Nonlinear Analysis with Frames. Part II: Lipschitz Reconstruction

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Problem Formulation

The phase retrieval problem

- Hilbert space $H = \mathbb{C}^n$, $\hat{H} = H/T^1$, frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \le k \le m} .$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \le k \le m} .$$

The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.

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  The frame is said *phase retrievable* (or that it gives phase retrieval) if α (or β) is injective.

- The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor.

Problem Formulation
Lipschitz Reconstruction

Our Problems Today: Assume $\mathcal{F}$ is phase retrievable.

- *Deterministic Analysis.*
  1. Are the nonliner maps α, β bi-Lipschitz with respect to appropriate metrics?
  2. Do they admit left inverses that are globally Lipschitz?
  3. What are the Lipschitz constants?
  4. Additionally, we want to understand the structure of Lipschitz bounds (to be defined shortly).

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- *Stochastic Analysis.*
  5. Cramer-Rao Lower Bounds.

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Metric Space Structures
Topological Structures

Let $H = \mathbb{C}^n$. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real φ with $x = e^{i\varphi} y$.

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Topologically:

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0,\infty) \times \mathbb{C}\mathbb{P}^{n-1}\right)$$

with

$$\mathring{\hat{\mathbb{C}}}^n = \hat{\mathbb{C}}^n \setminus \{0\} = (0,\infty) \times \mathbb{C}\mathbb{P}^{n-1}$$

a real analytic manifold of real dimension $2n-1$.

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a real analytic manifold of real dimension $2n-1$.
Another embedding is into the space of symmetric matrices $Sym(\mathbb{C}^n)$.
Specifically let

$$\mathcal{S}^{p,q}(H) = \{ T \in Sym(H) \ , \ T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs} \}$$

Then:

$$\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0} \quad , \quad \hat{x} \mapsto = xx^* \quad , \quad \text{is an embedding.}$$

Metric Space Structures

The matrix-norm induced metric structure

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the p-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y \rangle|^2}$$

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Lemma (BZ15)

1. $(d_p)_{1 \leq p \leq \infty}$ *are equivalent metrics and the identity map* $i : (\hat{H}, d_p) \to (\hat{H}, d_q)$, $i(x) = x$ *has Lipschitz constant*

$$Lip^d_{p,q,n} = \max(1, 2^{\frac{1}{q} - \frac{1}{p}}).$$

2. *The metric space* $(\hat{H}, d_p)$ *is isometrically isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the p-norm via* $\kappa_\beta : \hat{H} \to \mathcal{S}^{1,0} \ , \ \ x \mapsto \kappa_\beta(x) = xx^*$.

Metric Space Structures

The natural metric structure

Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ D_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

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2. *The metric space* $(\hat{H}, D_2)$ *is Lipschitz isomorphic to* $\mathcal{S}^{1,0}$ *endowed with the 2-norm via* $\kappa_\alpha : \hat{H} \to \mathcal{S}^{1,0}$, $x \mapsto \kappa_\alpha(x) = \frac{1}{\|x\|} x x^*$.

Metric Space Structures

Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map $i : (\hat{H}, D_p) \to (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \to (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

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Lipschitz Analysis

Lipschitz inversion: α

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\alpha : (\hat{H}, D_2) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{A_0}, \sqrt{B_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:*

$$A_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k \rangle| - |\langle y, f_k \rangle| \right|^2 \leq B_0 \min_{\varphi} \|x - e^{i\varphi} y\|_2^2.$$

2. *There is a Lipschitz map $\omega : (\mathbb{R}^m, \|\cdot\|_2) \rightarrow (\hat{H}, D_2)$ so that: (i) $\omega(\alpha(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\omega) \leq \frac{4+3\sqrt{2}}{\sqrt{A_0}} = \frac{8.24}{\sqrt{A_0}}$.*

Lipschitz Analysis

Lipschitz inversion: β

Theorem (BZ15)

Assume $\mathcal{F}$ is a phase retrievable frame for H. Then:

1. *The map $\beta : (\hat{H}, d_1) \to (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz. Let $\sqrt{a_0}, \sqrt{b_0}$ denote its Lipschitz constants: for every $x, y \in \hat{H}$:*

$$a_0 \|xx^* - yy^*\|_1^2 \leq \sum_{k=1}^{m} \left| |\langle x, f_k\rangle|^2 - |\langle y, f_k\rangle|^2 \right|^2 \leq b_0 \|xx^* - yy^*\|_1^2.$$

