

Scalable Frames and Convex Geometry

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ABSTRACT. The recently introduced and characterized scalable frames can be considered as those frames which allow for perfect preconditioning in the sense that the frame vectors can be rescaled to yield a tight frame. In this paper we define m -scalability, a refinement of scalability based on the number of non-zero weights used in the rescaling process, and study the connection between this notion and elements from convex geometry. Finally, we provide results on the topology of scalable frames. In particular, we prove that the set of scalable frames with “small” redundancy is nowhere dense in the set of frames.

1. Introduction

Frame theory is nowadays a standard methodology in applied mathematics and engineering. The key advantage of frames over orthonormal bases is the fact that frames are allowed to be redundant, yet provide stable decompositions. This is a crucial fact, for instance, for applications which require robustness against noise or erasures, or which require a sparse decomposition (cf. [3]).

Tight frames provide optimal stability, since these systems satisfy the Parseval equality up to a constant. Formulated in the language of numerical linear algebra, a tight frame is perfectly conditioned, since the condition number of its analysis operator is one. Thus, one key question is the following: Given a frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, $M \geq N$, say, can the frame vectors φ_k be modified so that the resulting system forms a tight frame? Again in numerical linear algebra terms, this question can be regarded as a request for perfect preconditioning [1, 4]. Since a frame is typically designed to accommodate certain requirements of an application, this modification process should be as careful as possible in order not to change the properties of the system too drastically.

One recently considered approach consists in multiplying each frame vector by a scalar/a weight. Notice that this process does not even disturb sparse decomposition properties at all, hence it might be considered ‘minimally invasive’. The formal definition was given in [8] by the authors and E.K. Tuley (see also [9]). In that paper, a frame, for which scalars exist so that the scaled frame forms a tight frame, was coined *scalable frame*. Moreover, in the infinite dimensional situation, various equivalent conditions for scalability were provided, and in the finite dimensional situation, a very intuitive geometric characterization was proven. In fact,

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this characterization showed that a frame is non-scalable, if the frame vectors do not spread ‘too much’ in the space. This seems to indicate that there exist relations to convex geometry.

Scalable frames were then also investigated in the papers [6] and [2]. In [6], the authors analyzed the problem by making use of the properties of so-called diagram vectors [7], whereas [2] gives a detailed insight into the set of weights which can be used for scaling.

The contribution of the present paper is three-fold. First, we refine the definition of scalability by calling a (scalable) frame m -scalable, if at most m non-zero weights can be used for the scaling. Second, we establish a link to convex geometry. More precisely, we prove that this refinement leads to a reformulation of the scalability question in terms of the properties of certain polytopes associated to a nonlinear transformation of the frame vectors. This nonlinear transformation is related but not equivalent to the diagram vectors used in the results obtained in [6]. Using this reformulation, we establish new characterizations of scalable frames using convex geometry, namely convex polytopes. Third, we investigate the topological properties of the set of scalable frames. In particular, we prove that in the set of frames in \mathbb{R}^N with M frame vectors the set of scalable frames is nowhere dense if $M < N(N+1)/2$. We wish to mention, that the results stated and proved in this paper were before announced in [10].

The paper is organized as follows. In Section 2, we introduce the required notions with respect to frames and their (m -)scalability as well as state some basic results. Section 3 is devoted to establishing the link to convex geometry and derive novel characterizations of scalable frames using this theory. Finally, in Section 4, we study the topology of the set of scalable frames.

2. Preliminaries

First of all, let us fix some notation. If X is any set whose elements are indexed by x_j , $j \in J$, and $I \subset J$, we define $X_I := \{x_i : i \in I\}$. Moreover, for the set $\{1, \dots, n\}$, $n \in \mathbb{N}$, we write $[n]$.

A set $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, $M \geq N$ is called a *frame*, if there exist positive constants A and B such that

$$(2.1) \quad A\|x\|^2 \leq \sum_{k=1}^M |\langle x, \varphi_k \rangle|^2 \leq B\|x\|^2$$

holds for all $x \in \mathbb{R}^N$. Constants A and B as in (2.1) are called *frame bounds* of Φ . The frame Φ is called *tight* if $A = B$ is possible in (2.1). In this case we have $A = \frac{1}{N} \sum_{k=1}^M \|\varphi_k\|^2$. A tight frame with $A = B = 1$ in (2.1) is called *Parseval frame*.

