Mathematical modeling by differential equations

Case study: Traffic flow

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Outline

1. Introduction
2. Case study: A simple model for traffic flow
3. Case study: Extension of traffic flow model
4. Numerical techniques
5. Criticism

Introduction

Introduction

1. State the problem.
2. Identify variables and effects.
3. Choose modeling method.
4. Formulate basic model.
5. Build on basic model.
6. Solve equations or run simulations.
7. Make conclusions and criticize model.

Outline

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- Types of possible models:
  - Cellular automata: Follow individual cars, give rules for motion
  - PDE: Ignore individual cars, consider average densities
- PDE approach: Formulate a model for the density $\rho(x, t)$, of cars per unit length, as a function of position x and time t, given the initial density $\rho(x, 0)$.
- Automata approach: Create a simulation of cars on a 1d lattice by specifying rules for their movement on the lattice.

Relevant variables / effects

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- position, x
- time, t
- density, $\rho = \rho(x, t)$
- velocity, $v = v(x, t)$
- flux, $J = J(x, t)$

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Effects to account for:

- length of cars
- spacing between cars
- length of roadway
- number of lanes
- merging
- on/off ramps
- weather
- accidents/construction

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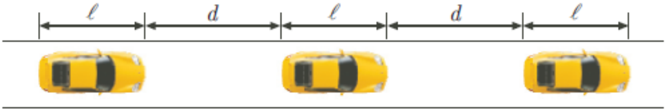
- Assume all cars have the same length L and they are evenly spaced at distance d. This is the *uniform distribution model*.



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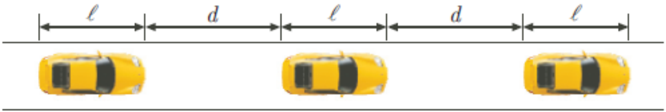
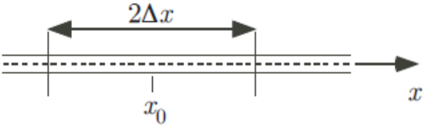


- Ignore all other effects.

Uniform distribution model: density

- Consider the interval $I_{x_0} = \{x_0 - \Delta x < x_0 < x_0 + \Delta x\}$ about the point x_0.



To estimate $\rho(x_0, t_0)$, count the number of cars in the interval and divide by its length:

$$\rho(x_0, t_0) \approx \frac{1}{2\Delta x}(\#\text{ of cars in the interval } I_{x_0} \text{ at time } t = t_0) \qquad (1)$$

- The assumption on Δx is that it is small enough so only cars in the vicinity matter but large enough so the interval contains many cars. The continuum view has $\Delta x \to 0$.

Uniform distribution model: density

- Recall the assumption is cars have length L and uniform spacing d. Using (1) we find that

$$\rho(x_0, t_0) = \lim_{\Delta x \to 0} \frac{1}{2\Delta x} \left(\frac{2\Delta x}{L + d} \right) = \frac{1}{L + d}$$

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  since the number of cars in the interval I_{x_0} is $\frac{2\Delta x}{L+d}$.
- Note that since $0 \leq d < \infty$ and L is finite that there is a maximum density:

$$0 < \rho \leq \rho_{max} = \frac{1}{L}.$$

- For instance, if $L = 17$ft and $d = 12$ft, then $\rho = 182 \frac{\text{cars}}{\text{mile}}$ and $\rho_{max} = 310.6 \frac{\text{cars}}{\text{mile}}$.

Uniform distribution model: flux

- Question: How many cars pass the point x_0 per unit time?
- To find the flux of cars through x_0 at time t_0, count the number of cars passing x_0 during the time interval

$$I_{t_0} = \{t_0 - \Delta t < t_0 < t_0 + \Delta t\}$$

and divide by the interval length. In other words,

$$J(x_0, t_0) \approx \frac{1}{2\Delta t}(\# \text{ of cars passing } x_0 \text{ during the time interval } I_{t_0}) \quad (2)$$

- The assumption on Δt is that it is small enough so only cars passing by near time t_0 are counted, but big enough so that many cars pass through. Again, in the continuum limit, $\Delta t \to 0$.