2. *There is a Lipschitz map $\psi : (\mathbb{R}^m, \|\cdot\|_2) \to (\hat{H}, d_1)$ so that: (i) $\psi(\beta(x)) = x$ for every $x \in \hat{H}$, and (ii) its Lipschitz constant is $Lip(\psi) \leq \frac{4+3\sqrt{2}}{\sqrt{a_0}} = \frac{8.24}{\sqrt{a_0}}$.*

Lipschiutz Analysis

Prior Works

Prior literature:

Lipschiutz Analysis
Prior Works

Prior literature:

- 2012: **B.**: Cramer-Rao lower bound in the real case; **Eldar&Mendelson** : map α in the real case

$$\|\alpha(x)-\alpha(y)\| \geq C\|x-y\|\|x+y\|.$$

- 2013: **Bandeira,Cahill,Mixon,Nelson**: improved the estimate of C. **B.**: β bi-Lipschitz in real and complex case.
- 2014: **B.&Yang**: Find the exact Lipschitz constant for α in the real case - the constants A_0, B_0; **B.&Z.**:constructed a Lipschitz left inverse for β; **B.**: lower Lipschitz constant A_0 connected to CRLB's for a non-AWGN model.
- 2015: **B.&Z.**: Proved α is bi-Lipschitz in the complex case; constructed a Lipschitz left inverse.

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Overview

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Overview

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1. Part 1: Injectivity $\longrightarrow$ bi-Lipschitz: Upper bounds are not too hard; lower bounds: relatively easy for β (the "square" map), but very hard for α.
2. Part 2: Left inverse construction is done in three steps:
   1. The left inverse is first extended to $\mathbb{R}^m$ into $Sym(H)$ using Kirszbraun's theorem;
   2. Then we show that $\mathcal{S}^{1,0}(H)$ is a Lipschitz retract in $Sym(H)$;
   3. The proof is concluded by composing the two maps.

Proofs
Part 1: Bi-Lipschitzianity for β

Key Remark (B.Bodmann,Casazza,Edidin - 2007): The nonlinear map β is the restrictrion of the linear map

$$\mathbb{A} : Sym(H) \to \mathbb{R}^m \quad , \quad \mathbb{A}(T) = (\langle Tf_k, f_k \rangle)_{1 \leq k \leq m}$$

Specifically: $\beta(x) = \mathbb{A}(xx^*)$.

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$$\begin{aligned} \|\beta(x) - \beta(y)\| = \|\mathbb{A}(xx^*) - \mathbb{A}(yy^*)\| &= \|\mathbb{A}(xx^* - yy^*)\| \\ &= \|xx^* - yy^*\| \|\mathbb{A}\left(\frac{xx^* - yy^*}{\|xx^* - yy^*\|}\right)\| \end{aligned}$$

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$$a_0 = \min_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\| > 0 \quad , \quad b_0 = \max_{T \in \mathcal{S}^{1,1}, \|T\|_1 = 1} \|\mathbb{A}(T)\|$$

Proofs

Part 2: Extension of the inverse for β

Assume $\beta : (\hat{H}, d_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_2)$ is bi-Lipschitz:

$$a_0 d_1(x,y)^2 \leq \|\beta(x) - \beta(y)\|^2 \leq b_0 d_1(x,y)^2$$

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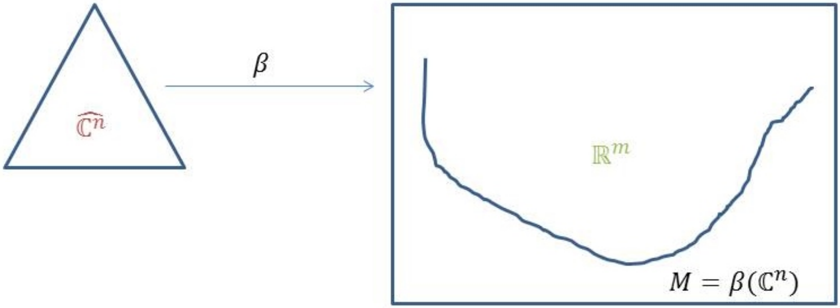
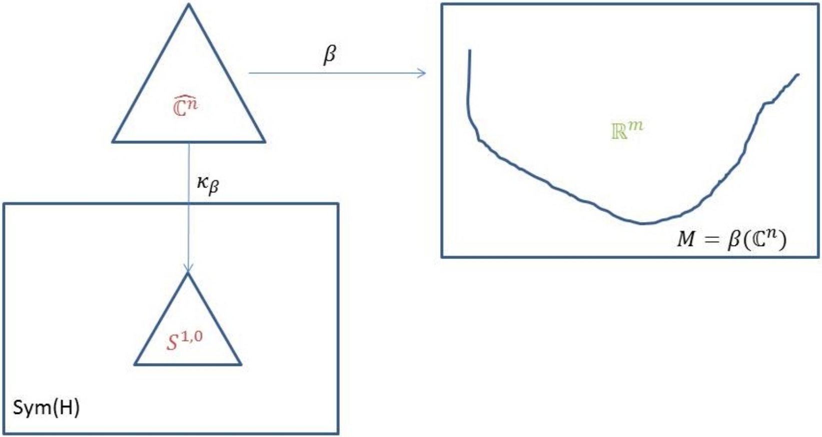
Let $M = \beta(\hat{H}) \subset \mathbb{R}^m$.



Proofs

Part 2: Extension of the inverse for β

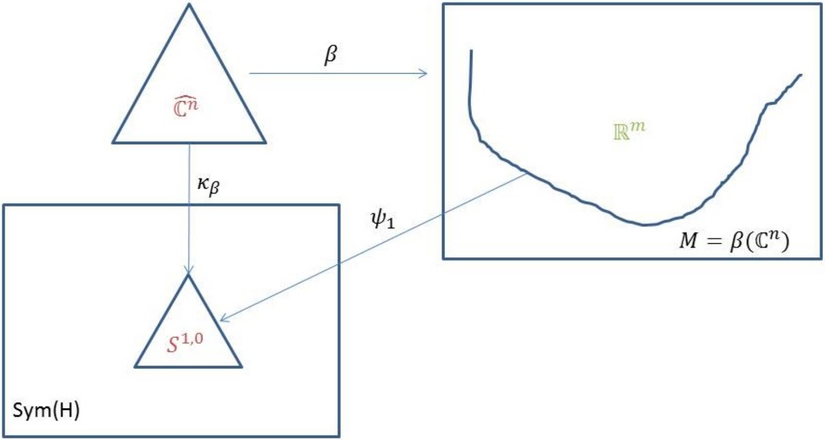
First identify $\hat{H}$ with $\mathcal{S}^{1,0}(H)$.



Proofs

Part 2: Extension of the inverse for β

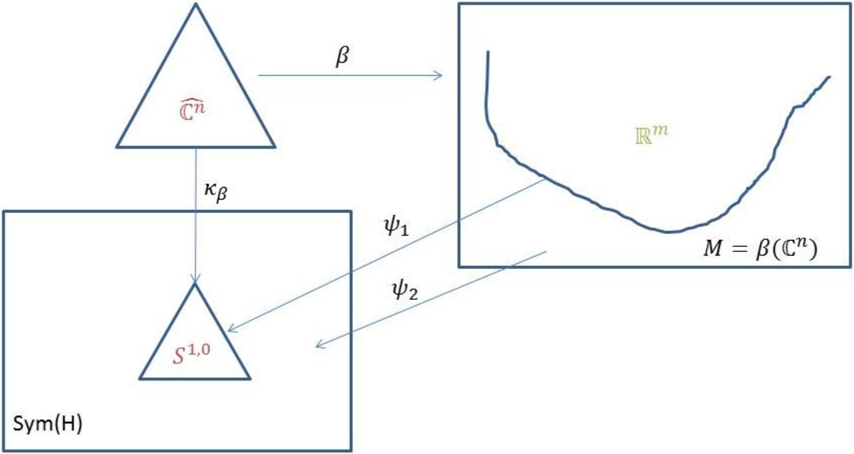
Then construct the local left inverse $\psi_1 : M \to \hat{H}$ with $Lip(\psi_1) = \frac{1}{\sqrt{a_0}}$.



