Nonlinear Analysis with Frames. Part I: Injectivity Results

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Problem Formulation

The phase retrieval problem

- Let $H = \mathbb{C}^n$ and $V \subset H$ a real subspace. The quotient space $\hat{H} = \mathbb{C}^n / T^1$, with classes induced by $x \sim y$ if there is real φ with $x = e^{i\varphi} y$. Set $\hat{V} = \{\hat{x}\ ,\ x \in V\}$.

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- Frame $\mathcal{F} = \{f_1, \cdots, f_m\} \subset \mathbb{C}^n$ and

$$\alpha : \hat{H} \to \mathbb{R}^m \quad , \quad \alpha(x) = (|\langle x, f_k \rangle|)_{1 \le k \le m} .$$

$$\beta : \hat{H} \to \mathbb{R}^m \quad , \quad \beta(x) = \left(|\langle x, f_k \rangle|^2\right)_{1 \le k \le m} .$$

The frame is said *phase retrievable with respect to V* (or that it gives phase retrieval for V) if α (or β) restricted to V is injective.

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  The frame is said *phase retrievable with respect to* V (or that it gives phase retrieval for V) if α (or β) restricted to V is injective.
- The general *phase retrieval problem* a.k.a. *phaseless reconstruction*: Decide when a given frame is phase retrievable, and, if so, find an algorithm to recover x from $y = \alpha(x)$ (or from $y = \beta(x)$) up to a global phase factor. Additionally find universal bounds on performance of any inversion algorithm.

Problem Formulation

Injectivity Results

- Our Problems Today: When is $\mathcal{F}$ phase retrievable.
- Want a general framework that covers both the real and complex case.

1. Obtain conditions when $V = \mathbb{R}^n$ (real case);
2. Obtain conditions when $V = \mathbb{C}^n$ (complex case)

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3. Finda minimal cardinals of phase retrievable frames.

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Topology of $\hat{V}$

Topological Structures

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Set $\hat{V} = \{\hat{x}\ ,\ x \in V\}$.
Topologically:

$$\hat{V} = \{0\} \cup ((0,\infty)] \times \mathbb{P}(V))$$

where $\mathbb{P}(V)$ denotes the projective space associated to V.
The interior subset

$$\mathring{\hat{V}} = \hat{V} \setminus \{0\} = ((0,\infty)] \times \mathbb{P}(V))$$

is a real analytic manifold of real dimension $1 + dim_{\mathbb{R}}\mathbb{P}(V)$.

Topology of $\hat{V}$
Topological Structures

- Complex case $V = \mathbb{C}^n$.

$$\hat{\mathbb{C}}^n = \{0\} \cup \left((0,\infty) \times \mathbb{CP}^{n-1}\right)$$

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a real analytic manifold of real dimension $2n-1$.

- Real case $V = \mathbb{R}^n$.

$$\hat{\mathbb{R}}^n = \{0\} \cup \left((0,\infty) \times \mathbb{RP}^{n-1} \right)$$

with

$$\mathring{\hat{\mathbb{R}}}^n = \hat{\mathbb{R}}^n \setminus \{0\} = (0,\infty) \times \mathbb{RP}^{n-1}$$

a real analytic manifold of real dimension n.

Topology of $\hat{V}$

Topological Structures

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Specifically let

$$\mathcal{S}^{p,q}(V) = \{T \in Sym(V)\ ,\ \ T \text{ has at most } p \text{ pos.eigs. and } q \text{ neg.eigs}\}$$

Then:

$$\kappa_\beta : \hat{V} \to \mathcal{S}^{1,0}\ \ ,\ \ \hat{x} \mapsto= xx^*\ \ ,\ \ \text{is an embedding.}$$

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$Sym(H)$ is a real Hilbert space with scalar product $\langle T, S\rangle_{HS} = trace\{TS\}$.
$\hat{V}$ is isomorphic (one-to-one and onto) to $\mathcal{S}^{1,0}(V)$.
Key Identity:

$$\beta(x)_k = |\langle x, f_k\rangle|^2 = \langle \kappa_\beta(\hat{x}), F_k\rangle_{HS}$$

where $F_k = f_k f_k^*$.

