Algebro-Geometric Techniques and Geometric Insights for Finite Frames

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Finite Frames: Early Motivations

- Optimization over spaces of finite frames for applications
- Weaver's formulation of the Kadison-Singer Conjecture
- Development of new techniques for infinite frames

Highly Redundant Definitions

A collection of vectors $\{f_i\}_{i=1}^N \subset \mathbb{R}^d$ is a **frame** if there are constants $0 < A \le B$ such that

$$A\|x\|^2 \le \sum_{i=1}^N |\langle x, f_i\rangle|^2 \le B\|x\|^2 \text{ for all } x \in \mathbb{R}^d.$$

Highly Redundant Definitions

If we can take $A = B$, then we say that the frame is **tight**.

Highly Redundant Definitions

If we have that $\|f_i\| = 1$ for all i in the N-set $[N]$, then we say that the frame is **unit-norm**.

Highly Redundant Definitions

If a finite frame is both unit-norm and tight, we say that it is a **FUNTF**.

Identification with Matrices

In this talk, we identify an indexed finite frame with the matrix

$$F = \begin{pmatrix} \langle f_1, e_1 \rangle & \langle f_2, e_1 \rangle & \cdots & \langle f_N, e_1 \rangle \\ \langle f_1, e_2 \rangle & \langle f_2, e_2 \rangle & \cdots & \langle f_N, e_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_1, e_d \rangle, & \langle f_2, e_d \rangle & \cdots & \langle f_N, e_d \rangle \end{pmatrix} \in M_{d,N}$$

where $\{e_i\}_{i \in [d]}$ is the standard orthonormal basis of $\mathbb{R}^d$.

Identification with Matrices

We also write $F = [f_1\ f_2 \cdots f_N]$. Letting F^* denote the transpose of this matrix, we have the **frame operator** of F

$$FF^* = \sum_{i=1}^{N} f_i f_i^* = \sum_{i=1}^{N} \begin{pmatrix} \langle f_i, e_1\rangle\langle f_i, e_1\rangle & \langle f_i, e_1\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_1\rangle\langle f_i, e_d\rangle \\ \langle f_i, e_2\rangle\langle f_i, e_1\rangle & \langle f_i, e_2\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_2\rangle\langle f_i, e_d\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_i, e_d\rangle\langle f_i, e_1\rangle & \langle f_i, e_d\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_d\rangle\langle f_i, e_d\rangle \end{pmatrix}$$

and the **Grammian** of F

$$F^*F = \begin{pmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & \cdots & \langle f_1, f_N\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \cdots & \langle f_2, f_N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_N, f_1\rangle & \langle f_N, f_2\rangle & \cdots & \langle f_N, f_N\rangle \end{pmatrix}.$$

Identification with Matrices

Lemma. *Let* $F \in M_{N\times d}$.

- F *is a tight frame with frame bounds* $A = B = \frac{N}{d}$ *if and only if*

$$F \in St_{d,N} = \{X \in M_{d,N} : XX^* = \frac{N}{d} I_d\}.$$

- F *is unit-norm if and only if*

$$F \in \mathbb{T}_{d,N} = \{X = [x_1 \cdots x_N] \in M_{d,N} : \|x_i\| = 1 \text{ for all } i \in [N]\}$$

- F *is a FUNTF if and only if*

$$F \in St_{d,N} \cap \mathbb{T}_{d,N} = \mathcal{F}_{d,N}.$$

Our Standard Examples

$$
\begin{aligned}
\Phi &= [\phi_1\ \phi_2\ \phi_3\ \phi_4\ \phi_5\ \phi_6] \\
&= \begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & 1 & 0 & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{6}}{6} \end{pmatrix}
\end{aligned}
$$

Our Standard Examples

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Spaces of FUNTFs as Algebraic Varieties

An **algebraic variety** is the zero set of a system of polynomials.

Spaces of FUNTFs as Algebraic Varieties

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix}$$ is in $St_{3,6}$ if and only if

$$\begin{aligned}
x_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{15}^2 + x_{16}^2 &= 2 \\
x_{21}^2 + x_{22}^2 + x_{23}^2 + x_{24}^2 + x_{25}^2 + x_{26}^2 &= 2 \\
x_{31}^2 + x_{32}^2 + x_{33}^2 + x_{34}^2 + x_{35}^2 + x_{36}^2 &= 2 \\
x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + x_{14}x_{24} + x_{15}x_{25} + x_{16}x_{26} &= 0 \\
x_{11}x_{31} + x_{12}x_{32} + x_{13}x_{33} + x_{14}x_{34} + x_{15}x_{35} + x_{16}x_{36} &= 0 \\
x_{21}x_{31} + x_{22}x_{32} + x_{23}x_{33} + x_{24}x_{34} + x_{25}x_{35} + x_{26}x_{36} &= 0
\end{aligned}$$

Spaces of FUNTFs as Algebraic Varieties

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix} \text{ is in } \mathbb{T}_{3,6} \text{ if and only if}$$

$$\begin{aligned} x_{11}^2 + x_{21}^2 + x_{31}^2 &= 1 \\ x_{12}^2 + x_{22}^2 + x_{32}^2 &= 1 \\ x_{13}^2 + x_{23}^2 + x_{33}^2 &= 1 \\ x_{14}^2 + x_{24}^2 + x_{34}^2 &= 1 \\ x_{15}^2 + x_{25}^2 + x_{35}^2 &= 1 \\ x_{16}^2 + x_{26}^2 + x_{36}^2 &= 1 \end{aligned}$$

Basic Question: Local Structure

What are the non-singular points of $\mathcal{F}_{d,N}$?

Basic Question: Local Structure

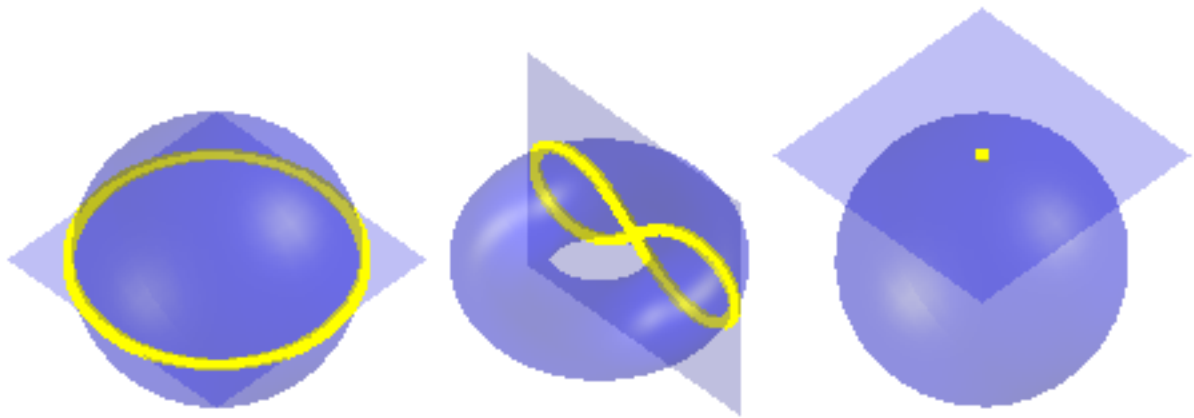
If $F \in \mathcal{F}_{d,N}$ is non-singular, then what is $T_F \mathcal{F}_{d,N}$?

From Global to Local

$$\mathcal{F}_{d,N} = \mathrm{St}_{d,N} \cap \mathbb{T}_{d,N}, \text{ so } T_F\mathcal{F}_{d,N} = T_F\mathrm{St}_{d,N} \cap T_F\mathbb{T}_{d,N}?$$

Transversal Intersections

Definition. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two smooth manifolds embedded in the same smooth manifold $\mathcal{K}$, and suppose $X \in \mathcal{M} \cap \mathcal{N}$. Then we say $\mathcal{M}$ and $\mathcal{N}$* **intersect transversally** *in $\mathcal{K}$ at X if $T_X\mathcal{K} = T_X\mathcal{M} + T_X\mathcal{N}$ (where $+$ is the Minkowski sum).*



Transversal Intersections

Proposition. *Suppose $\mathcal{M}$ and $\mathcal{N}$ intersect transversally in $\mathcal{K}$ at X. Then X is a nonsingular point of $\mathcal{M} \cap \mathcal{N}$ and*

$$T_X(\mathcal{M} \cap \mathcal{N}) = T_X\mathcal{M} \cap T_X\mathcal{N}.$$

Characterizations of Singular Points

We may therefore characterize the singular points of $\mathcal{F}_{d,N}$ as the points where the intersection of $\mathrm{St}_{d,N}$ and $\mathbb{T}_{N,d}$ fails to be transversal.

One important point is that $\mathrm{St}_{d,N}$ and $\mathbb{T}_{d,N}$ are both contained in the Hilbert-Schmidt sphere

$$\mathcal{S}_{d,N} = \{X \in M_{d,N} : \mathrm{trace}(XX^*) = N\}.$$

Transversality of the intersection is therefore relative to $\mathcal{S}_{d,N}$ instead of $M_{d,N}$.

Example

$$T_\Phi \mathcal{S}_{3,6} = \{Y \in M_{3,6} : \mathrm{tr}(Y^*\Phi) = 0\}$$

$$T_\Phi \mathbb{T}_{3,6} = \{Y \in M_{3,6} : \mathrm{diag}(Y^*\Phi) = \mathbf{0}_N\}$$

$$T_\Phi \mathrm{St}_{3,6} = \{\Phi Z \in M_{3,6} : Z \in M_{6,6},\ Z = -Z^*\}.$$

Example

$$\overbrace{X}^{\in T_\Phi \mathcal{S}_{3,6}} = \overbrace{\Phi Z}^{\in T_\Phi \mathrm{St}_{3,6}} + \overbrace{Y}^{\in T_\Phi \mathbb{T}_{3,6}}$$

$$\begin{array}{ccccccccccccccc}
x_1 & = & 0 & + & z_{21}\phi_2 & + & z_{31}\phi_3 & + & z_{41}\phi_4 & + & z_{51}\phi_5 & + & z_{61}\phi_6 & + & y_1 \\
x_2 & = & -z_{21}\phi_1 & + & 0 & + & z_{32}\phi_3 & + & z_{42}\phi_4 & + & z_{52}\phi_5 & + & z_{62}\phi_6 & + & y_2 \\
x_3 & = & -z_{31}\phi_1 & - & z_{32}\phi_2 & + & 0 & + & z_{43}\phi_4 & + & z_{53}\phi_5 & + & z_{63}\phi_6 & + & y_3 \\
x_4 & = & -z_{41}\phi_1 & - & z_{42}\phi_2 & - & z_{43}\phi_3 & + & 0 & + & z_{54}\phi_5 & + & z_{64}\phi_6 & + & y_4 \\
x_5 & = & -z_{51}\phi_1 & - & z_{52}\phi_2 & - & z_{53}\phi_3 & - & z_{54}\phi_4 & + & 0 & - & z_{65}\phi_6 & + & y_5 \\
x_6 & = & -z_{61}\phi_1 & - & z_{62}\phi_2 & - & z_{63}\phi_3 & - & z_{64}\phi_4 & - & z_{65}\phi_5 & + & 0 & + & y_6
\end{array}.$$