Proofs

Part 2: Extension of the inverse for β

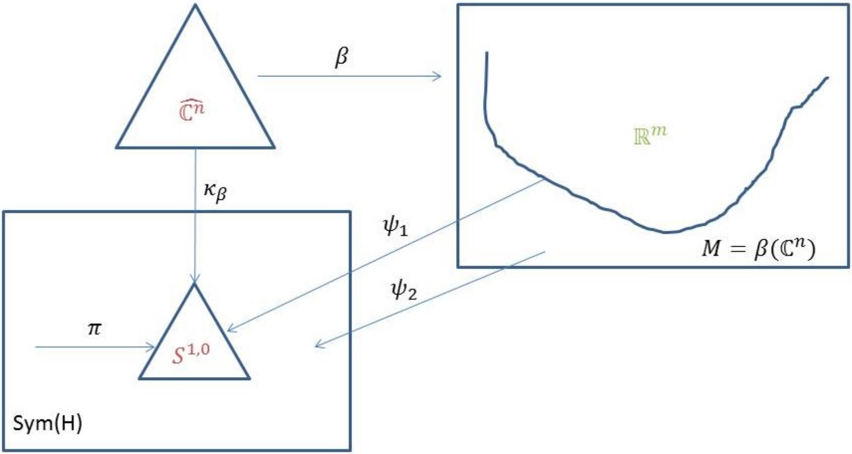
Use Kirszbraun's theorem to extend isometrically $\psi_2 : \mathbb{R}^m \to Sym(H)$.



Proofs

Part 2: Extension of the inverse for β

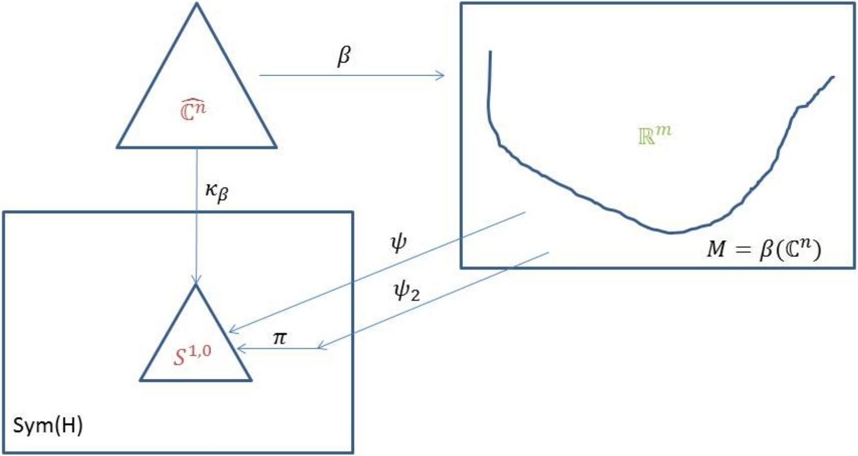
Construct a Lipschitz "projection" $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$.



Proofs

Part 2: Extension of the inverse for β

Compose the two maps to get $\psi : \mathbb{R}^m \to \mathcal{S}^{1,0}$, $\psi = \pi \circ \psi_2$.



Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

How to obtain $\pi : Sym(H) \to \mathcal{S}^{1,0}(H)$?

Proofs

Part 2: $\mathcal{S}^{1,0}(H)$ as Lipschitz retract in $Sym(H)$

Lemma

Consider the spectral decomposition of the self-adjoint operator A in $Sym(H)$, $A = \sum_{k=1}^{d} \lambda_{m(k)} P_k$. Then the map

$$\pi : Sym(H) \to \mathcal{S}^{1,0}(H) \quad , \quad \pi(A) = (\lambda_1 - \lambda_2) P_1$$

satisfies the following two properties:

1. *for $1 \leq p \leq \infty$, it is Lipschitz continuous from $(Sym(H), \|\cdot\|_p)$ to $(\mathcal{S}^{1,0}(H), \|\cdot\|_p)$ with Lipschitz constant less than or equal to $3 + 2^{1+\frac{1}{p}}$;*
2. *$\pi(A) = A$ for all $A \in \mathcal{S}^{1,0}(H)$.*

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Proof uses Weyl's inequality and spectral formula on a complex integration contour by Zwald & Blanchard (2006).

Proofs

Part 1: Bi-Lipschitzianity of α

The analysis requires a deeper understanding of local behavior.

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The analysis requires a deeper understanding of local behavior.