We will sometimes identify a frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ with the $N \times M$ matrix whose k th column is the vector φ_k . This matrix is called the *synthesis operator* of the frame. The adjoint Φ^T of Φ is called the *analysis operator*. Using the analysis operator, the relation (2.1) reads

$$A\|x\|^2 \leq \|\Phi^T x\|^2 \leq B\|x\|^2.$$

Hence, a frame Φ is tight if and only if some multiple of Φ^T is an isometry. The set of frames for \mathbb{R}^N with M elements will be denoted by $\mathcal{F}(M, N)$. We say that a frame $\Phi \in \mathcal{F}(M, N)$ is *degenerate* if one of its frame vectors is the zero-vector. If

$\mathcal{X}(M, N)$ is a set of frames in $\mathcal{F}(M, N)$, we denote by $\mathcal{X}^*(M, N)$ the set of the non-degenerate frames in $\mathcal{X}(M, N)$. For example, $\mathcal{F}^*(M, N)$ is the set of non-degenerate frames in $\mathcal{F}(M, N)$. For more details on frames, we refer the reader to [5, 3]

Let us recall the following definition from [8, Definition 2.1].

DEFINITION 2.1. A frame $\Phi = \{\varphi_k\}_{k=1}^M$ for \mathbb{R}^N is called *scalable*, respectively, *strictly scalable*, if there exist nonnegative, respectively, positive, scalars $c_1, \dots, c_M \in \mathbb{R}$ such that $\{c_k \varphi_k\}_{k=1}^M$ is a tight frame for \mathbb{R}^N . The set of scalable, respectively, strictly scalable, frames in $\mathcal{F}(M, N)$ is denoted by $\mathcal{SC}(M, N)$, respectively, $\mathcal{SC}_+(M, N)$.

In order to gain a better understanding of the structure of scalable frames we refine the definition of scalability.

DEFINITION 2.2. Let $M, N, m \in \mathbb{N}$ be given such that $N \leq m \leq M$. A frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}(M, N)$ is said to be *m-scalable*, respectively, *strictly m-scalable*, if there exists a subset $I \subseteq [M]$, $\#I = m$, such that Φ_I is a scalable frame, respectively, a strictly scalable frame for \mathbb{R}^N . We denote the set of *m-scalable* frames, respectively, strictly *m-scalable* frames in $\mathcal{F}(M, N)$ by $\mathcal{SC}(M, N, m)$, respectively, $\mathcal{SC}_+(M, N, m)$.

It is easily seen that for $m \leq m'$ we have that $\mathcal{SC}(M, N, m) \subset \mathcal{SC}(M, N, m')$. Therefore,

$$\mathcal{SC}(M, N) = \mathcal{SC}(M, N, M) = \bigcup_{m=N}^M \mathcal{SC}(M, N, m).$$

In the sequel, if no confusion can arise, we often only write \mathcal{F} , \mathcal{SC} , \mathcal{SC}_+ , $\mathcal{SC}(m)$, and $\mathcal{SC}_+(m)$ instead of $\mathcal{SC}(M, N)$, $\mathcal{SC}_+(M, N)$, $\mathcal{SC}(M, N, m)$, and $\mathcal{SC}_+(M, N, m)$, respectively. The notations \mathcal{F}^* , \mathcal{SC}^* , \mathcal{SC}_+^* , $\mathcal{SC}(m)^*$, and $\mathcal{SC}_+(m)^*$ are to be read analogously.

Note that for a frame $\Phi \in \mathcal{F}$ to be *m-scalable* it is necessary that $m \geq N$. In addition, $\Phi \in \mathcal{SC}(M, N)$ holds if and only if $T(\Phi) \in \mathcal{SC}(M, N)$ holds for one (and hence for all) orthogonal transformation(s) T on \mathbb{R}^N ; cf. [8, Corollary 2.6].