Example

$$
\begin{array}{lcccccccccccccc}
\langle x_1, \phi_1 \rangle & = & 0 & + & \frac{\sqrt{3}}{3} z_{21} & + & 0 & + & \frac{\sqrt{2}}{2} z_{41} & + & 0 & + & \frac{\sqrt{6}}{6} z_{61} \\
\langle x_2, \phi_2 \rangle & = & -\frac{\sqrt{3}}{3} z_{21} & + & 0 & + & \frac{\sqrt{3}}{3} z_{32} & + & 0 & + & \frac{\sqrt{3}}{3} z_{52} & + & 0 \\
\langle x_3, \phi_3 \rangle & = & 0 & - & \frac{\sqrt{3}}{3} z_{32} & + & 0 & + & 0 & + & 0 & - & \frac{\sqrt{6}}{3} z_{63} \\
\langle x_4, \phi_4 \rangle & = & -\frac{\sqrt{2}}{2} z_{41} & + & 0 & + & 0 & + & 0 & - & \frac{\sqrt{2}}{2} z_{54} & + & 0 \\
\langle x_5, \phi_5 \rangle & = & 0 & - & \frac{\sqrt{3}}{3} z_{52} & + & 0 & + & \frac{\sqrt{2}}{2} z_{54} & + & 0 & + & \frac{\sqrt{6}}{6} z_{65} \\
\langle x_6, \phi_6 \rangle & = & -\frac{\sqrt{6}}{6} z_{61} & + & 0 & + & \frac{\sqrt{6}}{3} z_{63} & + & 0 & - & \frac{\sqrt{6}}{6} z_{65} & + & 0
\end{array}
$$

Example

$$\begin{pmatrix}\langle x_1,\phi_1\rangle\\\langle x_2,\phi_2\rangle\\\langle x_3,\phi_3\rangle\\\langle x_4,\phi_4\rangle\\\langle x_5,\phi_5\rangle\\\langle x_6,\phi_6\rangle\end{pmatrix} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{3}z_{21} & 0 & \frac{\sqrt{2}}{2}z_{41} & 0 & \frac{\sqrt{6}}{6}z_{61} \\ -\frac{\sqrt{3}}{3}z_{21} & 0 & \frac{\sqrt{3}}{3}z_{32} & 0 & \frac{\sqrt{3}}{3}z_{52} & 0 \\ 0 & -\frac{\sqrt{3}}{3}z_{32} & 0 & 0 & 0 & -\frac{\sqrt{6}}{3}z_{63} \\ -\frac{\sqrt{2}}{2}z_{41} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2}z_{54} & 0 \\ 0 & -\frac{\sqrt{3}}{3}z_{32} & 0 & \frac{\sqrt{2}}{2}z_{54} & 0 & \frac{\sqrt{6}}{6}z_{65} \\ -\frac{\sqrt{6}}{6}z_{61} & 0 & \frac{\sqrt{6}}{3}z_{63} & 0 & -\frac{\sqrt{6}}{6}z_{65} & 0 \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}$$

Example

$$\begin{pmatrix} \langle x_1,\phi_1\rangle \\ \langle x_2,\phi_2\rangle \\ \langle x_3,\phi_3\rangle \\ \langle x_4,\phi_4\rangle \\ \langle x_5,\phi_5\rangle \\ \langle x_6,\phi_6\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \langle x_1,\phi_1\rangle \\ 0 & 0 & \frac{\sqrt{3}}{3} z_{32} & 0 & \frac{\sqrt{3}}{3} z_{52} & 0 \\ 0 & -\frac{\sqrt{3}}{3} z_{32} & 0 & 0 & 0 & -\frac{\sqrt{6}}{3} z_{63} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} z_{54} & 0 \\ 0 & -\frac{\sqrt{3}}{3} z_{32} & 0 & \frac{\sqrt{2}}{2} z_{54} & 0 & \frac{\sqrt{6}}{6} z_{65} \\ -\langle x_1,\phi_1\rangle & 0 & \frac{\sqrt{6}}{3} z_{63} & 0 & -\frac{\sqrt{6}}{6} z_{65} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{6}\langle x_1,\phi_1\rangle \\ 0 & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * \\ 0 & * & * & 0 & * & * \\ 0 & * & * & * & 0 & * \\ \sqrt{6}\langle x_1,\phi_1\rangle & * & * & * & * & 0 \end{pmatrix}$$

Example

$$\begin{pmatrix}\langle x_1,\phi_1\rangle\\ \langle x_2,\phi_2\rangle\\ \langle x_3,\phi_3\rangle\\ \langle x_4,\phi_4\rangle\\ \langle x_5,\phi_5\rangle\\ \langle x_6,\phi_6\rangle\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \langle x_1,\phi_1\rangle \\ 0 & 0 & 0 & 0 & \langle x_2,\phi_2\rangle & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\sqrt{6}}{3}z_{63} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2}z_{54} & 0 \\ 0 & -\langle x_2,\phi_2\rangle & 0 & \frac{\sqrt{2}}{2}z_{54} & 0 & \frac{\sqrt{6}}{6}z_{65} \\ -\langle x_1,\phi_1\rangle & 0 & \frac{\sqrt{6}}{3}z_{63} & 0 & -\frac{\sqrt{6}}{6}z_{65} & 0 \end{pmatrix}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}$$

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{6}\langle x_1,\phi_1\rangle \\ 0 & 0 & 0 & 0 & -\sqrt{3}\langle x_2,\phi_2\rangle & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & \sqrt{3}\langle x_2,\phi_2\rangle & * & * & 0 & * \\ \sqrt{6}\langle x_1,\phi_1\rangle & 0 & * & * & * & 0 \end{pmatrix}$$

Example

$$
\begin{pmatrix} \langle x_1,\phi_1\rangle \\ \langle x_2,\phi_2\rangle \\ \langle x_3,\phi_3\rangle \\ \langle x_4,\phi_4\rangle \\ \langle x_5,\phi_5\rangle \\ \langle x_6,\phi_6\rangle \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \langle x_1,\phi_1\rangle \\ 0 & 0 & 0 & 0 & \langle x_2,\phi_2\rangle & 0 \\ 0 & 0 & 0 & 0 & 0 & \langle x_3,\phi_3\rangle \\ 0 & 0 & 0 & 0 & \langle x_4,\phi_4\rangle & 0 \\ 0 & -\langle x_2,\phi_2\rangle & 0 & -\langle x_4,\phi_4\rangle & 0 & \frac{\sqrt{6}}{6} z_{65} \\ -\langle x_1,\phi_1\rangle & 0 & -\langle x_3,\phi_3\rangle & 0 & -\frac{\sqrt{6}}{6} z_{65} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
$$

$$
Z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{6}\langle x_1,\phi_1\rangle \\ 0 & 0 & 0 & 0 & -\sqrt{3}\langle x_2,\phi_2\rangle & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{6}}{2}\langle x_3,\phi_3\rangle \\ 0 & 0 & 0 & 0 & \sqrt{2}\langle x_4,\phi_4\rangle & 0 \\ 0 & \sqrt{3}\langle x_2,\phi_2\rangle & 0 & \sqrt{2}\langle x_4,\phi_4\rangle & 0 & * \\ \sqrt{6}\langle x_1,\phi_1\rangle & 0 & -\frac{\sqrt{6}}{2}\langle x_3,\phi_3\rangle & 0 & * & 0 \end{pmatrix}
$$

Example

$$
\begin{aligned}
z_{65} &= \sqrt{6}\left(\langle x_2, \phi_2\rangle + \langle x_4, \phi_4\rangle + \langle x_5, \phi_5\rangle\right) \\
z_{65} &= -\sqrt{6}\left(\langle x_1, \phi_1\rangle + \langle x_3, \phi_3\rangle + \langle x_6, \phi_6\rangle\right)
\end{aligned}
$$

Example

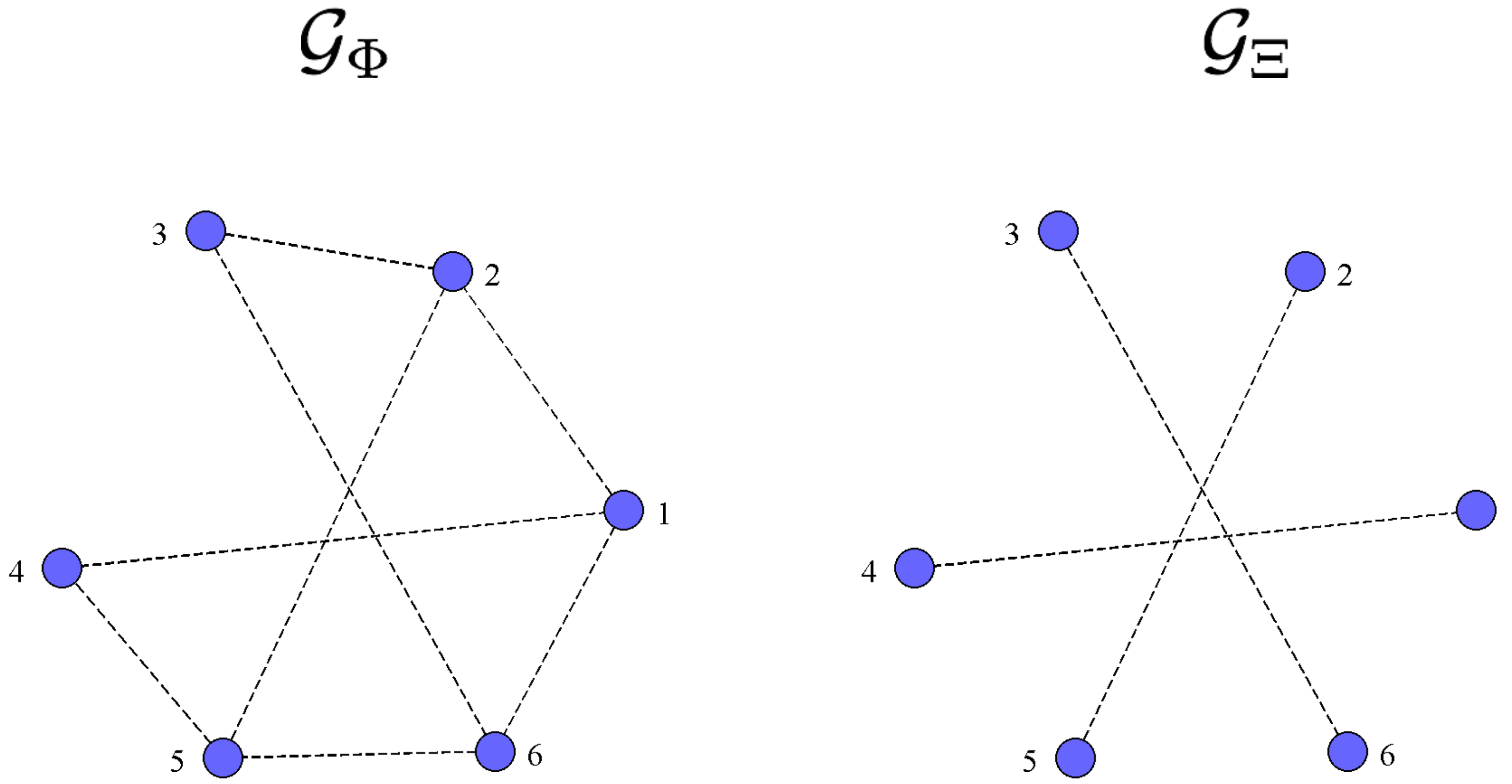
$$\mathrm{tr}(X^*\Phi) = \langle x_1, \phi_1 \rangle + \langle x_2, \phi_2 \rangle + \langle x_3, \phi_3 \rangle + \langle x_4, \phi_4 \rangle + \langle x_5, \phi_5 \rangle + \langle x_6, \phi_6 \rangle = 0$$