1. The *global lower* and *upper Lipschitz bounds*:

$$A_0 = \inf_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2} \ , \ B_0 = \sup_{x,y\in\hat{H}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

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2. The *type I local lower* and *upper Lipschitz bounds* at $z\in\hat{H}$:

$$A(z) = \lim_{r\to 0} \inf_{\substack{x,y\in\hat{H}\\ D_2(x,z)<r\\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}~,~B(z) = \lim_{r\to 0} \sup_{\substack{x,y\in\hat{H}\\ D_2(x,z)<r\\ D_2(y,z)<r}} \frac{\|\alpha(x)-\alpha(y)\|_2^2}{D_2(x,y)^2}$$

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3. The *type II local lower* and *upper Lipschitz bounds* at $z \in \hat{H}$:

$$\tilde{A}(z) = \lim_{r\to 0} \inf_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,z)^2} \ , \ \tilde{B}(z) = \lim_{r\to 0} \sup_{\substack{x\in\hat{H} \\ D_2(x,z)<r}} \frac{\|\alpha(x)-\alpha(z)\|_2^2}{D_2(x,y)^2}$$

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Part 1: Bi-Lipschitzianity of α

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Let $\varphi_1, \cdots, \varphi_m, \zeta \in \mathbb{R}^{2n}$, $\Phi_1, \cdots, \Phi_m \in Sym(\mathbb{R}^{2n})$, $J \in \mathbb{R}^{2n \times 2n}$ defined by:

$$\Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T, \varphi_k = \begin{bmatrix} real(f_k) \\ imag(f_k) \end{bmatrix}, J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \zeta = \begin{bmatrix} real(z) \\ imag(z) \end{bmatrix}$$

Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J\varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.

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Key relations: $\langle z, f_k \rangle = \langle \zeta, \varphi_k \rangle + i \langle \zeta, J\varphi_k \rangle$, $|\langle z, f_k \rangle| = \sqrt{\langle \Phi_k \zeta, \zeta \rangle}$.
Consider the following objects:

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k:\Phi_k \xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n}$$

Proofs

Lipschitz bounds for α

Theorem (BZ15)

Assume $\mathcal{F}$ is phase retrievable for $H = \mathbb{C}^n$ and A, B are its optimal frame bounds. Then:

1. *For every $0 \neq z \in \mathbb{C}^n$, $A(z) = \lambda_{2n-1}\left(\mathcal{S}(\zeta)\right)$ (the next to the smallest eigenvalue);*
2. $A_0 = A(0) > 0$;
3. *For every $z \in \mathbb{C}^n$, $\tilde{A}(z) = \lambda_{2n-1}\left(\mathcal{S}(\zeta) + \sum_{k:\langle z, f_k\rangle = 0} \Phi_k\right)$ (the next to the smallest eigenvalue);*
4. $\tilde{A}(0) = A$, *the optimal lower frame bound;*
5. *For every $z \in \mathbb{C}^n$, $B(z) = \tilde{B}(z) = \lambda_1\left(\mathcal{S}(\zeta) + \sum_{k:\langle z, f_k\rangle = 0} \Phi_k\right)$ (the largest eigenvalue);*
6. $B_0 = B(0) = \tilde{B}(0) = B$, *the optimal upper frame bound;*

Proofs

Lipschitz bounds for β

Theorem (cont'd)

7. *For every* $0 \neq z \in \mathbb{C}^n$, $a(z) = \tilde{a}(z) = \lambda_{2n-1}(\mathcal{R}(\zeta))/\|z\|^2$ *(the next to the smallest eigenvalue);*
8. *For every* $0 \neq z \in \mathbb{C}^n$, $b(z) = \tilde{b}(z) = \lambda_1(\mathcal{R}(\zeta))/\|z\|^2$ *(the largest eigenvalue);*
9. $a_0 = \min_{\|\xi\|=1} \lambda_{2n-1}(\mathcal{R}(\xi))$ *is also the largest constant to that* $\mathcal{R}(\xi) \geq a_0(\|\xi\|^2 I - J\xi\xi^T J^T)$;
10. $b(0) = \tilde{b}(0) = b_0 = \max_{\|\xi\|=1} \lambda_1(\mathcal{R}(\xi))$ *is also the* 4^{th} *power of the frame analysis operator norm* $T : (\mathbb{C}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_4)$*:* $b_0 = \|T\|^4_{B(l^2,l^4)} = \max_{\|x\|_2=1} \sum_{k=1}^m |\langle x, f_k\rangle|^4$;
11. $\tilde{a}(0)$ *is given by* $\tilde{a}(0) = \min_{\|z\|=1} \sum_{k=1}^m |\langle z, f_k\rangle|^4$.