If $M \geq N$, we have $\Phi \in \mathcal{SC}(M, N, N)$ if and only if Φ contains an orthogonal basis of \mathbb{R}^N . This completely characterizes the set $\mathcal{SC}(M, N, N)$ of *N-scalable* frames for \mathbb{R}^N consisting of M vectors. For frames with $M = N + 1$ vectors in \mathbb{R}^N we have the following result:

PROPOSITION 2.3. *Let $N \geq 2$ and $\Phi = \{\varphi_k\}_{k=1}^{N+1} \in \mathcal{F}^*$ with $\varphi_k \neq \pm\varphi_\ell$ for $k \neq \ell$. If $\Phi \in \mathcal{SC}_+(N+1, N, N)$ then $\Phi \notin \mathcal{SC}_+(N+1, N)$.*

PROOF. If $\Phi \in \mathcal{SC}_+(N+1, N, N)$, then Φ must contain an orthogonal basis. By applying some orthogonal transformation and rescaling the frame vectors, we can assume without loss of generality that $\{\varphi_k\}_{k=1}^N = \{e_k\}_{k=1}^N$ is the standard orthonormal basis of \mathbb{R}^N , and that $\varphi_{N+1} \neq \pm e_k$ for each $k = 1, 2, \dots, N$, with $\|\varphi_{N+1}\| = 1$. Thus, Φ can be written as $\Phi = [\text{Id}_N \quad \varphi_{N+1}]$, where Id_N is the $N \times N$ identity matrix.

Assume that there exists $\{\lambda_k\}_{k=1}^{N+1} \subset (0, \infty)$ such that $\tilde{\Phi} = \{\lambda_k \varphi_k\}_{k=1}^{N+1}$ is a tight frame, i.e. $\tilde{\Phi} \tilde{\Phi}^T = A \text{Id}_N$. Using a block multiplication this equation can be rewritten as

$$\Lambda + \lambda_{N+1}^2 \varphi_{N+1} \varphi_{N+1}^T = A \text{Id}_N$$

where $\Lambda = \text{diag}(\lambda_k^2)$ is the $N \times N$ diagonal matrix with λ_k^2 , $k = 1, \dots, N$, on its diagonal. Consequently,

$$\begin{aligned} \lambda_k^2 + \lambda_{N+1}^2 \varphi_{N+1,k}^2 &= A \quad \text{for } k = 1, \dots, N \quad \text{and} \\ \lambda_{N+1}^2 \varphi_{N+1,\ell} \varphi_{N+1,k} &= 0 \quad \text{for } k \neq \ell. \end{aligned}$$

But $\lambda_{N+1} > 0$ and so all but one entry in φ_{N+1} vanish. Since φ_{N+1} is a unit norm vector, we see that $\varphi_{N+1} = \pm e_k$ for some $k \in [N]$ which is contrary to the assumption, so Φ cannot be strictly $(N+1)$ -scalable. \square

3. Scalable Frames and Convex Polytopes

Our characterizations of m -scalable frames will be stated in terms of certain convex polytopes and, more generally, using tools from convex geometry. Therefore, we collect below some key facts and properties needed to state and prove our results. For a detailed treatment of convex geometry we refer to [11, 13, 14].

3.1. Background on Convex Geometry. In this subsection, let E be a real linear space, and let $X = \{x_i\}_{i=1}^M$ be a finite set in E . The *convex hull generated by* X is the compact convex subset of E defined by

$$\text{co}(X) := \left\{ \sum_{i=1}^M \alpha_i x_i : \alpha_i \geq 0, \sum_{i=1}^M \alpha_i = 1 \right\}.$$

The *affine hull generated by* X is defined by

$$\text{aff}(X) := \left\{ \sum_{i=1}^M \alpha_i x_i : \sum_{i=1}^M \alpha_i = 1 \right\}.$$

Hence, we have $\text{co}(X) \subset \text{aff}(X)$. Recall that for fixed $a \in \text{aff}(X)$, the set

$$V(X) := \text{aff}(X) - a = \{y - a : y \in \text{aff}(X)\}$$

is a subspace of E (which is independent of $a \in \text{aff}(X)$) and that one defines

$$\dim X := \dim \text{co}(X) := \dim \text{aff}(X) := \dim V(X).$$

We shall use Carathéodory's Theorem for convex polytopes (see, e.g., [13, Theorem 2.2.12]) in deciding whether a frame is scalable:

THEOREM 3.1 (Carathéodory). *Let $X = \{x_1, \dots, x_k\}$ be a finite subset of E with $d := \dim X$. Then for each $x \in \text{co}(X)$ there exists $I \subset [k]$ with $\#I = d + 1$ such that $x \in \text{co}(X_I)$.*

The *relative interior* of the polytope $\text{co}(X)$ denoted by $ri \text{co}(X)$, is the interior of $\text{co}(X)$ in the topology induced by $\text{aff}(X)$. We have that $ri \text{co}(X) \neq \emptyset$ as long as $\#X \geq 2$; cf. [13, Lemma 3.2.8]. Furthermore,

$$ri \text{co}(X) = \left\{ \sum_{i=1}^M \lambda_i x_i : \lambda_i > 0, \sum_{i=1}^M \lambda_i = 1 \right\},$$

see [14, Theorem 2.3.7]. Moreover, the interior of $\text{co}(X)$ in E is non-empty if and only if $\text{aff}(X) = E$.

The following lemma characterizes $\dim X$ in terms of $\dim \text{span } X$.

LEMMA 3.2. *Let X be a finite set of points in E . Put $m := \dim \text{span } X$. Then $\dim X \in \{m - 1, m\}$. Moreover, the following statements are equivalent:*

- (i) $\dim X = m - 1$.
- (ii) For all linearly independent $X' \subset X$ with $\dim \text{span } X' = m$ we have $X \setminus X' \subset \text{aff}(X')$.
- (iii) For some linearly independent $X' \subset X$ with $\dim \text{span } X' = m$ we have $X \setminus X' \subset \text{aff}(X')$.

PROOF. Let $X = \{x_1, \dots, x_k\}$. First of all, we observe that for a linearly independent set $X' = \{x_{i_1}, \dots, x_{i_m}\}$ as in (ii) or (iii) we have

$$\dim V(X') = \dim \text{span}\{x_{i_l} - x_{i_1} : l = 2, \dots, m\} = m - 1.$$

Therefore, $V(X') \subset V(X) \subset \text{span } X$ implies $m - 1 \leq \dim X \leq m$. Let us now prove the moreover-part of the lemma.

(i) \Rightarrow (ii). Assume that $\dim X = m - 1$ and let $X' = \{x_{i_1}, \dots, x_{i_m}\}$ be a linearly independent set as in (ii). From $\dim V(X) = \dim X = m - 1$ we obtain $V(X) = V(X')$. Therefore, for each $x_j \in X \setminus X'$ there exist $\mu_2, \dots, \mu_m \in \mathbb{R}$ such that

$$x_j - x_{i_1} = \sum_{i=2}^m \mu_i (x_i - x_{i_1}) = \sum_{i=2}^m \mu_i x_i - \left(\sum_{i=2}^m \mu_i \right) x_{i_1}.$$

And this implies

$$x_j = \left(1 - \sum_{i=2}^m \mu_i \right) x_{i_1} + \sum_{i=2}^m \mu_i x_i \in \text{aff}(X').$$

(ii) \Rightarrow (iii). This is obvious.

(iii) \Rightarrow (i). Let $X' = \{x_{i_1}, \dots, x_{i_m}\}$ be a linearly independent set as in (iii). If $x \in X \setminus X'$, then we have $x \in \text{aff}(X')$ by (iii). Consequently, there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ with $\sum_{l=1}^m \lambda_l = 1$ such that $x = \sum_{l=1}^m \lambda_l x_{i_l}$. Hence, we obtain

$$x - x_{i_1} = \sum_{l=1}^m \lambda_l x_{i_l} - \left(\sum_{l=1}^m \lambda_l \right) x_{i_1} = \sum_{l=1}^m \lambda_l (x_{i_l} - x_{i_1}) \in V(X').$$

This implies $V(X) = V(X')$ and hence (i). \square

In the sequel we will have to deal with a special case of the situation in Lemma 3.2, where X is a set of rank-one orthogonal projections acting on a real or complex Hilbert space \mathcal{H} . In this case, E is the set consisting of the selfadjoint operators in \mathcal{H} which is a real linear space.