Correlation Networks

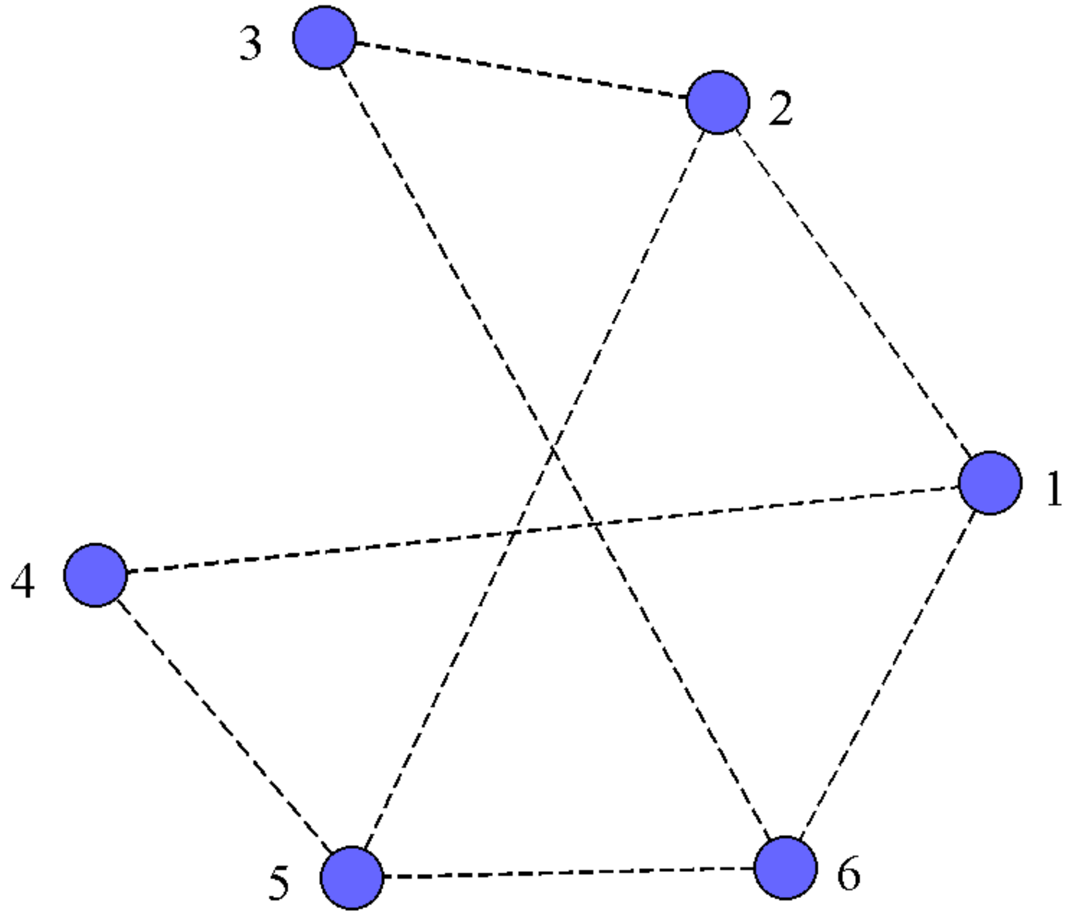
Definition. *For a frame* $F = [f_1\ f_2\ \cdots\ f_N] \in M_{d,N}$*, the* **correlation network** *is the symmetric graph* $\mathcal{G}_F = (V, E)$ *with vertices* $V = \{1, 2, \ldots, N\}$ *and edge set*

$$E = \{(i,j) : \langle f_i, f_j \rangle \neq 0\}.$$

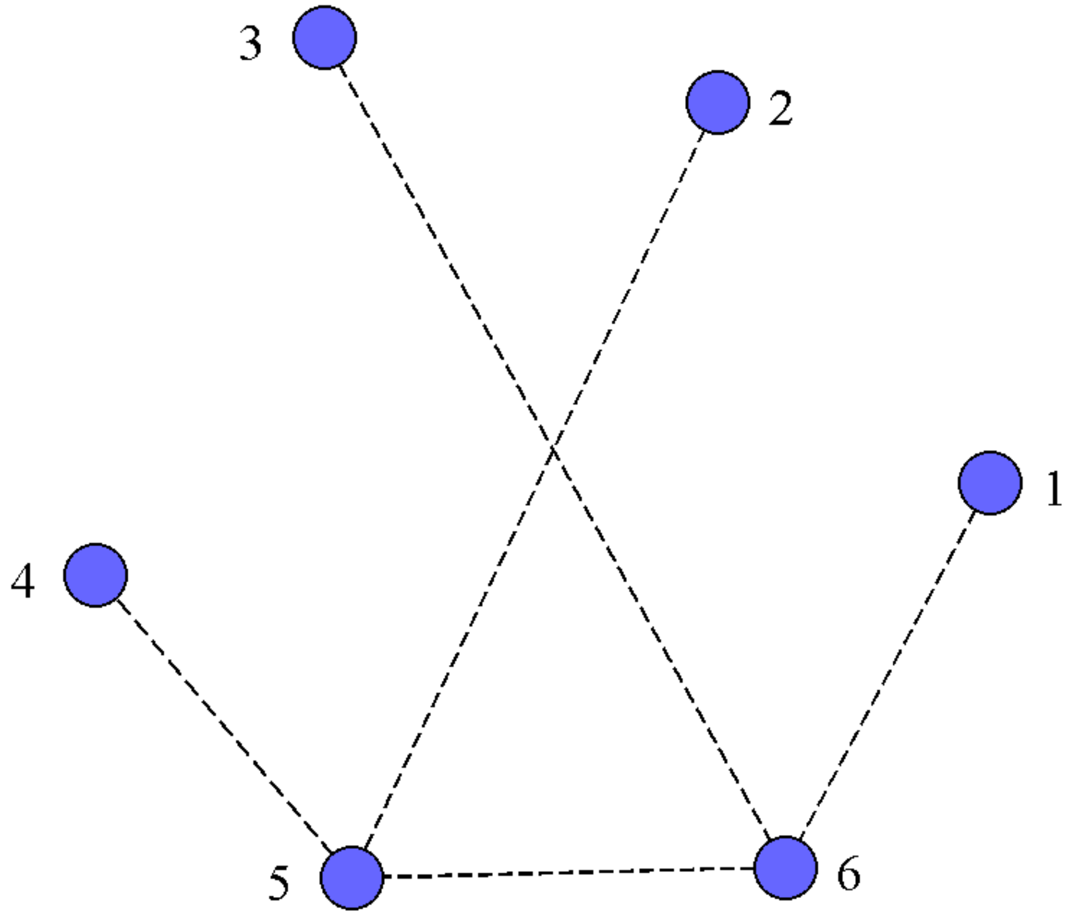
Example



Example



Example



Orthodecomposable Frames

Proposition. *The correlation network $\mathcal{G}_F$ is connected if and only if F cannot be partitioned into two non-trivial subsets of matrices with orthogonal column spaces. If F does admit such a partition, we say that F is* **orthodecomposable**.

Locally Transversal Intersections of Tori and Stiefel Manifolds

Theorem. *Suppose $N \geq d \geq 2$. The manifolds $\mathbb{T}_{d,N}$ and $St_{d,N}$ intersect transversally in $\mathcal{S}_{d,N}$ at $F \in \mathcal{F}_{d,N}$ if and only if F is not orthodecomposable. Moreover, the local dimension of $\mathcal{F}_{d,N}$ around such an F is given by*

$$(d-1)N + \left(dN - \binom{d+1}{2}\right) - (dN-1) = (d-1)N - \binom{d+1}{2} + 1.$$

When FUNTF Spaces are Manifolds

Proposition. *The variety $\mathcal{F}_{d,N}$ is a manifold if and only if N and d are relatively prime.*

Basic Question: Local Coordinates

Can we write down local parameterizations of $\mathcal{F}_{d,N}$?

FUNTF Equations are Locally Solvable

Idea: Freely articulate $N - d$ vectors in the frame and then the remaining d vectors react to preserve tightness.

FUNTF Equations are Locally Solvable

$$\widetilde{F} = [\widetilde{f}_1 \ \widetilde{f}_2 \ \widetilde{f}_3 \ \widetilde{f}_4 \ \widetilde{f}_5 \ \widetilde{f}_6] = [\widetilde{F}_1 \widetilde{F}_2]$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}\widetilde{F}^* = \widetilde{F}_1\widetilde{F}_1^* + \widetilde{F}_2\widetilde{F}_2^* = 2I_3$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2\widetilde{F}_2^* = \overbrace{2I_3 - \widetilde{F}_1\widetilde{F}_1^*}^{\widetilde{H}}$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2 \widetilde{F}_2^* \widetilde{H}^{-1} = I_3$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2^* \widetilde{H}^{-1} = \widetilde{F}_2^{-1}$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2^* \widetilde{H}^{-1} \widetilde{F}_2 = I_3$$

FUNTF Equations are Locally Solvable

$$\begin{aligned} \widetilde{f}_4^* H^{-1} \widetilde{f}_4 &= 1 \\ \widetilde{f}_4^* \widetilde{f}_4 &= 1 \end{aligned}$$

FUNTF Equations are Locally Solvable

$$\begin{aligned} f_4^*(I_4 - F_1 F_1^*)^{-1} g &= 0 \\ f_4^* g &= 0 \\ \Rightarrow \widetilde{f}_4^* g &= \theta \end{aligned}$$

FUNTF Equations are Locally Solvable

$$
\begin{aligned}
\widetilde{f}_4^* \widetilde{H}^{-1} \widetilde{f}_4 &= 1 \\
\widetilde{f}_4^* \widetilde{f}_4 &= 1 \\
\widetilde{f}_4^* g &= \theta \\
\widetilde{f}_4 &= x H^{-1} f_4 + y f_4 + \theta g
\end{aligned}
$$

Solving a System of Two Quadratics and a Linear Equation

$$
\begin{aligned}
(f_4^* H^{-1} \widetilde{H}^{-1} H^{-1} f_4) x^2 + 2 f_4^* H^{-1} \widetilde{H}^{-1} (y f_4 + \theta g) x + (y f_4 + \theta g)^* \widetilde{H}^{-1} (y f_4 + \theta g) - 1 &= 0 \\
(f_4^* H^{-2} f_4) x^2 + 2 y f_4^* H^{-1} f_4 x + y^2 + \theta^2 - 1 &= 0
\end{aligned}
$$

Solving a System of Two Quadratics and a Linear Equation

Proposition. *The system*

$$\begin{aligned} \alpha_2 x^2 + \alpha_1 x + \alpha_0 &= 0 \\ \beta_2 x^2 + \beta_1 x + \beta_0 &= 0 \end{aligned}$$

where α_2 and β_2 are non-zero admits a solution if and only if the Bézout determinant vanishes:

$$(\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_1\beta_0 - \alpha_0\beta_1) - (\alpha_2\beta_0 - \alpha_0\beta_2)^2 = 0.$$

Solving a System of Two Quadratics and a Linear Equation

This yields a quartic in the variable y, which has an explicit, but complicated solution.

Eigensteps: Cleaner Local Coordinates

Let $\widetilde{\lambda} : M_{d,d} \to \mathbb{R}^d$ be such that

$$\widetilde{\lambda}(X) = \begin{pmatrix} \lambda_1(X) \\ \lambda_2(X) \\ \vdots \\ \lambda_d(X) \end{pmatrix}$$

where $\{\lambda_i(X)\}_{i=1}^d$ are the eigenvalues of X counting multiplicity, and put in non-increasing order.

