p < \infty$. 
Connection to Calderón-Zygmund theory

Note that: \( S_{1,0}^0 \subset S_{1,\delta}^0 \subset S_{1,1}^0 \).

**Theorem**

The class \( S_{1,1}^0 \) is the largest one such that \( T_\sigma \) has a Calderón-Zygmund kernel.

That is,

\[
T_\sigma(f)(x) = \int K(x, y)f(y) \, dy,
\]

where \( K(x, y) \) satisfies

\[
|\partial_x^\alpha \partial_y^\beta K(x, y)| \lesssim |x - y|^{-n - |\alpha| - |\beta|}.
\]

In particular, \( T_\sigma : L^p \to L^p \iff T_\sigma : L^2 \to L^2 \).

\( S_{1,\delta}^0 : L^2 \to L^2, 0 \leq \delta < 1 \) but \( S_{1,1}^0 : L^2 \not\to L^2 \).
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Some examples

1. Let $a_k \in C^\infty$ and $|\partial_x^\alpha a_k(x)| \lesssim 1$. Define the PDO

$$T = \sum_{|k| \leq m} a_k(x) \partial_x^k.$$ 

Then: $T = T_\sigma$, where

$$\sigma(x, \xi) = \sum_{|k| \leq m} a_k(x)(i\xi)^k.$$ 

We have: $\sigma \in S_{1,0}^m$.

2. Let $|\partial_x^\alpha a_k(x)| \lesssim 2^{|\alpha|} \psi(\xi)$ supported in $1/2 \leq |\xi| \leq 2$. Define

$$\sigma(x, \xi) = \sum_{k=1}^\infty a_k(x)\psi(2^{-k}\xi).$$

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$$\sigma(x, \xi) = \sum_{k=1}^{\infty} a_k(x)\psi(2^{-k}\xi).$$ 

We have: $\sigma \in S^0_{1,1}$. 
3. The heat operator

\[ L = \partial_t - \sum_{k=1}^{n} \partial^2_{x_k^2} \]

has an approximate inverse \( T = T_\sigma \) \((LT \sim I)\) and

\[ \sigma \in S_{1/2,0}^{-1}. \]
The classes $S^0_{\rho,\rho}$

Motivation
Kumano-go, Nagase-Shinkai (‘70s): applications to parabolic and semi-elliptic operators

Theorem (Calderón-Vaillancourt, 1970)
If $\sigma \in S^0_{0,0}$, then $T_\sigma : L^2 \to L^2$ (but not on $L^p$, $p \neq 2$, in general).

Recall that
$$\sigma \in S^0_{0,0} \iff |\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \lesssim 1.$$ 

Theorem (Cordes, 1975)
If $\sigma \in S^0_{\rho,\rho}$, $0 \leq \rho < 1$, then $T_\sigma : L^2 \to L^2$. 
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The classes $S^m_{\rho,0}$

**Theorem (Fefferman-Stein, 1972)**

If $\sigma \in S^m_{\rho,0}, 0 < \rho < 1, -(1 - \rho)n/2 < m \leq 0$, then $T_\sigma : L^2 \to L^2$.

**Theorem (Fefferman, 1973)**

If $\sigma \in S^{-(1-\rho)n/2}_{\rho,0}, 0 < \rho \leq 1$, then $T_\sigma : L^\infty \to BMO$.

Fefferman’s result uses the fact (due to Hörmader, ’70s) that

$$S^0_{\rho,\delta} : L^2 \to L^2, 0 < \delta < \rho \leq 1.$$
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Fefferman’s result uses the fact (due to Hörmader, ’70s) that

$$S^0_{\rho,\delta} : L^2 \rightarrow L^2, 0 < \delta < \rho \leq 1.$$
Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. A symbol $\sigma(x, \xi, \eta)$ belongs to the bilinear Hörmander class $BS^{m}_{\rho,\delta}$ if

$$|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} \partial_{\eta}^{\gamma} \sigma(x, \xi, \eta)| \lesssim (1 + |\xi| + |\eta|)^{m+\delta|\alpha| - \rho(|\beta| + |\gamma|)}$$

Associated to such a symbol we have a bilinear $\psi$DO:

$$T_{\sigma}(f, g)(x) = \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$ 

Bilinear $\psi$DOs generalize the product of two functions $f \cdot g$.