COROLLARY 3.3. *Let X be a finite set consisting of rank-one orthogonal projections acting on a Hilbert space \mathcal{H} . Then we have*

$$\dim X = \dim \text{span } X - 1.$$

PROOF. Let $X = \{P_1, \dots, P_k\}$, $m := \dim \text{span } X$, and let $X' \subset X$ be a linearly independent subset of X such that $\dim \text{span } X' = m$. Without loss of generality assume that $X' = \{P_1, \dots, P_m\}$. Let $j \in \{m + 1, \dots, k\}$. Then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that $P_j = \sum_{i=1}^m \lambda_i P_i$. This implies

$$1 = \text{Tr } P_j = \text{Tr} \left(\sum_{i=1}^m \lambda_i P_i \right) = \sum_{i=1}^m \lambda_i \text{Tr}(P_i) = \sum_{i=1}^m \lambda_i,$$

which shows that $P_j \in \text{aff}(X')$. The statement now follows from Lemma 3.2. \square

3.2. Scalability in Terms of Convex Combinations of Rank-One Matrices. Here, and for the rest of this paper, for a frame $\Phi = \{\varphi_i\}_{i=1}^M$ in $\mathcal{F}(M, N)$ we set

$$X_\Phi := \{\varphi_i \varphi_i^T : i \in [M]\}.$$

This is a subset of the space of all real symmetric $N \times N$ -matrices which we shall denote by S_N . We shall also denote the set of positive multiples of the identity by $I_+ := \{\alpha \text{Id}_N : \alpha > 0\}$.

PROPOSITION 3.4. *For a frame $\Phi \in \mathcal{F}(M, N)$ the following statements are equivalent:*

- (i) Φ is scalable, respectively, strictly scalable.
- (ii) $I_+ \cap \text{co}(X_\Phi) \neq \emptyset$, respectively, $I_+ \cap \text{ri co}(X_\Phi) \neq \emptyset$.

PROOF. Assume that the frame $\Phi = \{\varphi_i\}_{i=1}^M$ is scalable. Then there exist non-negative scalars c_1, \dots, c_M such that

$$\sum_{i=1}^M c_i \varphi_i \varphi_i^T = \text{Id}.$$

Put $\alpha := \sum_{i=1}^M c_i$. Then $\alpha > 0$ and with $\lambda_i := \alpha^{-1} c_i$ we have

$$\sum_{i=1}^M \lambda_i \varphi_i \varphi_i^T = \alpha^{-1} \text{Id} \quad \text{and} \quad \sum_{i=1}^M \lambda_i = 1.$$

Hence $\alpha^{-1} \text{Id} \in \text{co}(X_\Phi)$. The converse direction is obvious. \square

As pointed out earlier, for $m \leq m'$ we have $\mathcal{SC}(m) \subset \mathcal{SC}(m')$. Given $\Phi \in \mathcal{SC}(M, N) = \mathcal{SC}(M)$, there exists $m \leq M$ such that $\Phi \in \mathcal{SC}(m)$, e.g., we can always take $m = M$. However, the next result gives a ‘‘canonical’’ integer $m = m_\Phi$ that is in a way ‘‘optimal’’.

PROPOSITION 3.5. *For a frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$, put $m = m_\Phi := \dim \text{span } X_\Phi$. Then the following statements are equivalent:*

- (i) Φ is scalable.
- (ii) Φ is m -scalable.

PROOF. Clearly, (ii) implies (i). Conversely, let $\Phi = \{\varphi_i\}_{i=1}^M$ be scalable. After possibly removing zero vectors from the frame and thereby reducing M (which does not affect the value of m), we may assume that Φ is unit-norm. By Proposition 3.4, there exists $\alpha > 0$ such that $\alpha \text{Id}_N \in \text{co}(X_\Phi)$. Therefore, from Theorem 3.1 it follows that there exists $I \subset [M]$ with $\#I = \dim X_\Phi + 1$ such that $\alpha \text{Id}_N \in \text{co}(X_{\Phi_I})$. Hence, Φ_I is scalable by Proposition 3.4. And since $\dim X_\Phi = \dim \text{span } X_\Phi - 1$ by Corollary 3.3, the claim follows. \square

As $X_\Phi \subset S_N$ and $\dim S_N = N(N+1)/2$, we immediately obtain the following corollary.