Eigensteps: Cleaner Local Coordinates

Define the **eigensteps map**, $\lambda : M_{d,N} \to M_{d,N}$, by

$$\lambda(X) = [\widetilde{\lambda}(X_1 X_1^*) \, \widetilde{\lambda}(X_2 X_2^*) \cdots \widetilde{\lambda}(X X^*)]$$

where $X_i = [x_1 \, x_2 \cdots x_N]$ for all i

Example

$$\lambda(\Phi) = \begin{pmatrix} 1 & (3+\sqrt{3})/3 & (3+\sqrt{6})/3 & 2 & 2 & 2 \\ 0 & (3-\sqrt{3})/3 & 1 & (6+\sqrt{6})/6 & 2 & 2 \\ 0 & 0 & (3-\sqrt{6})/3 & (6-\sqrt{6})/6 & 1 & 2 \end{pmatrix}$$

Eigensteps

Lemma. $\Delta_{d,N} = \lambda(\mathcal{F}_{d,N})$ *is a polytope.*

Lifting Eigensteps

Proposition. *Let* $\Delta^\circ_{3,6}$ *denote the interior of* $\Delta^\circ_{3,6}$. *There are sequences of vector-valued functions* $v_k : \Delta^\circ_{3,6} \to \mathbb{R}^3$ *and* $w_k : \Delta^\circ_{3,6} \to \mathbb{R}^3$, *and a sequence of matrix-valued functions* $W_k : \Delta^\circ_{3,6} \to M_{3,3}$ *such that whenever* $F \in \mathcal{F}_{3,6}$ *satisfies the condition that* $\lambda(F) \in \Delta^\circ_{3,6}$, *then there are sequences of orthogonal matrices* V_k, P_k, *and* Q_k *so that when we define the sequences*

$$\begin{aligned} U_1(U,\mu) &= U \\ \phi_1(U,\mu) &= U_1(U,\mu)e_1 \\ \phi_{k+1}(U,\mu) &= U_k(U,\mu)V_kP_k^*v_k(\mu) \\ U_{k+1}(U,\mu) &= U_k(U,\mu)V_kP_k^*W_k(\mu)Q_k \end{aligned}$$

for $k = 1, 2, 3, 4, 5$ *and all* $(U,\mu) \in \mathcal{O}(3) \times \Delta^\circ_{3,6}$, *then*

$$\Phi(U,\mu) = [\phi_1(U,\mu)\ \phi_2(U,\mu)\ \phi_3(U,\mu)\ \phi_4(U,\mu)\ \phi_5(U,\mu)\ \phi_6(U,\mu)] \in \mathcal{F}_{d,N}$$

satisfies $\lambda(\Phi(U,\mu)) = \mu$ *and* $\Phi(\mathcal{Q}(F), \lambda(F)) = F$ *where* $\mathcal{Q}(F)$ *is* Q *from the* QR *decomposition of the first* 3 *columns of* F.

Example: Coordinates from Eigensteps

k	$v_k(\lambda)$	$w_k(\lambda)$	$W_k(\lambda)$
1	$\begin{pmatrix} \sqrt{-\frac{(\lambda_{11}-\lambda_{12})(\lambda_{11}-\lambda_{22})(\lambda_{11}-\lambda_{32})}{(\lambda_{11}-\lambda_{21})(\lambda_{11}-\lambda_{31})}} \\ \sqrt{-\frac{(\lambda_{21}-\lambda_{12})(\lambda_{21}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{(\lambda_{12}-\lambda_{11})(\lambda_{12}-\lambda_{21})(\lambda_{12}-\lambda_{31})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}} \\ \sqrt{\frac{(\lambda_{22}-\lambda_{11})(\lambda_{22}-\lambda_{21})}{(\lambda_{21}-\lambda_{11})}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{v_{11}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{11}} & \frac{v_{11}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{11}} & 0 \\ \frac{v_{21}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{21}} & \frac{v_{21}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{21}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
2	$\begin{pmatrix} \sqrt{-\frac{(\lambda_{12}-\lambda_{13})(\lambda_{12}-\lambda_{23})(\lambda_{12}-\lambda_{33})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}} \\ \sqrt{-\frac{(\lambda_{22}-\lambda_{13})(\lambda_{22}-\lambda_{23})(\lambda_{22}-\lambda_{33})}{(\lambda_{22}-\lambda_{12})(\lambda_{22}-\lambda_{32})}} \\ \sqrt{-\frac{(\lambda_{32}-\lambda_{13})(\lambda_{32}-\lambda_{23})(\lambda_{32}-\lambda_{33})}{(\lambda_{32}-\lambda_{12})(\lambda_{32}-\lambda_{22})}} \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{(\lambda_{13}-\lambda_{12})(\lambda_{13}-\lambda_{22})(\lambda_{13}-\lambda_{32})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}} \\ \sqrt{\frac{(\lambda_{23}-\lambda_{12})(\lambda_{23}-\lambda_{22})(\lambda_{23}-\lambda_{32})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}} \\ \sqrt{\frac{(\lambda_{33}-\lambda_{12})(\lambda_{33}-\lambda_{22})(\lambda_{33}-\lambda_{32})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}} \end{pmatrix}$	$\begin{pmatrix} \frac{v_{12}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{12}} \\ \frac{v_{22}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{22}} \\ \frac{v_{32}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{32}} \end{pmatrix}$
3	$\begin{pmatrix} \sqrt{-\frac{(\lambda_{13}-\lambda_{14})(\lambda_{13}-\lambda_{24})(\lambda_{13}-\lambda_{34})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}} \\ \sqrt{-\frac{(\lambda_{23}-\lambda_{14})(\lambda_{23}-\lambda_{24})(\lambda_{23}-\lambda_{34})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}} \\ \sqrt{-\frac{(\lambda_{33}-\lambda_{14})(\lambda_{33}-\lambda_{24})(\lambda_{33}-\lambda_{34})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}} \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{(\lambda_{14}-\lambda_{13})(\lambda_{14}-\lambda_{23})(\lambda_{14}-\lambda_{33})}{(\lambda_{14}-\lambda_{24})(\lambda_{14}-\lambda_{34})}} \\ \sqrt{\frac{(\lambda_{24}-\lambda_{13})(\lambda_{24}-\lambda_{23})(\lambda_{24}-\lambda_{33})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}} \\ \sqrt{\frac{(\lambda_{34}-\lambda_{13})(\lambda_{34}-\lambda_{23})(\lambda_{34}-\lambda_{33})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}} \end{pmatrix}$	$\begin{pmatrix} \frac{v_{13}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{13}} \\ \frac{v_{23}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{23}} \\ \frac{v_{33}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{33}} \end{pmatrix}$
4	$\begin{pmatrix} \sqrt{-\frac{(\lambda_{24}-\lambda_{15})(\lambda_{24}-\lambda_{25})(\lambda_{24}-\lambda_{35})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}} \\ \sqrt{-\frac{(\lambda_{34}-\lambda_{15})(\lambda_{34}-\lambda_{25})(\lambda_{34}-\lambda_{35})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \sqrt{\frac{(\lambda_{25}-\lambda_{24})(\lambda_{25}-\lambda_{34})}{(\lambda_{25}-\lambda_{35})}} \\ \sqrt{\frac{(\lambda_{35}-\lambda_{14})(\lambda_{35}-\lambda_{24})(\lambda_{35}-\lambda_{34})}{(\lambda_{35}-\lambda_{15})(\lambda_{35}-\lambda_{25})}} \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{v_{14}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{24}} & \frac{v_{14}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{24}} & 0 \\ \frac{v_{24}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{34}} & \frac{v_{24}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{34}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$
5	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

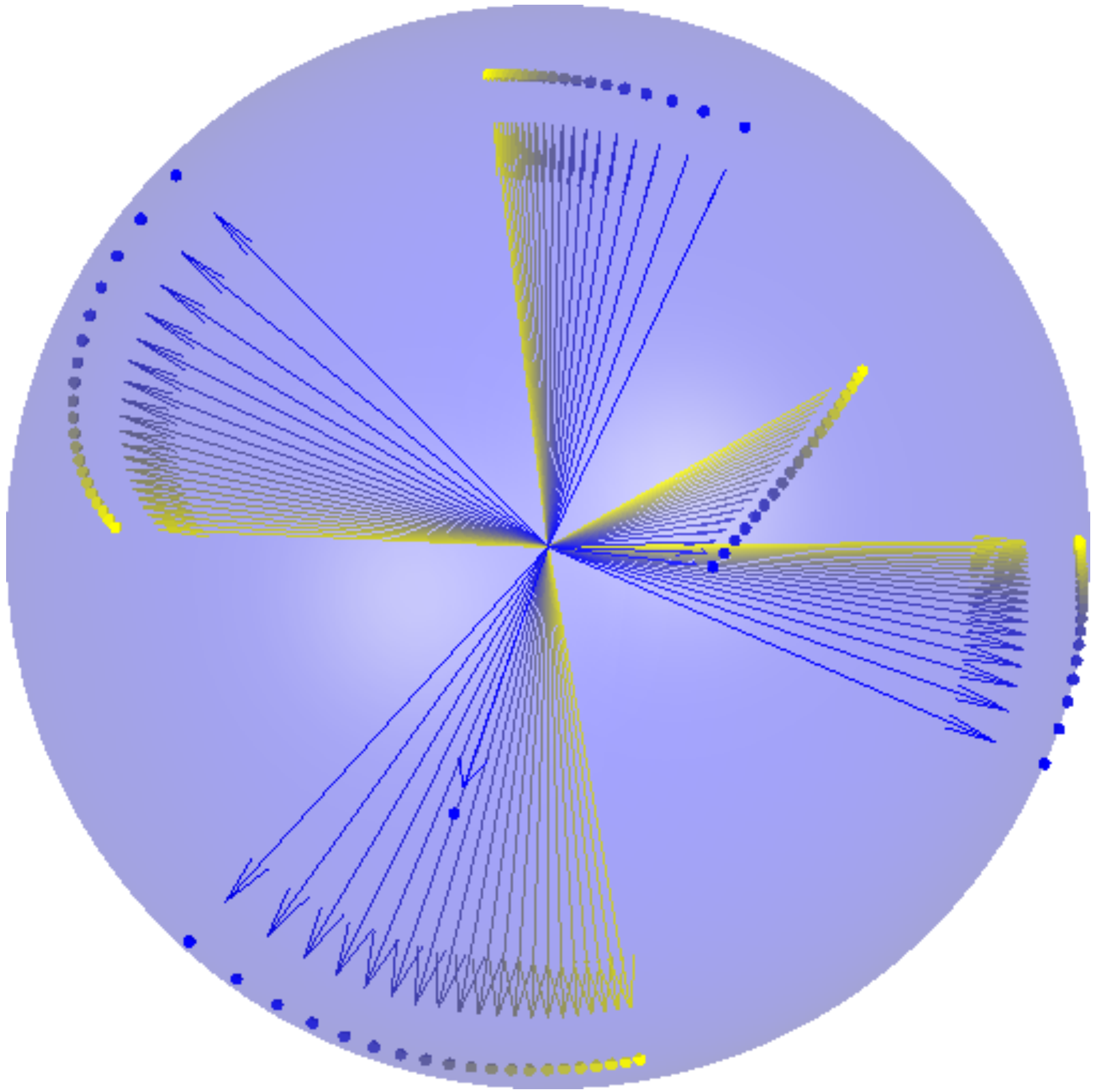
Example: Coordinates from Eigensteps

$$\mathcal{Q}(\Phi) \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix},$$

Example: Coordinates from Eigensteps

k	V_k	P_k	Q_k
1	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
2	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
3	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
4	$-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
5	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Eigenstep Trajectories