Metric Space Structures

The matrix-norm induced metric and the natural metric structures

Fix $1 \leq p \leq \infty$. The *matrix-norm induced distance*

$$d_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ \ d_p(\hat{x}, \hat{y}) = \|xx^* - yy^*\|_p$$

with the p-norm of the singular values. In the case $p = 2$ we obtain

$$d_2(x, y) = \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y\rangle|^2}$$

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Fix $1 \leq p \leq \infty$. The *natural metric*

$$D_p : \hat{H} \times \hat{H} \to \mathbb{R} \ , \ D_p(\hat{x}, \hat{y}) = \min_{\varphi} \|x - e^{i\varphi} y\|_p$$

with the usual p-norm on $\mathbb{C}^n$. In the case $p = 2$ we obtain

$$D_2(\hat{x}, \hat{y}) = \sqrt{\|x\|^2 + \|y\|^2 - 2|\langle x, y\rangle|}$$

Metric Space Structures
Distinct Structures

Two different structures: topologically equivalent, BUT the metrics are NOT equivalent:

Lemma (BZ15)

The identity map $i : (\hat{H}, D_p) \to (\hat{H}, d_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Likewise, the identity map $i : (\hat{H}, d_p) \to (\hat{H}, D_p)$, $i(x) = x$ is continuous but it is not Lipschitz continuous. Hence the induced topologies on $(\hat{H}, D_p)$ and $(\hat{H}, d_p)$ are the same, but the corresponding metrics are not Lipschitz equivalent.

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Classes $\mathcal{S}^{p,q}$

General properties; Witt's decomposition

The following lemma summarizes basic properties of $\mathcal{S}^{p,q}$.

Lemma (Bal13)

1. *For any* $p_1 \leq p_2$ *and* $q_1 \leq q_2$, $\mathcal{S}^{p_1,q_1} \subset \mathcal{S}^{p_2,q_2}$;
2. *For any nonnegative integers* p, q *the following disjoint decomposition holds true*
$$\mathcal{S}^{p,q} = \cup_{r=0}^{p} \cup_{s=0}^{q} \mathring{\mathcal{S}}^{r,s} \tag{3.1}$$
where by convention $\mathring{\mathcal{S}}^{p,q} = \emptyset$ *for* $p + q > n$.
3. *For any* $p, q \geq 0$,
$$-\mathcal{S}^{p,q} = \mathcal{S}^{q,p} \tag{3.2}$$
4. *For any linear operator* $T : H \to H$ *(symmetric or not, invertible or not) and nonnegative integers* p, q,
$$T^* \mathcal{S}^{p,q} T \subset \mathcal{S}^{p,q} \tag{3.3}$$

Classes $\mathcal{S}^{p,q}$

General properties; Witt's decomposition

Lemma (cont'd)

5. *(Witt's decomposition) For any nonnegative integers p, q, r, s,*

$$\mathcal{S}^{p,q} + \mathcal{S}^{r,s} = \mathcal{S}^{p,q} - \mathcal{S}^{s,r} = \mathcal{S}^{p+r,q+s} \tag{3.4}$$

$$\mathring{\mathcal{S}}^{p,q} = \{T \in \mathcal{S}^{p,q} \text{ have exactly } p \text{ positive eigs and } q \text{ negative eigs}\}$$

Classes $\mathcal{S}^{p,q}$

Class $\mathcal{S}^{1,0}$

Lemma (Space $\mathcal{S}^{1,0}$)

The following hold true:

1. $\mathring{\mathcal{S}}^{1,0} = \{xx^* \ , \ x \in H, x \neq 0\}$;
2. $\mathcal{S}^{1,0} = \{xx^* \ , \ x \in H\} = \{0\} \cup \{xx^* \ , \ x \in H, x \neq 0\}$;
3. *The set* $\mathring{\mathcal{S}}^{1,0}$ *is a real analytic manifold in* $Sym(n)$ *of real dimension* $2n-1$. *As a real manifold, its tangent space at* $X = xx^*$ *is given by*

$$T_X\mathring{\mathcal{S}}^{1,0} = \left\{ [\![x,y]\!] := \frac{1}{2}(xy^* + yx^*) \ , \ y \in \mathbb{C}^n \right\}. \tag{3.5}$$

The $\mathbb{R}$*-linear embedding* $\mathbb{C}^n \mapsto T_X\mathring{\mathcal{S}}^{1,0}$ *given by* $y \mapsto [\![x,y]\!]$ *has null space* $\{iax \ , \ a \in \mathbb{R}\}$.