COROLLARY 3.6. *For $M \geq N(N+1)/2$ we have*

$$\mathcal{SC}(M, N) = \mathcal{SC}\left(M, N, \frac{N(N+1)}{2}\right).$$

3.3. Convex Polytopes Associated with m -Scalable Frames. Let $\Phi = \{\varphi_k\}_{k=1}^M$ be a frame for \mathbb{R}^N . Then the analysis operator of the scaled frame $\{c_k \varphi_k\}_{k=1}^M$ is given by $C\Phi^T$, where C is the diagonal matrix with the values c_k on its diagonal. Hence, the frame Φ is scalable if and only if

$$(3.1) \quad \Phi C^2 \Phi^T = A \text{Id}_N,$$

where $A > 0$. Similarly, Φ is m -scalable if and only if (3.1) holds with $C = \text{diag}(c)$, where $c \in [0, \infty)^M$ such that $\|c\|_0 \leq m$. Here, we used the so-called “zero-norm” (which is in fact not a norm), defined by

$$\|x\|_0 := \#\{k \in [n] : x_k \neq 0\}, \quad x \in \mathbb{R}^n.$$

Comparing corresponding entries from left- and right-hand sides of (3.1), it is seen that for a frame to be m -scalable it is necessary and sufficient that there exists a vector $u = (c_1^2, c_2^2, \dots, c_M^2)^T$ with $\|u\|_0 \leq m$ which is a solution of the following linear system of $\frac{N(N+1)}{2}$ equations in M unknowns:

$$(3.2) \quad \begin{cases} \sum_{j=1}^M \varphi_j(k)^2 y_j = A & \text{for } k = 1, \dots, N, \\ \sum_{j=1}^M \varphi_j(\ell) \varphi_j(k) y_j = 0 & \text{for } \ell, k = 1, \dots, N, k > \ell. \end{cases}$$

Subtraction of equations in the first system in (3.2) leads to the new *homogeneous* linear system

$$(3.3) \quad \begin{cases} \sum_{j=1}^M (\varphi_j(1)^2 - \varphi_j(k)^2) y_j = 0 & \text{for } k = 2, \dots, N, \\ \sum_{j=1}^M \varphi_j(\ell) \varphi_j(k) y_j = 0 & \text{for } \ell, k = 1, \dots, N, k > \ell. \end{cases}$$

It is not hard to see that we have not lost information in the last step, hence Φ is m -scalable if and only if there exists a nonnegative vector $u \in \mathbb{R}^M$ with $\|u\|_0 \leq m$ which is a solution to (3.3). In matrix form, (3.3) reads

$$F(\Phi)u = 0,$$

where the $(N-1)(N+2)/2 \times M$ matrix $F(\Phi)$ is given by

$$F(\Phi) = (F(\varphi_1) \quad F(\varphi_2) \quad \dots \quad F(\varphi_M)),$$

where $F : \mathbb{R}^N \rightarrow \mathbb{R}^d$, $d := (N-1)(N+2)/2$, is defined by

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{N-1}(x) \end{pmatrix}, \quad F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \quad F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix},$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \dots, N-1$.

Summarizing, we have just proved the following proposition.

PROPOSITION 3.7. *A frame Φ for \mathbb{R}^N is m -scalable, respectively, strictly m -scalable if and only if there exists a nonnegative $u \in \ker F(\Phi) \setminus \{0\}$ with $\|u\|_0 \leq m$, respectively, $\|u\|_0 = m$.*

We will now utilize the above reformulation to characterize m -scalable frames in terms of the properties of convex polytopes of the type $\text{co}(F(\Phi_I))$, $I \subset [M]$. One of the key tools will be Farkas' lemma (see, e.g., [11, Lemma 1.2.5]).