Uniform distribution model: flux

- Assume that cars move to the right with constant speed v.
- During time $2\Delta t$, cars move a distance equal to $2v\Delta t$, hence the number of cars passing a point x_0 during this time is $\rho \times \text{distance} = \frac{2v\Delta t}{L+d}$. Using (2), the flux about x_0 at time t_0 is

$$J(x_0, t_0) = \lim_{\Delta t \to 0} \frac{1}{2\Delta t} \left(\frac{2v\Delta t}{L+d} \right) = \frac{v}{L+d}.$$

- If for instance $L = 17\text{ft}$ and $v = 70\text{mph}$, then $J = 5435 \frac{\text{cars}}{\text{mile}}$.
- Note also that

$$J = \rho v.$$

Outline

Balance law for density

- The previous model is extended to allow for changes in density. This is accomplished by using a control volume (CV) and accounting for all the ways the density can change in the volume.
- Assumptions: cars cannot enter through the top or bottom (no ramps), cars enter from the left and exit from the right (one way traffic).

Balance law for density

- Consider the CV consisting of the interval I_x during the time interval I_t. The idea is simple, the net change of cars in I_x during I_t is equal to the net flux of cars through the boundaries during this time. In other words:

$$\{\# \text{ of cars in } I_x \text{ during time } t_0 + \Delta t\} - \{\# \text{ cars in } I_x \text{ during time } t_0 - \Delta t\}$$
$$= \{\# \text{ of cars that enter through } x_0 - \Delta x \text{ during } I_t\}$$
$$- \{\# \text{ of cars that leave through } x_0 + \Delta x \text{ during } I_t\}$$

Balance law for density

- In terms of density and flux:

$$
\begin{aligned}
2\Delta x[\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t)] \\
= 2\Delta t\,[J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)]
\end{aligned}
$$

- The goal is, as before, to take $\Delta x, \Delta t \to 0$ to deduce the continuum limit. This is accomplished by expanding ρ and J in Taylor series and considering the leading order terms.

Balance law for density

- By Taylor expanding we find that (about the point (x_0, t_0)):

$$\begin{aligned}
&2\Delta x[\rho + \Delta t \rho_t + \frac{1}{2}(\Delta t)^2 \rho_{tt} + O((\Delta t)^3) \\
&\qquad - \rho - (-\Delta t \rho_t) - \frac{1}{2}(\Delta t)^2 \rho_{tt} + O((\Delta t)^3)] \\
&= 2\Delta t[J - \Delta x J_x + \frac{1}{2}(\Delta x)^2 J_{xx} + O((\Delta x)^3) \\
&\qquad - J - \Delta x J_x - \frac{1}{2}(\Delta x)^2 J_{xx} + O((\Delta x)^3)]
\end{aligned}$$

- This reduces to

$$2\Delta x[2\Delta t \rho_t + O((\Delta t)^3)] = 2\Delta t[-2\Delta x J_x + O((\Delta x)^3)]$$

or as $\Delta x, \Delta t \to 0$,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}. \tag{3}$$

Balance law for density

- If the flux function J is known, then (3) tells us how ρ evolves. This is usually deduced empirically.
- Instead focus on v. Assume $J = \rho v$ as before, where $v = v(x,t)$ is the average velocity of the cars at (x,t).
- The PDE model for traffic flow then reads:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) & = 0 \\ \rho(x,0) & = f(x) \end{cases}$$

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- PDE can now be solved analytically or numerically.
- BUT, the PDE model is an approximation! Discrete traffic flow has been modeled by continuous density.
- Low densities are therefore a problem. We don't expect the PDE to hold when there are few cars on the road.
- The velocity v is still needed. We need a constitutive law relating v to ρ. This can be obtained by examining the physics or determined from empirical evidence.

Determining velocity: empirical data

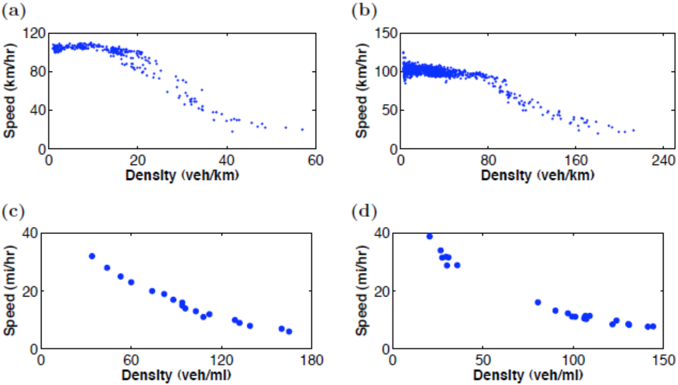

Figure 5.6 The velocity as a function of the density as measured for different roadways. Shown is (a) a highway near Toronto, (b) a freeway near Amsterdam, (c) the Lincoln Tunnel, and (d) the Merritt Parkway. Data for (a) and (b) are from Aerde and Rakha [1995], and (c) and (d) are from Greenberg [1959].