Classes $\mathcal{S}^{p,q}$

Class $\mathcal{S}^{1,1}$

Lemma (Space $\mathcal{S}^{1,1}$)

The following hold true:

1. $\mathcal{S}^{1,1} = \mathcal{S}^{1,0} - \mathcal{S}^{1,0} = \mathcal{S}^{1,0} + \mathcal{S}^{0,1} = \{[\![x,y]\!]\ ,\ x,y \in H\}$;
2. *For any vectors* $x,y,u,v \in H$,

$$xx^* - yy^* = [\![x+y, x-y]\!] = [\![x-y, x+y]\!] \quad (3.6)$$

$$[\![u,v]\!] = \frac{1}{4}(u+v)(u+v)^* - \frac{1}{4}(u-v)(u-v)^* \quad (3.7)$$

Additionally, for any $T \in \mathcal{S}^{1,1}$ *let* $T = a_1\ e_1e_1^* - a_2e_2e_2^*$ *be its spectral factorization with* $a_1, a_2 \geq 0$ *and* $\langle e_i, e_j\rangle = \delta_{i,j}$*. Then*

$$T = [\![\sqrt{a_1}e_1 + \sqrt{a_2}e_2, \sqrt{a_1}e_1 - \sqrt{a_2}e_2]\!].$$

Classes $\mathcal{S}^{p,q}$

Class $\mathcal{S}^{1,1}$

Lemma (Space $\mathcal{S}^{1,1}$-cont'd)

3. *The set* $\mathring{\mathcal{S}}^{1,1}$ *is a real analytic manifold in* $Sym(n)$ *of real dimension* $4n-4$*. Its tangent space at* $X = [\![x, y]\!]$ *is given by*

$$T_X\mathring{\mathcal{S}}^{1,1} = \{[\![x,u]\!] + [\![y,v]\!] = \frac{1}{2}(xu^* + ux^* + yv^* + vy^*)\ ,\ u,v \in \mathbb{C}^n\}.$$

The $\mathbb{R}$*-linear embedding* $\mathbb{C}^n \times \mathbb{C}^n \mapsto T_X\mathring{\mathcal{S}}^{1,1}$ *given by* $(u,v) \mapsto [\![x,u]\!] + [\![y,v]\!]$ *has null space* $\{a(ix,0) + b(0,iy) + c(y,-x) + d(iy,ix)\ ,\ a,b,c,d \in \mathbb{R}\}$.

Classes $\mathcal{S}^{p,q}$

Class $\mathcal{S}^{1,1}$

Lemma (Space $\mathcal{S}^{1,1}$-cont'd)

4. *Let $T = [\![u,v]\!] \in \mathcal{S}^{1,1}$. Then its eigenvalues and p-norms are:*

$$
\begin{aligned}
a_+ &= \frac{1}{2}\left(real(\langle u,v\rangle) + \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2} \right) \geq 0 \\
a_- &= \frac{1}{2}\left(real(\langle u,v\rangle) - \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2} \right) \leq 0 \\
\|T\|_1 &= \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2} \\
\|T\|_2 &= \sqrt{\frac{1}{2}\left(\|u\|^2\|v\|^2 + (real(\langle u,v\rangle))^2 - (imag(\langle u,v\rangle))^2 \right)} \\
\|T\|_\infty &= \frac{1}{2}\left(|real(\langle u,v\rangle)| + \sqrt{\|u\|^2\|v\|^2 - (imag(\langle u,v\rangle))^2} \right)
\end{aligned}
$$

Classes $\mathcal{S}^{p,q}$

Class $\mathcal{S}^{1,1}$

Lemma (Space $\mathcal{S}^{1,1}$-cont'd)