Table of Contents

CRLB
Stochastic Models

Consider the measurement process:

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where $(\mu_k)_k, (\nu_k)_k$ are two noise processes.

CRLB
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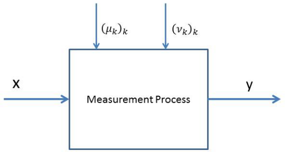
An *estimator*: $\omega : \mathbb{R}^m \to H$. *Unbiased* if: $\mathbb{E}[\omega(y); x] = x$.
Fix a direction $z_0 \in H$ and fix the global phase so that $\langle x, z_0\rangle > 0$. We want universal performance bounds of any unbiased estimator.

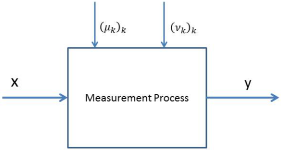
CRLB

Methodology



CRLB

Methodology



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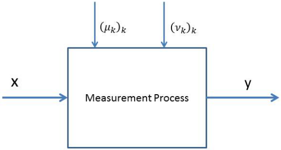
CRLB

Methodology



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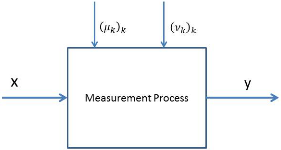
CRLB
Methodology



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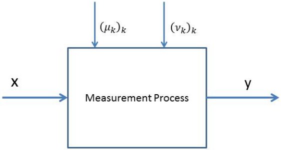
CRLB
Methodology



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Step 4: Identifiability

Determine CRLB based injectivity conditions.

CRLB
Fisher Info Matrix for the AWGN Model

The AWGN model:

$$y_k = |\langle x, f_k \rangle|^2 + \nu_k \quad , \quad \nu_k \sim \mathbb{N}(0, \sigma^2) \ , \ 1 \leq k \leq m.$$

CRLB

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- The likelihood function:

$$p(y; x, \sigma^2) = \prod_{k=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(y_k - |\langle x, f_k \rangle|^2\right)^2} = \prod_{k=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_k - \langle \Phi_k \xi, \xi \rangle)^2}$$

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$$\mathbb{I} = \mathbb{E}\left[(\nabla_x \, log \, p(y;x))(\nabla_x \, log \, p(y;x))^*\right]$$

- $\mathbb{I}^{AWGN,real}(x) = \frac{4}{\sigma^2}\sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k f_k^T = \frac{4}{\sigma^2}\sum_{k=1}^{m} (f_k f_k^T) x x^T (f_k f_k^T)$

CRLB

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CRLB

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CRLB

The Cramer-Rao Lower Bound for AWGN Model

Fix $z_0 \in \mathbb{C}^n$, $\|z_0\| = 1$, let $\zeta_0 = [real(z_0)\ imag(z_0)]^T$ and set

$$\Omega_{z_0} = \{\xi \in \mathbb{R}^{2n}\ ,\ \langle \xi, \zeta_0 \rangle) \geq 0, \langle \xi, J\zeta_0 \rangle) = 0\}.$$

Let $\Pi_{z_0} = 1 - J\zeta_0\zeta_0^* J^*$ with J the symplectic form matrix.

Theorem

Assume the measurement model $y_k = |\langle x, f_k \rangle|^2 + \nu_k$ *with* ν_k *i.i.d.* $\mathbb{N}(0, \sigma^2)$, *and* $\xi \in \mathring{\Omega}_{z_0}$. *Then the covariance of any unbiased estimtor* $\omega : \mathbb{R}^m \to \mathbb{C}^n$ *is bounded below by*

$$Cov[\omega(y); \xi] \geq \left(\Pi_{z_0} \mathbb{I}^{AWGN}(\xi) \Pi_{z_0}\right)^{\dagger}.$$

In particular: $\mathbb{E}[\|\omega(y) - \xi\|^2; \xi] \geq trace\left\{\left(\Pi_{z_0} \mathbb{I}^{AWGN}(\xi) \Pi_{z_0}\right)^{\dagger}\right\}.$

CRLB

Fisher Info Matrix for the Non-AWGN Model

Consider the Non-AWGN model:

$$y_k = |\langle x, f_k \rangle + \mu_k|^2 \quad , \quad \mu_k \sim \mathbb{CN}(0, \rho^2) \ , \ 1 \leq k \leq m.$$