Frame Homotopy Problem

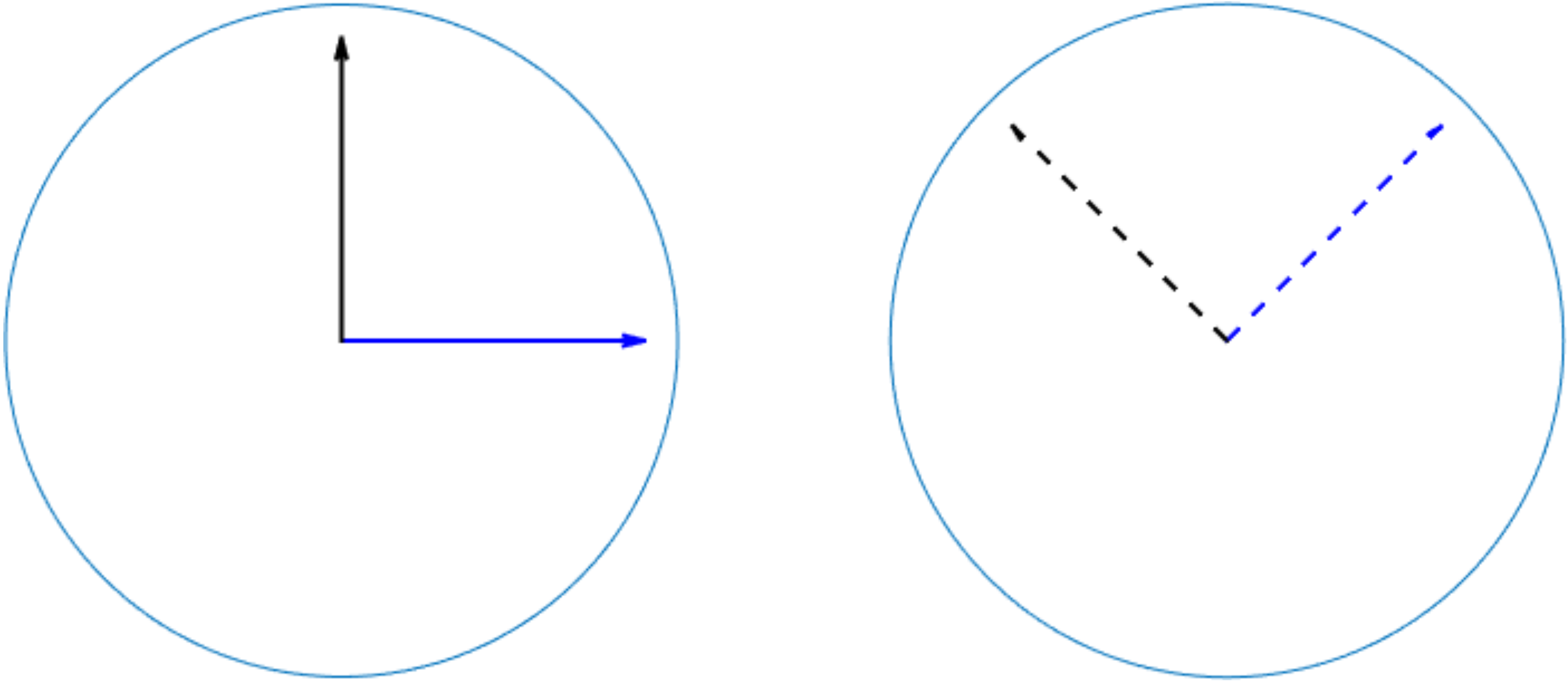
Find all pairs (N, d) such that $\mathcal{F}_{d,N}$ path-connected.

Idea: Identify "Hubs" by Eigensteps and Connect

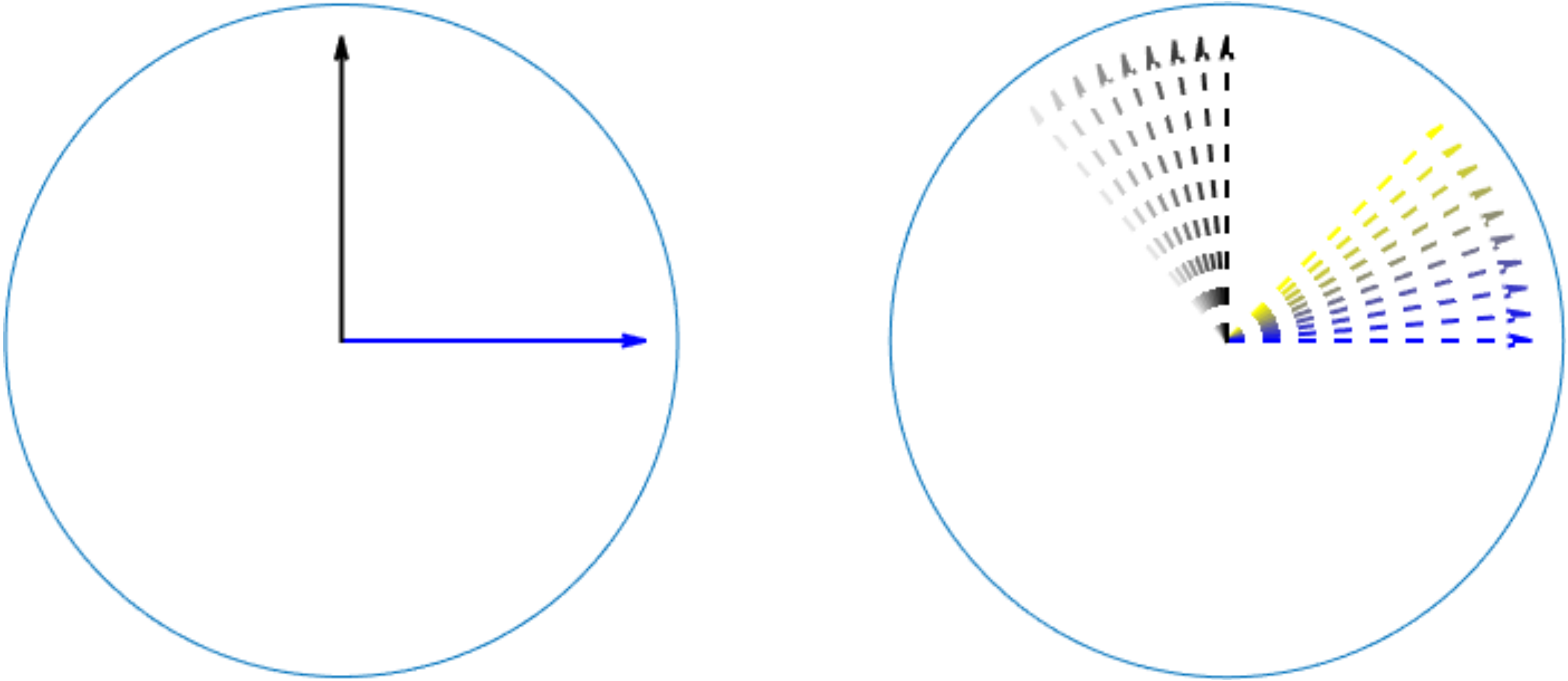
The "hub" for $\mathcal{F}_{3,6}$ is the union of two orthonormal bases indicated by the eigensteps

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Swapping Vectors between ONBs