5. *Let $T = xx^* - yy^* \in \mathcal{S}^{1,1}$. Then its eigenvalues and p-norms are:*

$$
\begin{aligned}
a_+ &= \frac{1}{2}\left(\|x\|^2 - \|y\|^2 + \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x, y\rangle|^2}\right) \geq 0 \\
a_- &= \frac{1}{2}\left(\|x\|^2 - \|y\|^2 - \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x, y\rangle|^2}\right) \leq 0 \\
\|T\|_1 &= \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x, y\rangle|^2} \\
\|T\|_2 &= \sqrt{\|x\|^4 + \|y\|^4 - 2|\langle x, y\rangle|^2} \\
\|T\|_\infty &= \frac{1}{2}\left(|\|x\|^2 - \|y\|^2| + \sqrt{(\|x\|^2 + \|y\|^2)^2 - 4|\langle x, y\rangle|^2}\right)
\end{aligned}
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Realification

Realification of H

First we describe the realification of H and V. Consider the $\mathbb{R}$-linear map $\mathbf{j} : \mathbb{C}^n \to \mathbb{R}^{2n}$ defined by

$$\mathbf{j}(x) = \begin{bmatrix} real(x) \\ imag(x) \end{bmatrix}$$

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Let J denote the folowing orthogonal antisymmetric $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \quad (4.8)$$

where I_n denotes the identity matrix of order $n \times n$. Note the transpose $J^T = -J$, the square $J^2 = -I_{2n}$ and the inverse $J^{-1} = -J$.

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Note: $\mathbf{j}(ix) = J\mathbf{j}(x)$ for every $x \in H$.

Realification

Realification of frame vectors

Each vector f_k of the frame set $\mathcal{F} = \{f_1, \cdots, f_m\}$ gets mapped into a vector in $\mathbb{R}^{2n}$ denoted by φ_k, and a symmetric operator in $\mathcal{S}^{2,0}(\mathbb{R}^{2n})$ denoted by Φ_k:

$$\varphi_k = \mathbf{j}(f_k) = \left[\begin{array}{c} real(f_k) \\ imag(f_k) \end{array} \right] \quad , \quad \Phi_k = \varphi_k \varphi_k^T + J \varphi_k \varphi_k^T J^T \tag{4.9}$$

Note that when $f_k \neq 0$:

- The symmetric form Φ_k has rank 2 and belongs to $\mathring{\mathcal{S}}^{2,0}$.
- Its spectrum has two distinct eigenvalues: $\|\varphi_k\|^2 = \|f_k\|^2$ with multiplicity 2, and 0 with multiplicity $2n - 2$.
- Furthermore, $\frac{1}{\|\varphi_k\|^2}\Phi_k$ is a rank 2 projection.

Realification

Relationships

Let $\xi = \mathbf{j}(x)$ and $\eta = \mathbf{j}(y)$ denote the realifications of vectors $x, y \in \mathbb{C}^n$. Then a bit of algebra shows that

$$
\begin{aligned}
\langle x, f_k \rangle &= \langle \xi, \varphi_k \rangle + i \langle \xi, J\varphi_k \rangle \\
\langle F_k, xx^* \rangle_{HS} = trace\left(F_k xx^*\right) = |\langle x, f_k \rangle|^2 &= \langle \Phi_k \xi, \xi \rangle = trace\left(\Phi \xi \xi^T\right) \\
&= \langle \Phi_k, \xi\xi^T \rangle_{HS} \\
\langle F_k, [\![x, y]\!] \rangle_{HS} = trace\left(F_k [\![x, y]\!]\right) &= real(\langle x, f_k \rangle \langle f_k, y \rangle) = \langle \Phi_k \xi, \eta \rangle \\
&= (trace(\Phi_k [\![\xi, \eta]\!]) = \langle \Phi_k, [\![\xi, \eta]\!] \rangle_{H}
\end{aligned}
$$

where $F_k = [\![f_k, f_k]\!] = f_k f_k^* \in \mathcal{S}^{1,0}(H)$.