CRLB

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- The likelihood function:

$$p(y;x) = \frac{1}{\rho^{2m}} exp\left\{ -\frac{1}{\rho^2}\left(\sum_{k=1}^{m} y_k + \sum_{k=1}^{m} |\langle x, f_k\rangle|^2\right)\right\} \prod_{k=1}^{m} I_0\left(\frac{2|\langle x, f_k\rangle|\sqrt{y_k}}{\rho^2}\right)$$

CRLB

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- With realification, log-likelihood:

$$\begin{aligned} log\ p(y;\xi) &= -2m \log \rho + \sum_{k=1}^m log\ I_0\left(\frac{2\sqrt{y_k\langle \Phi_k\xi, \xi\rangle}}{\rho^2}\right) - \frac{1}{\rho^2}\sum_{k=1}^m y_k - \\ &- \frac{1}{\rho^2}\sum_{k=1}^m \langle \Phi_k\xi, \xi\rangle. \end{aligned}$$

CRLB

Likelihood and Derivations for Non-AWGN

Key Estimate:

Lemma

For the Non-AWGN model in this paper and for each k,

$$\mathbb{E}\left[\frac{I_1}{I_0}\left(\frac{2\sqrt{y_k\langle\Phi_k\xi,\xi\rangle}}{\rho^2}\right)\sqrt{\frac{y_k}{\langle\Phi_k\xi,\xi\rangle}}\right]=1.$$

CRLB

Fisher Info Matrix for the Non-AWGN Model

Theorem

The Fisher information matrix for the Non-AWGN model is given by

$$\mathbb{I}^{nonAWGN}(\xi) = \frac{4}{\rho^4}\sum_{k=1}^{m}\left(G_1\left(\frac{\langle\Phi_k\xi,\xi\rangle}{\rho^2}\right)-1\right)\Phi_k\xi\xi^*\Phi_k$$
$$= \frac{4}{\rho^2}\sum_{k=1}^{m}G_2\left(\frac{\langle\Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle\Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k$$

where

$$G_1(a) = \frac{e^{-a}}{8a^3}\int_0^\infty \frac{I_1^2(t)}{I_0(t)}t^3 e^{-\frac{t^2}{4a}}dt \quad , \quad G_2(a) = a(G_1(a)-1)$$

CRLB

The Cramer-Rao Lower Bound for the Non-AWGN Model

Fix $z_0 \in \mathbb{C}^n$, $\|z_0\| = 1$, let $\zeta_0 = [real(z_0)\ imag(z_0)]^T$ and set

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Theorem

Assume the measurement model $y_k = |\langle x, f_k \rangle + \mu_k|^p$ with μ_k i.i.d. $\mathbb{CN}(0, \rho^2)$, and $\xi \in \mathring{\Omega}_{z_0}$. Then the covariance of any unbiased estimtor $\omega : \mathbb{R}^m \to \mathbb{C}^n$ is bounded below by

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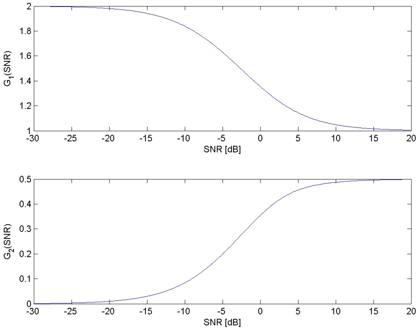
In particular: $\mathbb{E}[\|\omega(y) - \xi\|^2; \xi] \geq trace\left\{\left(\Pi_{z_0} \mathbb{I}(\xi) \Pi_{z_0}\right)^{\dagger}\right\}.$

CRLB

Comparisons for Asymptotic Regimes



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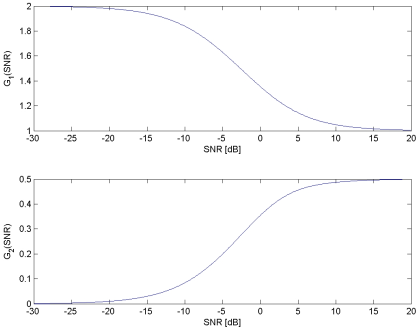