Swapping Vectors between ONBs



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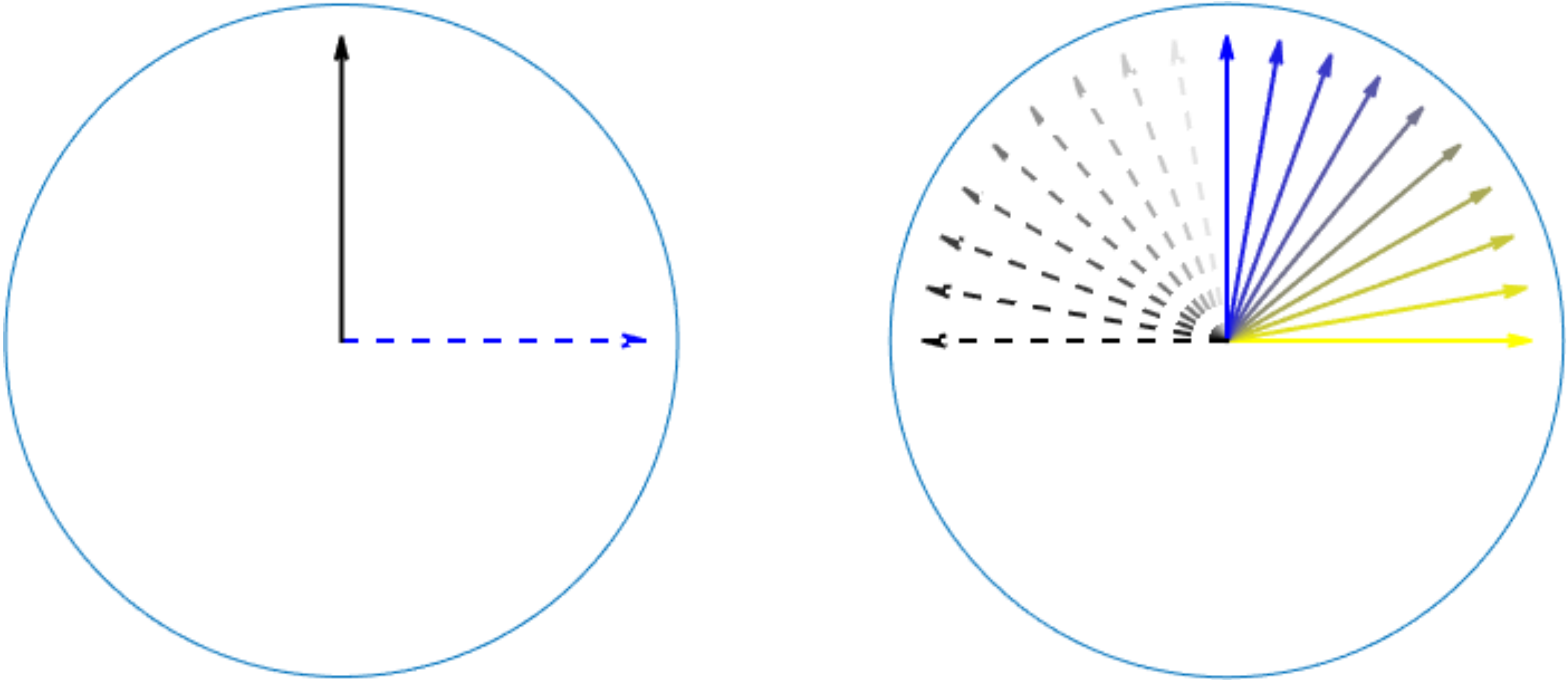
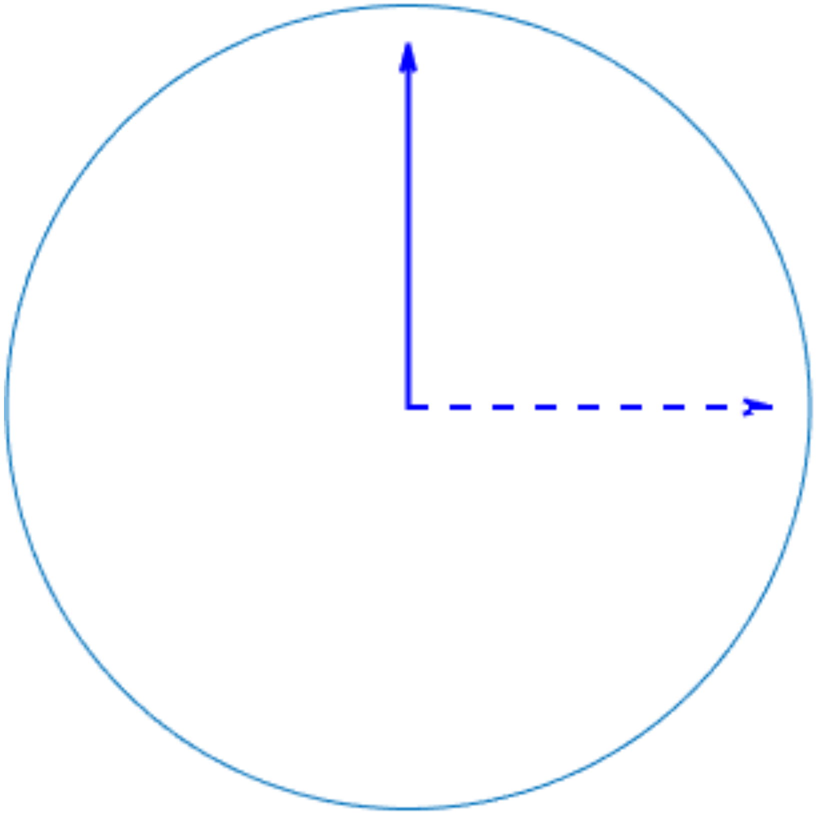
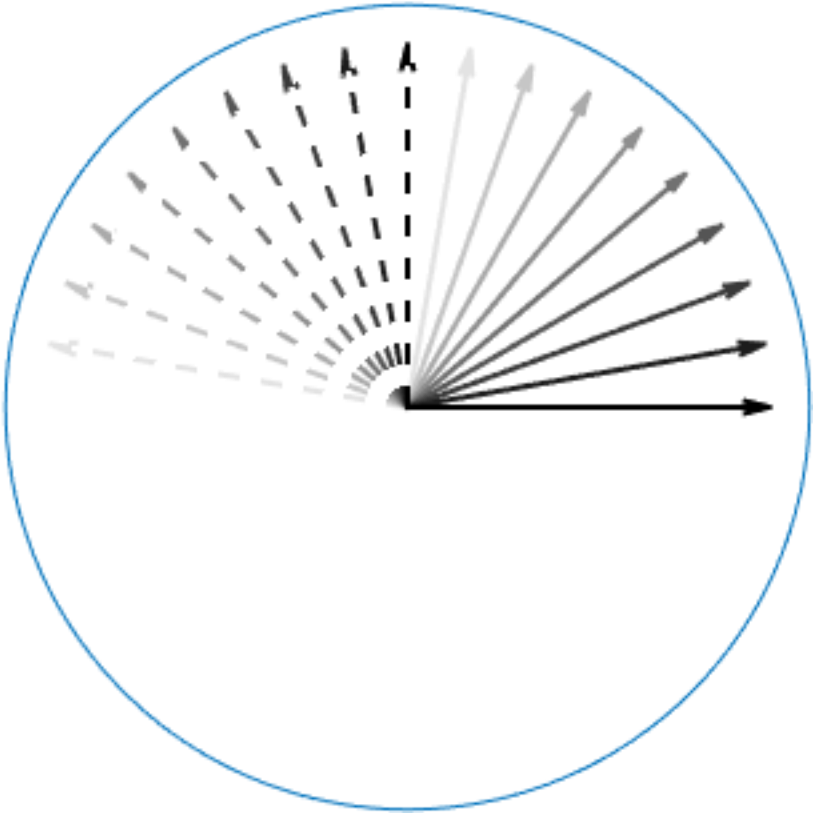
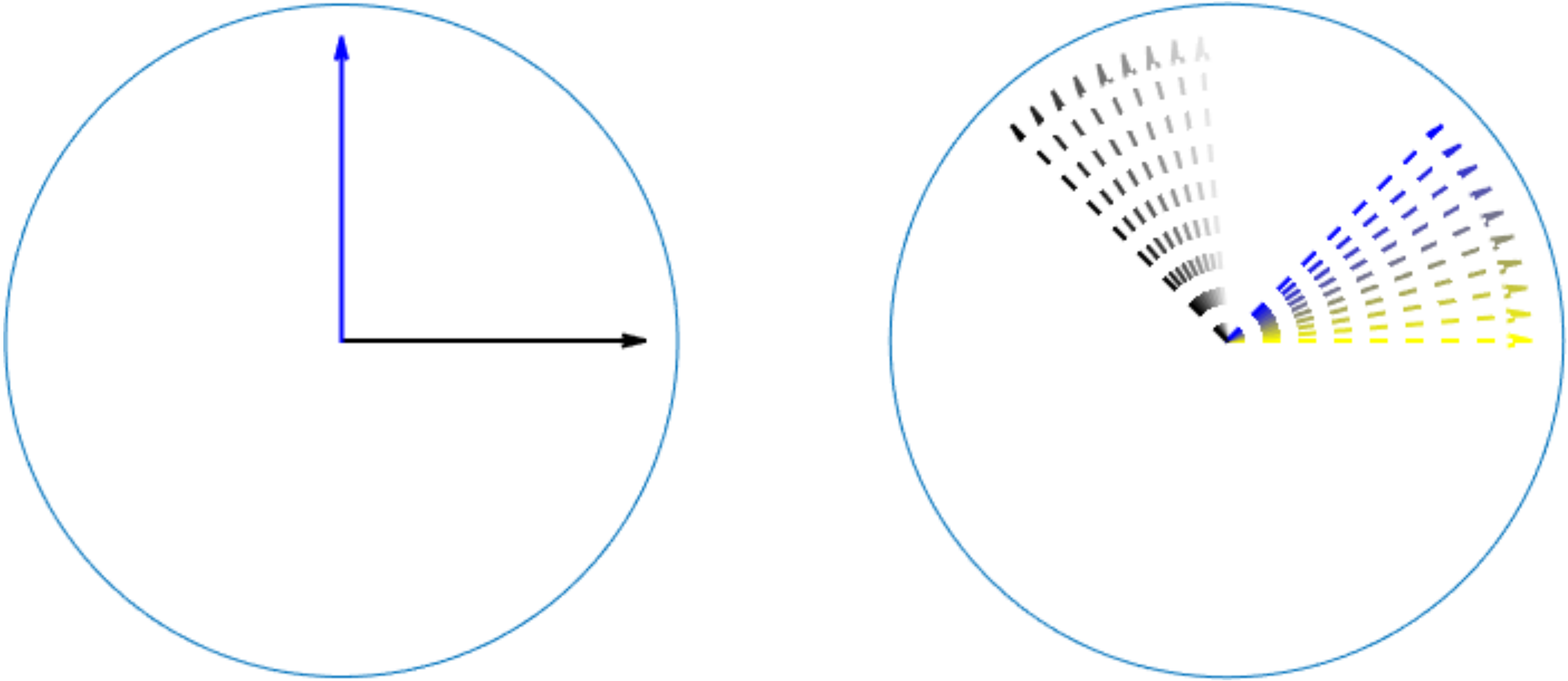


Swapping Vectors between ONBs





Swapping Vectors between ONBs



Basic Question: Are FUNTF Varieties Irreducible?

Proposition. *Suppose V is an algebraic variety such that*

(i) *the set of non-singular points of V is path-connected, and*

(ii) *the set of non-singular points is dense in V.*

Then V is an irreducible algebraic variety.

Nonsingular Points are Dense

Proposition. *A frame $F \in \mathcal{F}_{3,6}$ is orthodecomposable if and only if there are two distinct frame vectors f_i and f_j that are parallel.*

Proof. If F is orthodecomposable, then there is a partition of F into F_1 and F_2 so that the linear spans of the vectors in F_1 and F_2 (denoted V_1 and V_2) are non-trivial orthogonal subspaces of $\mathbb{R}^3$. Consequently, either V_1 or V_2 has dimension equal to 1, and hence by the tight frame bound condition either F_1 or F_2 consists of two parallel vectors.

On the other hand, assuming that there are vectors f_i and f_j which are parallel, the tight frame bound condition requires that all other vectors in F are orthogonal to f_i and f_j. Consequently, F is orthodecomposable. □

Connectivity of the Non-singular FUNTFs

Proposition. *If $F \in \mathcal{F}_{3,6}$ is orthodecomposable, then it is a union of two orthonormal bases.*

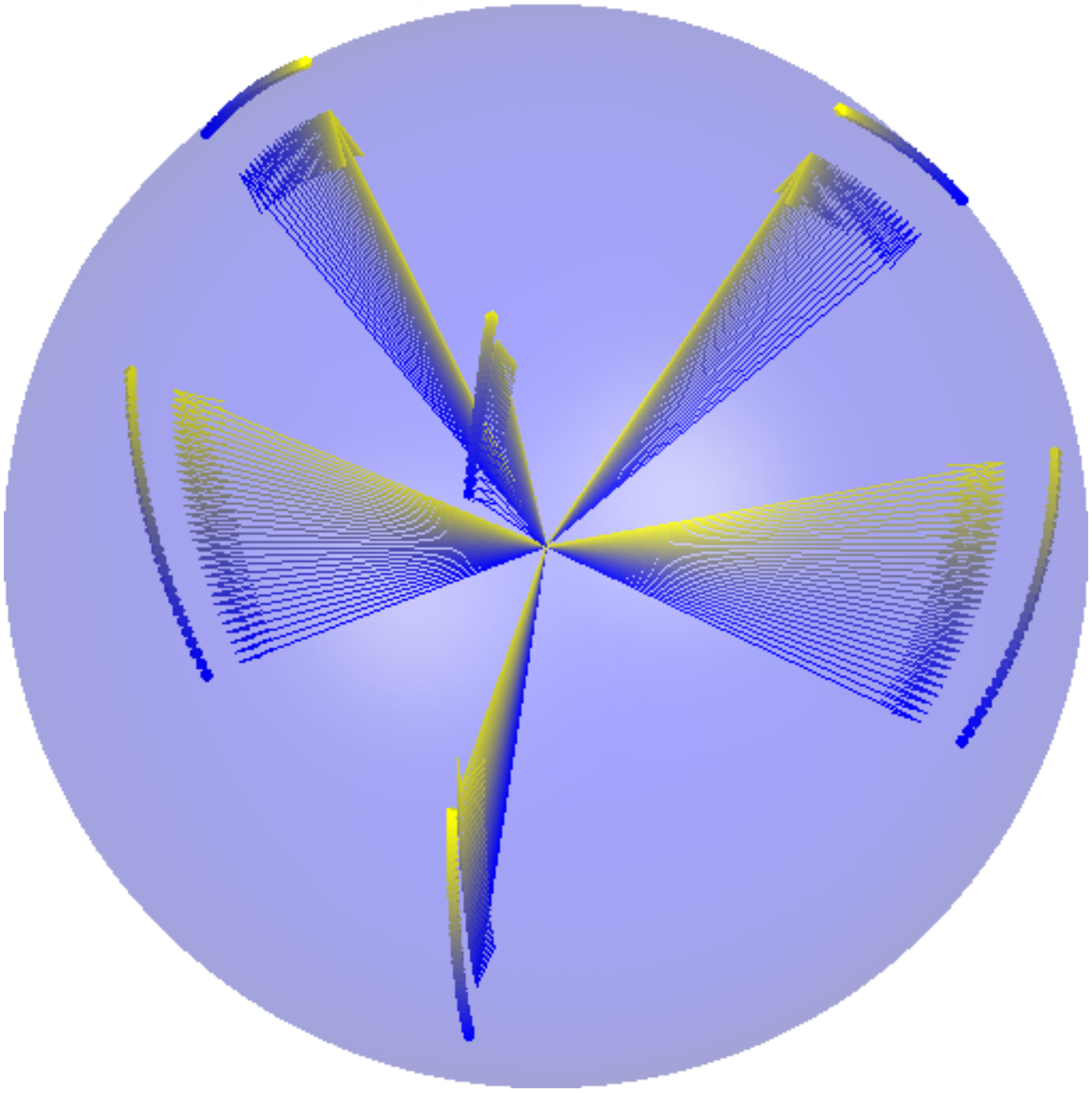
Proposition. *The non-singular points of $\mathcal{F}_{3,6}$ are dense in $\mathcal{F}_{3,6}$.*