LEMMA 3.8 (Farkas' Lemma). *For every real $N \times M$ -matrix A exactly one of the following cases occurs:*

- (i) *The system of linear equations $Ax = 0$ has a nontrivial nonnegative solution $x \in \mathbb{R}^M$ (i.e., all components of x are nonnegative and at least one of them is strictly positive.)*
- (ii) *There exists $y \in \mathbb{R}^N$ such that $A^T y$ is a vector with all entries strictly positive.*

In our first main result we use the notation $\text{co}(A)$ for a matrix A which we simply define as the convex hull of the set of column vectors of A .

THEOREM 3.9. *Let $M \geq m \geq N \geq 2$, and let $\Phi = \{\varphi_k\}_{k=1}^M$ be a frame for \mathbb{R}^N . Then the following statements are equivalent:*

- (i) *Φ is m -scalable, respectively, strictly m -scalable,*
- (ii) *There exists a subset $I \subset [M]$ with $\#I = m$ such that $0 \in \text{co}(F(\Phi_I))$, respectively, $0 \in \text{ri co}(F(\Phi_I))$.*
- (iii) *There exists a subset $I \subset [M]$ with $\#I = m$ for which there is no $h \in \mathbb{R}^d$ with $\langle F(\varphi_k), h \rangle > 0$ for all $k \in I$, respectively, with $\langle F(\varphi_k), h \rangle \geq 0$ for all $k \in I$, with at least one of the inequalities being strict.*

PROOF. (i) \Leftrightarrow (ii). This equivalence follows directly if we can show the following equivalences for $\Psi \subset \Phi$:

$$(3.4) \quad \begin{aligned} 0 \in \text{co}(F(\Psi)) &\iff \ker F(\Psi) \setminus \{0\} \text{ contains a nonnegative vector and} \\ 0 \in \text{ri co}(F(\Psi)) &\iff \ker F(\Psi) \text{ contains a positive vector.} \end{aligned}$$

The implication " \Rightarrow " is trivial in both cases. For the implication " \Leftarrow " in the first case let $I \subset [M]$ be such that $\Psi = \Phi_I$, $I = \{i_1, \dots, i_m\}$, and let $u = (c_1, \dots, c_m)^T \in \ker F(\Psi)$ be a non-zero nonnegative vector. Then $A := \sum_{k=1}^m c_k > 0$ and with $\lambda_k := c_k/A$, $k \in [m]$, we have $\sum_{k=1}^m \lambda_k = 1$ and $\sum_{k=1}^m \lambda_k F(\varphi_{i_k}) = A^{-1}F(\Psi)u = 0$. Hence $0 \in \text{co}(F(\Psi))$. The proof for the second case is similar.

(ii) \Leftrightarrow (iii). In the non-strict case this follows from (3.4) and Lemma 3.8. In the strict case this is a known fact; e.g., see [14, Lemma 3.6.5]. \square

We now derive a few consequences of the above theorem. A given vector $v \in \mathbb{R}^d$ defines a hyperplane by

$$H(v) = \{y \in \mathbb{R}^d : \langle v, y \rangle = 0\},$$

which itself determines two open convex cones $H^-(v)$ and $H^+(v)$, defined by

$$H^-(v) = \{y \in \mathbb{R}^d : \langle v, y \rangle < 0\} \quad \text{and} \quad H^+(v) = \{y \in \mathbb{R}^d : \langle v, y \rangle > 0\}.$$

Using these notations we can restate the equivalence (i) \Leftrightarrow (iii) in Theorem 3.9 as follows:

PROPOSITION 3.10. *Let $M \geq N \geq 2$, and let m be such that $N \leq m \leq M$. Then a frame $\Phi = \{\varphi_k\}_{k=1}^M$ for \mathbb{R}^N is m -scalable if and only if there exists a subset $I \subset [M]$ with $\#I = m$ such that $\bigcap_{i \in I} H^+(F(\varphi_i)) = \emptyset$.*

REMARK 3.11. In the case of strict m -scalability we have the following necessary condition: If Φ is strictly m -scalable, then there exists a subset $I \subset [M]$ with $\#I = m$ such that $\bigcap_{i \in I} H^-(F(\varphi_i)) = \emptyset$.