Form 1: Low SNR

$$\mathbb{I}^{nonAWGN}(\xi) = \frac{4}{\rho^4}\sum_{k=1}^m \left(G_1\left(\frac{\langle\Phi_k\xi,\xi\rangle}{\rho^2}\right)-1\right)\Phi_k\xi\xi^*\Phi_k \approx \frac{4}{\rho^4}\sum_{k=1}^m \Phi_k\xi\xi^*\Phi_k$$

CRLB

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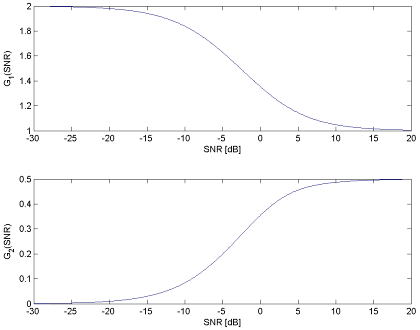

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Form 2: High SNR

$$\mathbb{I}^{nonAWGN}(\xi) = \frac{4}{\rho^2}\sum_{k=1}^m G_2\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k \approx \frac{2}{\rho^2}\sum_{k=1}^m \frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k$$

CRLB

AWGN vs. non-AWGN: The Identifiability Problem

Recall $\mathbb{I}^{AWGN,cplx}(\xi) = \frac{4}{\sigma^2}\sum_{k=1}^{m}\Phi_k\xi\xi^*\Phi_k$. Let B be frame upper bound.

Lemma

$$\frac{\sigma^2}{\rho^4}\left(G_1(\frac{B\|\xi\|^2}{\rho^2}) - 1\right)\mathbb{I}^{AWGN,cplx} \leq \mathbb{I}^{nonAWGN}(\xi) \leq \frac{\sigma^2}{\rho^4}\mathbb{I}^{AWGN,cplx}$$

CRLB

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Theorem

The following are equivalent:

1. *The frame $\mathcal{F}$ is phase retrievable;*
2. *For every $0 \neq \xi \in \mathbb{R}^{2n}$, $rank(\mathbb{I}^{nonAWGN}(\xi)) = 2n-1$;*
3. *For every $0 \neq \xi \in \mathbb{R}^{2n}$, $rank(\mathbb{I}^{AWGN,cplx}(\xi)) = 2n-1$;*

CRLB
Other nonlinear maps

Consider the model:

$$z_k = |\langle x, f_k\rangle + \mu_k|^p \quad , \quad 1 \leq k \leq m$$

where $p \neq 0$ and $(\mu_1, \cdots, \mu_m)$ are i.i.d. $\mathbb{CN}(0, \rho^2)$.

CRLB
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where $p \neq 0$ and $(\mu_1, \cdots, \mu_m)$ are i.i.d. $\mathbb{CN}(0, \rho^2)$.

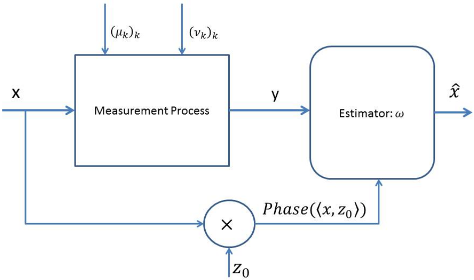
It turns out the Fisher information matrix is the same as before:

$$\begin{aligned}
\mathbb{I}^{nonAWGN,p\neq 0}(\xi) &= \mathbb{I}^{nonAWGN,p=2}(\xi) \\
&= \frac{4}{\rho^4}\sum_{k=1}^{m}\left(G_1\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)-1\right)\Phi_k\xi\xi^*\Phi_k \\
&= \frac{4}{\rho^2}\sum_{k=1}^{m}G_2\left(\frac{\langle \Phi_k\xi,\xi\rangle}{\rho^2}\right)\frac{1}{\langle \Phi_k\xi,\xi\rangle}\Phi_k\xi\xi^*\Phi_k
\end{aligned}$$

CRLB

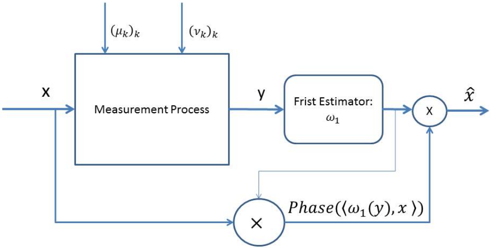
Oracle-based Estimator

Current estimator:



CRLB
Oracle-based Estimator

A more natural estimator is given by:



Open Problem: What is the CRLB in this case?

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