Connectivity of the Non-singular FUNTFs

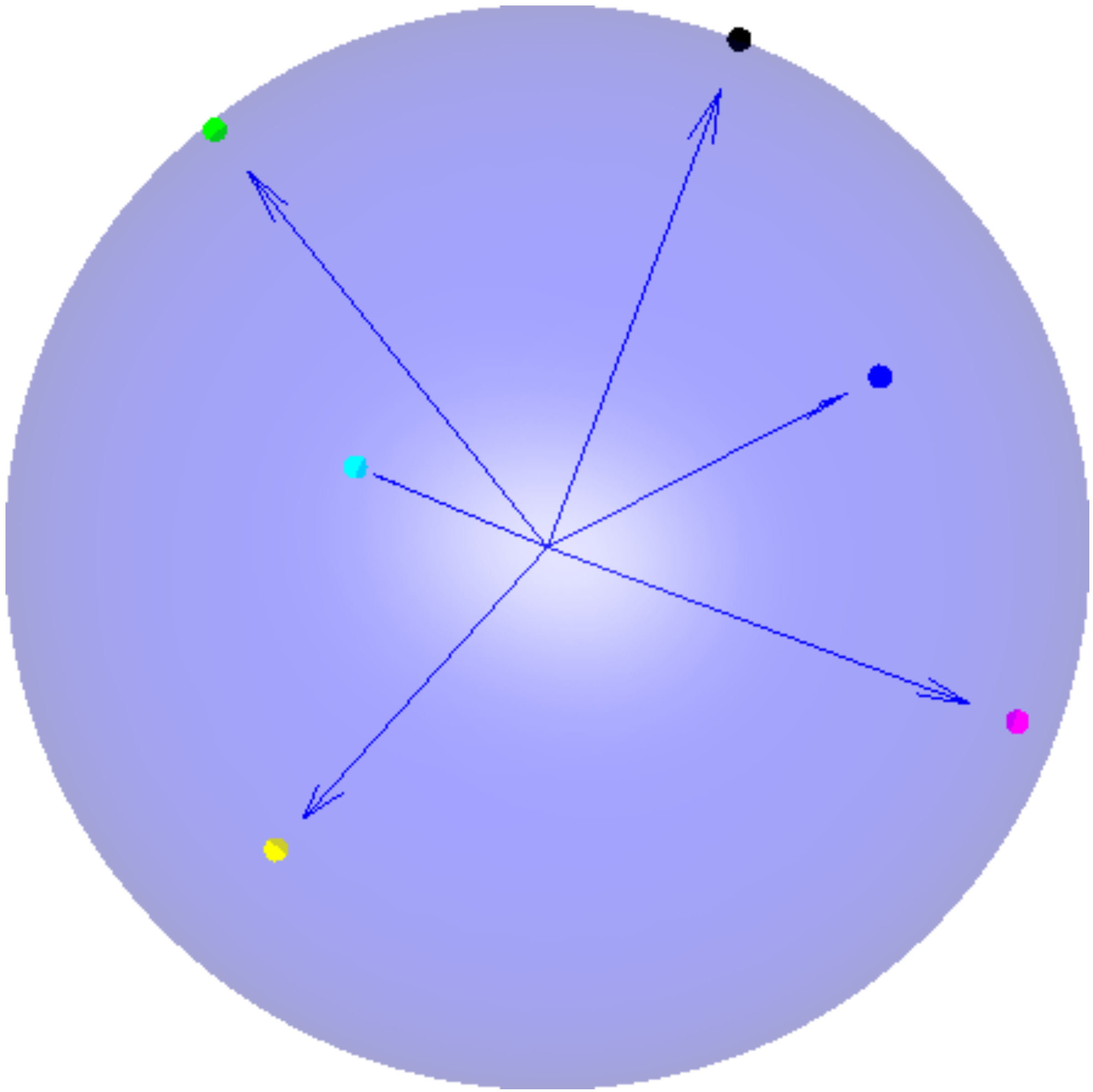
Connect to a frame with eigensteps

$$\begin{pmatrix} 1 & 3/2 & 3/2 & 2 & 2 & 2 \\ 0 & 1/2 & 3/2 & 3/2 & 2 & 2 \\ 0 & 0 & 0 & 1/2 & 1 & 2 \end{pmatrix}$$