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Injectivity Results
Notations

The following objects play an important role in subsequent theory:

$$R : \mathbb{C}^n \to Sym(\mathbb{C}^n) \quad , \quad R(x) = \sum_{k=1}^{m} |\langle x, f_k \rangle|^2 f_k f_k^* \ , \ x \in \mathbb{C}^n \tag{5.10}$$

$$\mathcal{R} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{R}(\xi) = \sum_{k=1}^{m} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n} \tag{5.11}$$

$$\mathcal{S} : \mathbb{R}^{2n} \to Sym(\mathbb{R}^{2n}) \quad , \quad \mathcal{S}(\xi) = \sum_{k:\Phi_k \xi \neq 0} \frac{1}{\langle \Phi_k \xi, \xi \rangle} \Phi_k \xi \xi^T \Phi_k \ , \ \xi \in \mathbb{R}^{2n} \tag{5.12}$$

$$\mathcal{Z} : \mathbb{R}^{2n} \to \mathbb{R}^{2n \times m} \quad , \quad \mathcal{Z}(\xi) = \left[\begin{array}{c|c|c} \Phi_1 \xi & \cdots & \Phi_m \xi \end{array} \right] \ , \ \xi \in \mathbb{R}^{2n} \tag{5.13}$$

Note $\mathcal{R} = \mathcal{Z}\mathcal{Z}^T$.

Injectivity Results
Induced Linear operator

Recall the key identity:

$$|\langle x, f_k\rangle|^2 = trace\,(F_k X) = \langle F_k, X\rangle_{HS}$$

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Thus the nonlinear map β induces a linear map on the real vector space $Sym(\mathbb{C}^n)$ of symmetric forms over $\mathbb{C}^n$:

$$\mathbb{A} : Sym(\mathbb{C}^n) \to \mathbb{R}^m \quad , \quad \mathbb{A}(T) = (\langle T, F_k\rangle_{HS})_{1\leq k\leq m} = (\langle Tf_k, f_k\rangle)_{1\leq k\leq m}$$

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Similarly it induces a linear map on $Sym(\mathbb{R}^{2n})$ the space of symmetric forms over $\mathbb{R}^{2n} = \mathbf{j}(\mathbb{C}^n)$ that is denoted by $\mathcal{A}$:

$$\begin{aligned} \mathcal{A} : Sym(\mathbb{R}^{2n}) \to \mathbb{R}^m \quad , \quad \mathcal{A}(T) &= (\langle T, \Phi_k \rangle_{HS})_{1 \le k \le m} \\ &= (\langle T\varphi_k, \varphi_k \rangle + \langle TJ\varphi_k, J\varphi_k \rangle)_{1 \le k \le m} \end{aligned}$$

Injectivity Results

General Form

Necessary and sufficient condition for injectivity that works in both the real and the complex case:

Theorem (HMW11,BCMN13a,Bal13a)

Let $H = \mathbb{C}^n$ and let V be a real vector space that is also a subset of H, $V \subset H$. Denote $\mathcal{V} = \mathbf{j}(V)$ the realification of V. Assume $\mathcal{F}$ is a frame for V. The following are equivalent:

1. *The frame $\mathcal{F}$ is phase retrievable with respect to V;*
2. $\ker \mathbb{A} \cap (\mathcal{S}^{1,0}(V) - \mathcal{S}^{1,0}(V)) = \{0\}$;
3. $\ker \mathbb{A} \cap \mathcal{S}^{1,1}(V) = \{0\}$;
4. $\ker \mathbb{A} \cap (\mathcal{S}^{2,0}(V) \cup \mathcal{S}^{1,1}(V) \cup \mathcal{S}^{0,2}) = \{0\}$;
5. *There do not exist vectors $u, v \in V$ with $[\![u, v]\!] \neq 0$ so that*

$$real\left(\langle u, f_k\rangle\langle f_k, v\rangle\right) = 0 \quad , \quad \forall 1 \leq k \leq m$$

Injectivity Results

General Form - cont'd

Theorem (cont'd)