Determining velocity

- Assume the velocity is constant,

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- This perhaps is too unrealistic, except maybe low densities.

Determining velocity

- Assume a linear velocity,

$$v = v(\rho) = a - b\rho.$$

  This is known as the Greenshields model. It is commonly written as $v = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$.

- From the Merritt Parkway and Lincoln Tunnel data, we can use least squares to find that $v_{max} = 36.821$mph and $\rho_{max} = 166.4226\frac{\text{cars}}{\text{mile}}$ are the constants of best fit for the linear velocity.

Determining velocity

- Using this form of v in our PDE we get that now

$$\rho_t + c(\rho)\rho_x = 0,$$

  where $c(\rho) = v_{max}\left(1 - \frac{2\rho}{\rho_{max}}\right)$.
- Other more complicated velocity fits can be considered.

Outline

Numerical Solution

Return to the model with a constant velocity on a finite length highway:

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \;\; \text{for } 0 < x < L, t > 0 \tag{4}$$

with an initial condition

$$\rho(x,0) = \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}$$

and a boundary condition

$$\rho(0,t) = g(t) \;\; .$$

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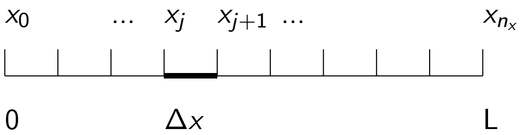
Question?

Why is boundary specified at $x = 0$ and not at $x = L$?

Discretized problem

- Goal: Given $g(t)$ and a, numerically compute the density at time T.
- Basic Idea: Discretize the time interval $[0, T]$ into intervals of length Δt and the spatial interval $[0, l]$ into intervals of length Δx and use a finite difference to approximate the derivative.



Finite differences

Recall the definition of the derivative

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{5}$$

Let $x = x_j$ and $h = \Delta x$. Without the limit, we have forward difference of the first derivative

$$\frac{f(x_{j+1}) - f(x_j)}{\Delta x} \,. \tag{6}$$

The backward difference of the first derivative is

$$\frac{f(x_j) - f(x_{j-1})}{\Delta x} \,. \tag{7}$$

Finite differences

Let us return to our model,

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \quad \text{for } 0 < x < L, t > 0 \tag{8}$$

Denote $\rho(x_j, t_n) = \rho_j^n$.

Finite differences

Let us return to our model,

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Denote $\rho(x_j, t_n) = \rho_j^n$.
Use the forward difference to approximate the time derivative and the backward difference to approximate the spatial derivative:

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + a\frac{\rho_j^n - \rho_{j-1}^n}{\Delta x} = 0 \ . \tag{9}$$

Thus we have approximations of the two derivatives, which will approach continuous equations in the limit as $\Delta x \to 0$ and $\Delta t \to 0$.

Finite differences

Thus our numerical scheme is

$$\rho_j^{n+1} = \rho_j^n + \frac{a\Delta t}{\Delta x}(\rho_j^n - \rho_{j-1}^n) \tag{10}$$

Example 1

- Cars move at $a = 30$ mph or $1/120$ miles per second.
- The highway is 3 miles long $L = 3$.
- Let's run the model for 3 minutes which means $T = 180s$.
- $\Delta x = 0.1$ mile, $\Delta t = 0.1$ s
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What does this boundary condition mean in our real world scenario?

Example 2

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$$g(t) = \begin{cases} 1 & \text{if } 0 < t < 60, t \geq 120 \\ 0 & \text{if } 60 \leq t < 120 \end{cases}$$

Outline

Criticism

- We have ignored many effects mentioned earlier, for instance, intersections, weather, traffic lights, etc.
- Perhaps our guess on velocity was too restrictive. A form such as $v = F(\rho, \rho_x)$ may be better.
- For highway travel, merging and multilane effects become important.
- The models presented seem to work only for light to heavy traffic situations.

References

[1] M. Holmes
Introduction to the Foundations of Applied Mathematics.
Springer (2009)