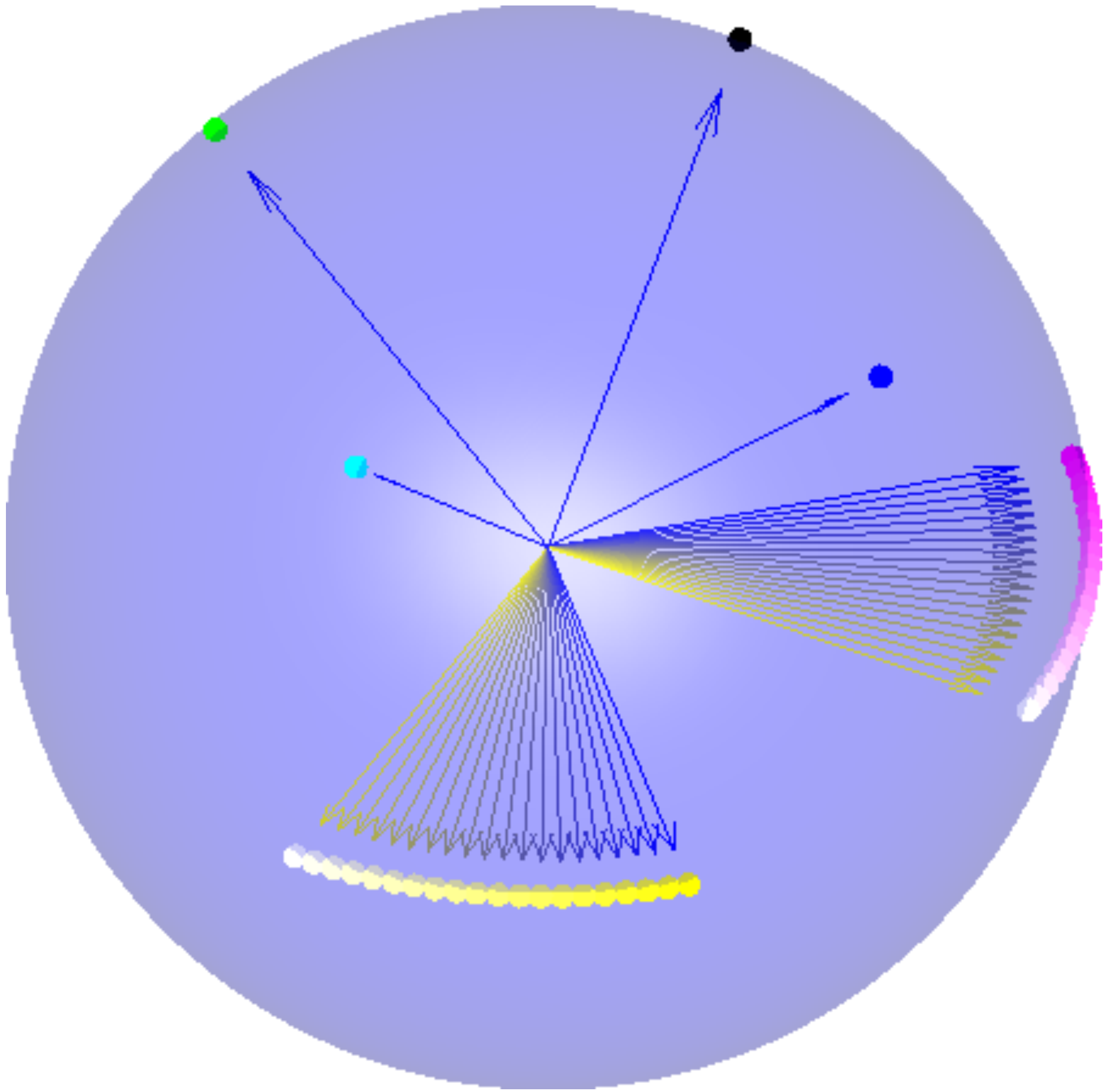
Connectivity of the Non-singular FUNTFs



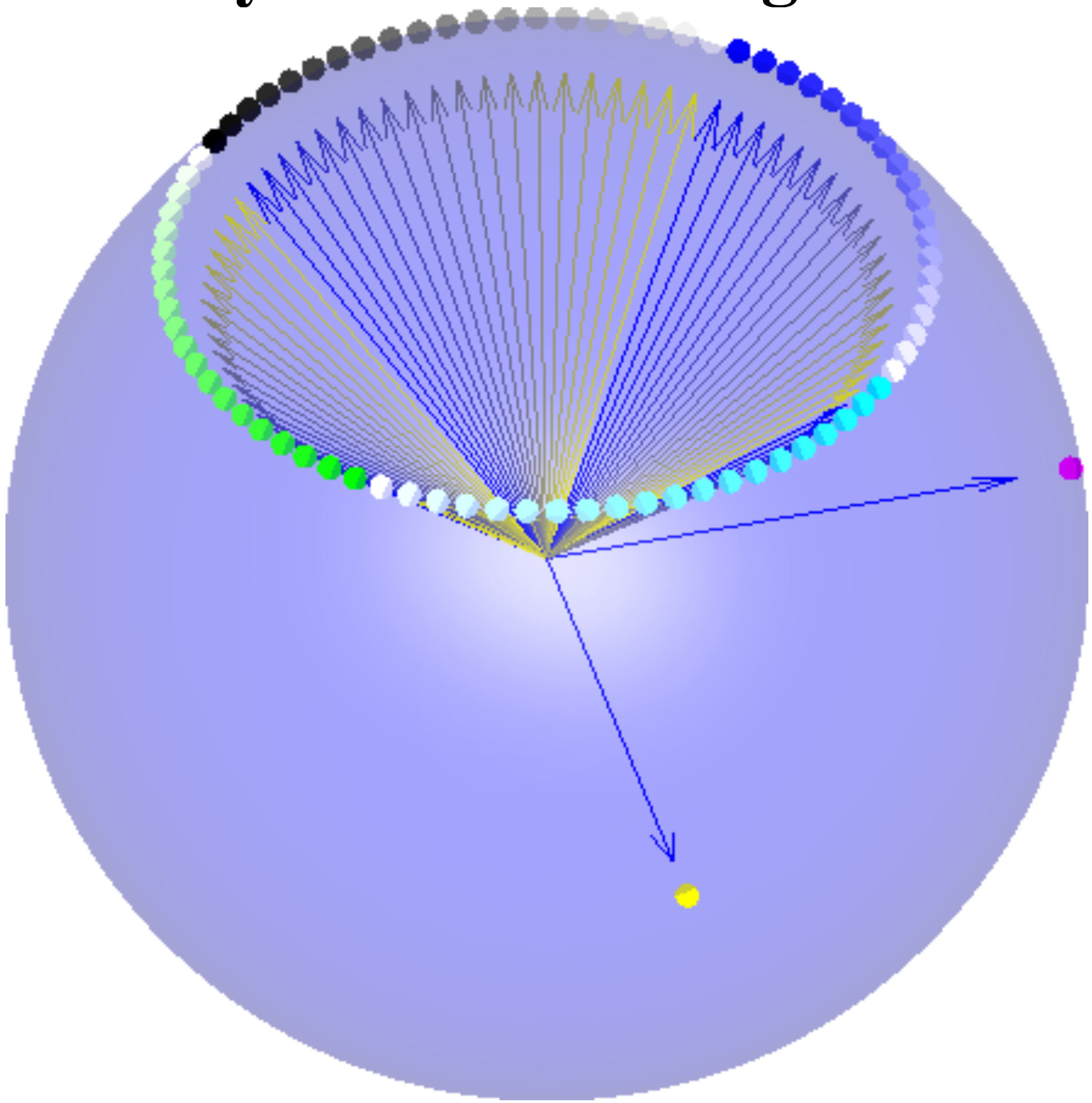
Connectivity of the Non-singular FUNTFs



Connectivity of the Non-singular FUNTFs



Connectivity of the Non-singular FUNTFs



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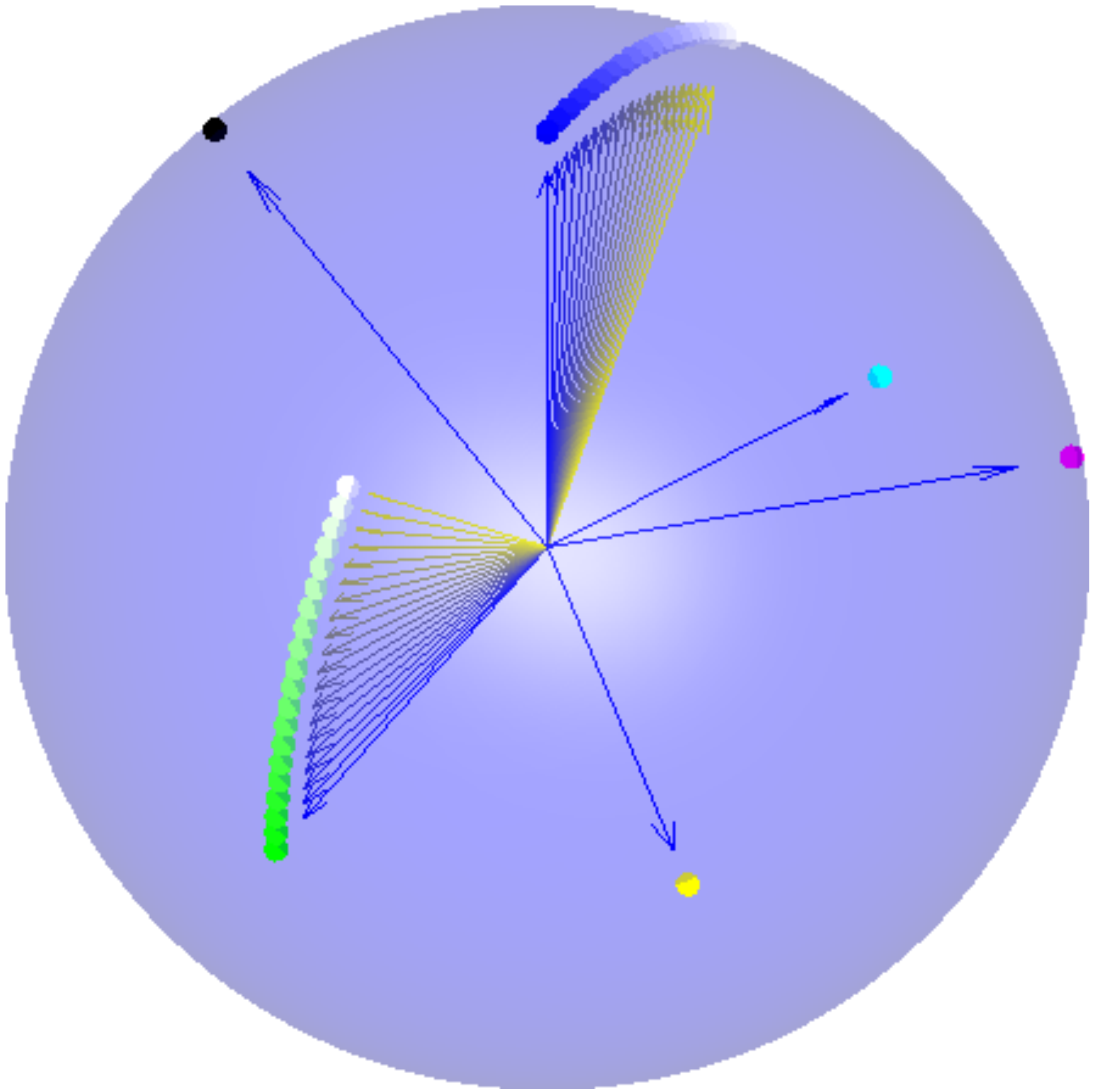
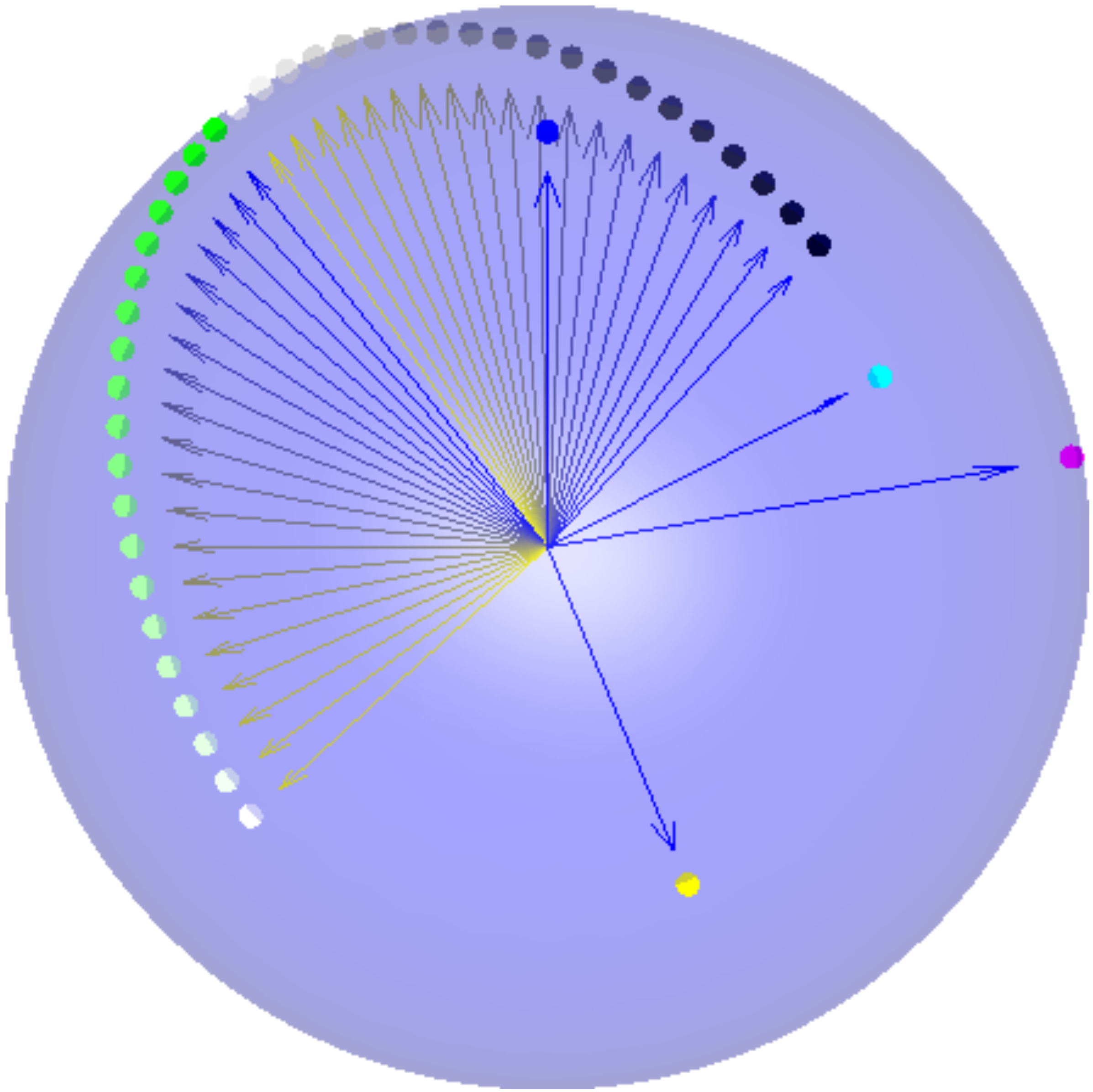
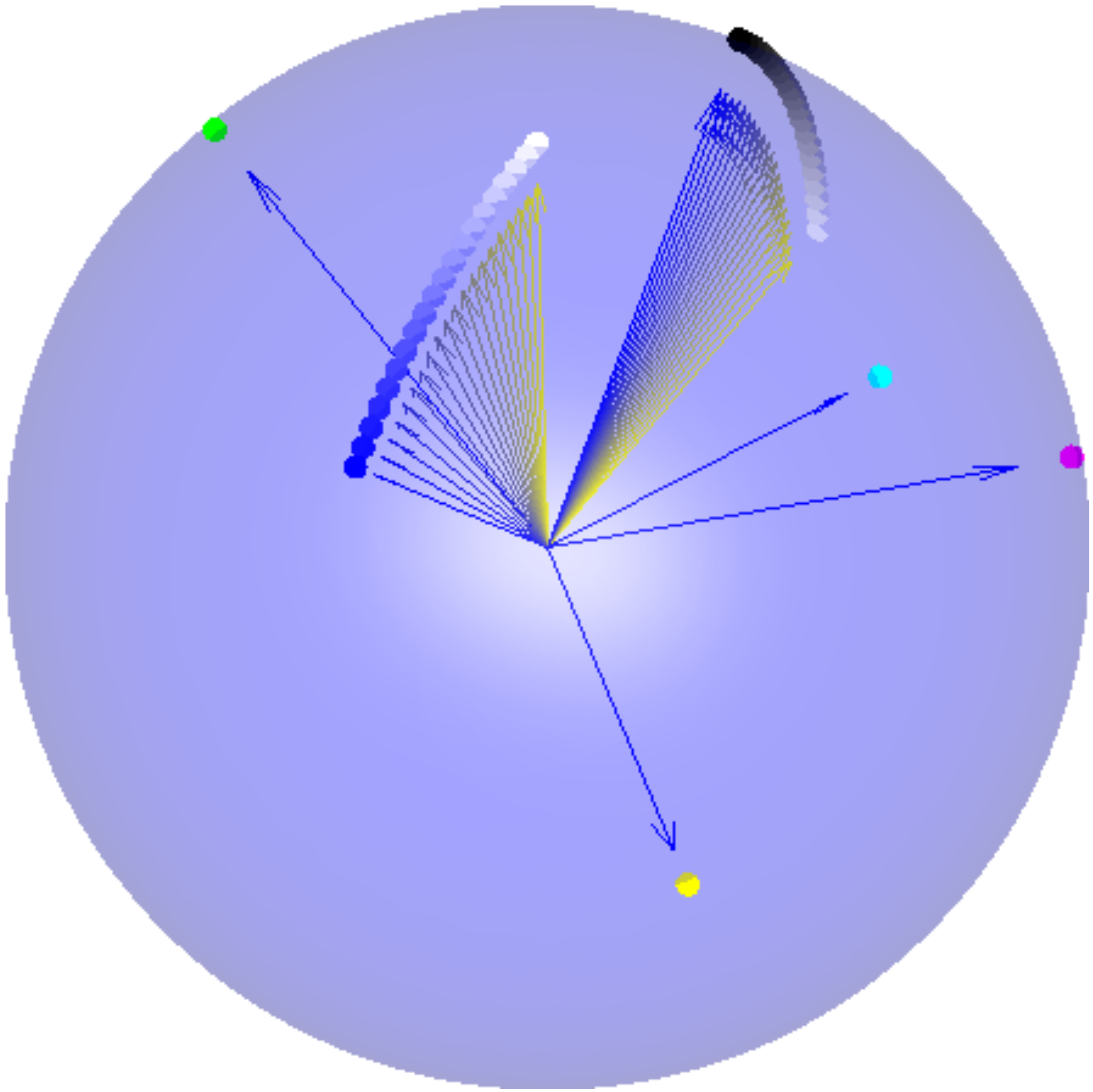


Connectivity of the Non-singular FUNTFs



Connectivity of the Non-singular FUNTFs



Open Problem: Describe the Local Geometry Around OD Frames

Open Problem: Are there any FUNTF Varieties such that All the Members are Permutation Equivalent to Frames which Map to the Interior of the Eigensteps Polytope?

Open Problem: Compute the Homotopy of the FUNTF Varieties

Open Problem: Construct the FUNTF Varieties as Configuration Spaces

Idea

Set $\varphi(f) = ff^*$ and note that $\varphi(S)$ is diffeomorphic to $\mathbb{RP}^{d-1}$.

FUNTFs correspond to chains in $M_{d,d}$ from 0 to $\frac{N}{d}I_d$ with links in $\varphi(\mathcal{S}^{d-1})$.