**Question**

Do the results for linear $\psi$DOs go through in the bilinear case?
Let $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. A symbol $\sigma(x, \xi, \eta)$ belongs to the bilinear Hörmander class $BS^m_{\rho, \delta}$ if

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*Do the results for linear $\psi$DOs go through in the bilinear case?*
Some examples

1. Let \( \xi, \eta \in \mathbb{R} \) and \( \sigma(\xi, \eta) = \xi^k \eta^l (1 + |\xi|^2 + |\eta|^2)^{-1/2} \).
   We have: \( \sigma \in BS_{1,0}^{k+l} \).

2. Let \( \sigma(\xi, \eta) = \varphi(\xi, \eta)(1 + |\xi|^2 + |\eta|)^{-1} \), where \( \varphi \) is a smooth function such that \( \varphi = 1 \) away from the set \( \{ (\xi, \eta) : \eta = 0 \} \).
   We have: \( \sigma \in BS_{-1/2,0}^{-1} \).

3. Similarly, we have

\[
\varphi(\xi, \eta)(1 + |\xi + \eta|^2 + |\xi|^2 + |\eta|)^{-1} \in BS_{1,0}^{-2}.
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2. Let $\sigma(\xi, \eta) = \varphi(\xi, \eta)(1 + |\xi|^2 + |\eta|)^{-1}$, where $\varphi$ is a smooth function such that $\varphi = 1$ away from the set $\{(\xi, \eta) : \eta = 0\}$. We have: $\sigma \in BS^{-1}_{1/2,0}$.

3. Similarly, we have

$$\varphi(\xi, \eta)(1 + |\xi + \eta|^2 + |\xi|^2 + |\eta|)^{-1} \in BS^{-2}_{1,0}.$$
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Bilinear $\psi$DOs: why?

1. Multilinear operators as intermediate tools to study specific linear and nonlinear operators (Coifman-Meyer, ’70s)
2. Commutator estimates to study the regularity of solutions of nonlinear PDEs (Kato-Ponce, ’88)
3. Proof of Calderón’s conjecture on the boundedness of the bilinear Hilbert transform. This question was posed in connection with the Cauchy integral on Lipschitz curves and the so-called Calderón commutators (Lacey-Thiele, ’97; Grafakos-Li, ’01)
4. Bilinear pseudodifferential operators with non-smooth symbols (Gilbert-Nahmod, Muscalu-Tao-Thiele, ’99)
5. Systematic study of multilinear singular integrals (Grafakos-Torres, ’99)
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The bilinear Coifman-Meyer classes: $BS_{1,\delta}^0$, $0 \leq \delta < 1$

Theorem (Coifman-Meyer ’78; Grafakos-Torres ’02; B.-Torres ’03)

If $\sigma \in BS_{1,0}^0$, then $T_\sigma : L^p \times L^q \to L^r$, \( 1/p + 1/q = 1/r < 2 \).

Theorem (B.-Oh, ’10)

If $\sigma \in BS_{1,\delta}^0$, $0 \leq \delta < 1$, then $T_\sigma : L^p \times L^q \to L^r$, \( 1/p + 1/q = 1/r < 2 \).

Tools: Littlewood-Paley theory; elementary symbols.
The bilinear Coifman-Meyer classes: $BS^{0}_{1,\delta}, 0 \leq \delta < 1$

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Tools: Littlewood-Paley theory; elementary symbols.
The class $BS_{1,1}^0$ is the largest one to produce bilinear Calderón-Zygmund kernels.

That is,

$$T_{\sigma}(f, g)(x) = \int \int K(x, y, z)f(y)g(z) \, dydz,$$

and $K(x, y, z)$ satisfies appropriate smoothness-decay estimates.

Both previous $\psi$DO boundedness results on the Coifman-Meyer classes follow once we can establish a transposition symbolic calculus.

If $\sigma \in \text{BS}_{\rho,\delta}^m$, $0 \leq \delta < \rho \leq 1$, then $T_{\sigma^*j} = T_{\sigma_j}$ with $\sigma^*j \in \text{BS}_{\rho,\delta}^m$, $j = 1, 2$. 
Theorem (Grafakos-Torres, '02)

*The class $BS^0_{1,1}$ is the largest one to produce bilinear Calderón-Zygmund kernels.*

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Theorem (B.-Maldonado-Naibo-Torres, '10)

*If $\sigma \in BS^m_{\rho, \delta}, 0 \leq \delta < \rho \leq 1$, then $T^{*j}_\sigma = T_{\sigma^{*j}}$ with $\sigma^{*j} \in BS^m_{\rho, \delta}, j = 1, 2.$*
Theorem (B.-Torres, ’04)

*There exists a symbol in $BS^0_{\rho,\rho}$ such that $T : L^2 \times L^2 \not\to L^1$.***