6. $\ker \mathcal{A} \cap (\mathcal{S}^{1,0}(\mathcal{V}) - \mathcal{S}^{1,0}(\mathcal{V})) = \{0\}$;
7. $\ker \mathcal{A} \cap \mathcal{S}^{1,1}(\mathcal{V}) = \{0\}$;
8. *There do not exist vectors $\xi, \eta \in \mathcal{V}$, with $[\![\xi, \eta]\!] \neq 0$ so that*

$$\langle \Phi_k \xi, \eta \rangle = 0 \quad , \quad \forall 1 \leq k \leq m$$

Injectivity Results

Real Case

Theorem (BCE06,Bal12a)

(The real case) Assume $\mathcal{F} \subset \mathbb{R}^n$. The following are equivalent:

1. *$\mathcal{F}$ is phase retrievable for $V = \mathbb{R}^n$;*
2. *$R(x) = \sum_{k=1}^{m} |\langle x, f_k\rangle|^2 f_k f_k^T$ is invertible for every $x \in \mathbb{R}^n$, $x \neq 0$;*
3. *There do not exist vectors $u, v \in \mathbb{R}^n$ with $u \neq 0$ and $v \neq 0$ so that*
$$\langle u, f_k\rangle\langle f_k, v\rangle = 0 \quad , \quad \forall 1 \leq k \leq m$$
4. *For any disjoint partition of the frame set $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, either $\mathcal{F}_1$ spans $\mathbb{R}^n$ or $\mathcal{F}_2$ spans $\mathbb{R}^n$.*

Injectivity Results

Real Case-cont'd

Recall a set $\mathcal{F} \subset \mathbb{C}^n$ is called *full spark* if any subset of n vectors is linearly independent.

Corollary (BCE06)

Assume $\mathcal{F} \subset \mathbb{R}^n$. Then

1. *If $\mathcal{F}$ is phase retrievable for $\mathbb{R}^n$ then $m \geq 2n-1$;*
2. *If $m = 2n-1$, then $\mathcal{F}$ is phase retrievable if and only if $\mathcal{F}$ is full spark;*

Injectivity Results
Complex Case

Theorem (BCMN13a,Bal13a)

(The complex case) The following are equivalent:

1. $\mathcal{F}$ *is phase retrievable for* $H = \mathbb{C}^n$;
2. $\mathrm{rank}(\mathcal{Z}(\xi)) = 2n - 1$ *for all* $\xi \in \mathbb{R}^{2n}$, $\xi \neq 0$;
3. $\dim \ker \mathcal{R}(\xi) = 1$ *for all* $\xi \in \mathbb{R}^{2n}$, $\xi \neq 0$;
4. *There do not exist* $\xi, \eta \in \mathbb{R}^{2n}$, $\xi \neq 0$ *and* $\eta \neq 0$ *so that* $\langle J\xi, \eta \rangle = 0$ *and*

$$\langle \Phi_k \xi, \eta \rangle = 0 \quad , \quad \forall 1 \leq k \leq m$$

Injectivity Results
Cardinality

In terms of cardinality, here is what we know:

Theorem (Mil67,HMW11,BH13,Bal13b,MV13,CEHV13,KE14,Viz15)

MW11 *If $\mathcal{F}$ is a phase retrievable frame for $\mathbb{C}^n$ then*

$$m \geq 4n - 2 - 2b + \begin{cases} 2 & \text{if } n \text{ odd and } b = 3 \bmod 4 \\ 1 & \text{if } n \text{ odd and } b = 2 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

where $b = b(n)$ denotes the number of 1's in the binary expansion of $n-1$.

BH13 *For any positive integer n there is a frame with $m = 4n - 4$ vectors so that $\mathcal{F}$ is phase retrievable for $\mathbb{C}^n$;*

Injectivity Results
Cardinality-cont'd

Theorem

- EHV13 *If $m \geq 4n-4$ then a (Zariski) generic frame is phase retrievable on $\mathbb{C}^n$;*
- Bal13b *The set of phase retrievable frames is open in $\mathbb{C}^n \times \cdots \times \mathbb{C}^n$. In particular phase retrievable property is stable under small perturbations.*
- EHV13 *If $n = 2^k + 1$ and $m \leq 4m - 5$ then $\mathcal{F}$ cannot be phase retrievable for $\mathbb{C}^n$.*
- Viz15 *For $n = 4$ there is a frame with $m = 11 < 4n - 4 = 12$ vectors that is phase retrievable.*