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, ’11)

*If $\sigma \in BS^0_{\rho,\rho}$, $0 \leq \rho < 1$, then $BS^0_{\rho,\rho} : L^2 \times L^2 \not\to L^1$.***

Theorem (B.-Torres, ’04)

*If $\sigma \in BS^0_{0,0}$ and $\partial_\xi^\alpha \sigma \in L_\xi^\infty L_\eta^1 L_\xi^2$, $\partial_\eta^\alpha \sigma \in L_\eta^\infty L_\eta^1 L_\xi^2$, then $T : L^2 \times L^2 \to L^1$.***

Tool: almost orthogonality.
The bilinear Calderón-Vaillancourt classes: $BS^0_{\rho,\rho}$

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If $\sigma \in BS_{\rho,\rho}^0$, $0 \leq \rho < 1$, then $BS_{\rho,\rho}^0 : L^2 \times L^2 \not\to L^1$.

Theorem (B.-Torres, ’04)

If $\sigma \in BS_{0,0}^0$ and $\partial_\xi^\alpha \sigma \in L^\infty_\xi L^1_\eta L^2_\eta$, $\partial_\eta^\alpha \sigma \in L^\infty_\eta L^1_\eta L^2_\xi$, then $T : L^2 \times L^2 \to L^1$.

Tool: almost orthogonality.
The bilinear Calderón-Vaillancourt classes: $BS^0_{\rho,\rho}$

Theorem (B.-Torres, ’04)

There exists a symbol in $BS^0_{0,0}$ such that $T : L^2 \times L^2 \not\to L^1$.  

Theorem (B.-Bernicot-Maldonado-Naibo-Torres, ’11)

If $\sigma \in BS^0_{\rho,\rho}$, $0 \leq \rho < 1$, then $BS^0_{\rho,\rho} : L^2 \times L^2 \not\to L^1$.  

Theorem (B.-Torres, ’04)

If $\sigma \in BS^0_{0,0}$ and $\partial_\xi^{\alpha} \sigma \in L_\infty L_\xi^1 L_\eta^2$, $\partial_\eta^{\alpha} \sigma \in L_\infty L_\eta^1 L_\xi^2$, then $T : L^2 \times L^2 \to L^1$.  

Tool: almost orthogonality.
Theorem (B.-Gröchenig-Heil-Okoudjou, ’05)

If $\sigma \in BS_{0,0}^0$, then $T : L^2 \times L^2 \to M^{1,\infty} \supseteq L^1$

An instructive statement (not completely correct):

$$f \in M^{p,q} \sim f \in L^p \text{ and } \hat{f} \in L^q$$
Fefferman’s result in the bilinear case

Although the classes $BS^0_{\rho,\delta}$ fail to be bounded on products of Lebesgue spaces, we have surprisingly

**Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)**

If $\sigma \in BS^{n(\rho-1)}_{\rho,0}$, $0 \leq \rho < \frac{1}{2}$, then $T_\sigma : L^\infty \times L^\infty \rightarrow BMO$.

The crucial observation in the proof:

**Theorem (B.-Bernicot-Maldonado-Naibo-Torres, '11)**

If $\lambda$ is a symbol such that

$$\sup_{|\beta| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1} \sup_{\xi, y \in \mathbb{R}^n} \| \partial_\xi^\alpha \partial_y^\beta \lambda(y, \xi - \cdot, \cdot) \|_{L^2} < \infty,$$

then $T_\lambda : L^2 \times L^2 \rightarrow L^2$. 
Fefferman’s result in the bilinear case

Although the classes $BS_{\rho,\delta}^0$ fail to be bounded on products of Lebesgue spaces, we have surprisingly

**Theorem (B.-Bernicot-Maldonado-Naibo-Torres, ’11)**

If $\sigma \in BS_{\rho,0}^n(\rho^{-1})$, $0 \leq \rho < \frac{1}{2}$, then $T_\sigma : L^\infty \times L^\infty \rightarrow BMO$.

The crucial observation in the proof:

**Theorem (B.-Bernicot-Maldonado-Naibo-Torres, ’11)**

If $\lambda$ is a symbol such that

$$\sup_{|\beta| \leq \lfloor \frac{n}{2} \rfloor + 1} \sup_{\xi, y \in \mathbb{R}^n} \| \partial_\xi^\alpha \partial_y^\beta \lambda(y, \xi - \cdot, \cdot) \|_{L^2} < \infty,$$

then $T_\lambda : L^2 \times L^2 \rightarrow L^2$. 
Thank you!