REMARK 3.12. When $M \geq d + 1 = N(N + 1)/2$, we can use properties of the convex sets $H^\pm(F(\varphi_k))$ to give an alternative proof of Corollary 3.6. For this, let the frame $\Phi = \{\varphi_k\}_{k=1}^M$ for \mathbb{R}^N be scalable. Then, by Proposition 3.10 we have that $\bigcap_{k=1}^M H^+(F(\varphi_k)) = \emptyset$. Now, Helly's theorem (see, e.g., [11, Theorem 1.3.2]) implies that there exists $I \subset [M]$ with $\#I = d + 1$ such that $\bigcap_{i \in I} H^+(F(\varphi_i)) = \emptyset$. Exploiting Proposition 3.10 again, we conclude that Φ is $(d + 1)$ -scalable.

The following result is an application of Proposition 3.10 which provides a simple condition for $\Phi \notin \mathcal{SC}(M, N)$.

PROPOSITION 3.13. *Let $\Phi = \{\varphi_k\}_{k=1}^M$ be a frame for \mathbb{R}^N , $N \geq 2$. If there exists an isometry T such that $T(\Phi) \subset \mathbb{R}^{N-2} \times \mathbb{R}_+^2$, then Φ is not scalable. In particular, Φ is not scalable if there exist $i, j \in [N]$, $i \neq j$, such that $\varphi_k(i)\varphi_k(j) > 0$ for all $k \in [M]$.*

PROOF. Without loss of generality, we may assume that $\Phi \subset \mathbb{R}^{N-2} \times \mathbb{R}_+^2$, cf. [8, Corollary 2.6]. Let $\{e_k\}_{k=1}^d$ be the standard ONB for \mathbb{R}^d . Then for each $k \in [M]$ we have that

$$\langle e_d, F(\varphi_k) \rangle = \varphi_k(N-1)\varphi_k(N) > 0.$$

Hence, $e_d \in \bigcap_{i \in [M]} H^+(F(\varphi_i))$. By Proposition 3.10, Φ is not scalable. \square

The characterizations stated above can be recast in terms of the convex cone $C(F(\Phi))$ generated by $F(\Phi)$. We state this result for the sake of completeness. But first, recall that for a finite subset $X = \{x_1, \dots, x_M\}$ of \mathbb{R}^d the *polyhedral cone generated by X* is the closed convex cone $C(X)$ defined by

$$C(X) = \left\{ \sum_{i=1}^M \alpha_i x_i : \alpha_i \geq 0 \right\}.$$

Let C be a cone in \mathbb{R}^d . The *polar cone* of C is the closed convex cone C° defined by

$$C^\circ := \{x \in \mathbb{R}^N : \langle x, y \rangle \leq 0 \text{ for all } y \in C\}.$$

The cone C is said to be *pointed* if $C \cap (-C) = \{0\}$, and *blunt* if the linear space generated by C is \mathbb{R}^N , i.e. $\text{span } C = \mathbb{R}^N$.

COROLLARY 3.14. *Let $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*$, and let $N \leq m \leq M$ be fixed. Then the following conditions are equivalent:*

- (i) Φ is strictly m -scalable .
- (ii) There exists $I \subset [M]$ with $\#I = m$ such that $C(F(\Phi_I))$ is not pointed.
- (iii) There exists $I \subset [M]$ with $\#I = m$ such that $C(F(\Phi_I))^\circ$ is not blunt.
- (iv) There exists $I \subset [M]$ with $\#I = m$ such that the interior of $C(F(\Phi_I))^\circ$ is empty.

PROOF. (i) \Leftrightarrow (ii). By Proposition 3.7, Φ is strictly m -scalable if and only if there exist $I \subset [M]$ with $\#I = m$ and a nonnegative $u \in \ker F(\Phi_I) \setminus \{0\}$ with $\|u\|_0 = m$. By [13, Lemma 2.10.9], this is equivalent to the cone $C(F(\Phi_I))$ being not pointed. This proves that (i) is equivalent to (ii).

(ii) \Leftrightarrow (iii). This follows from the fact that the polar of a pointed cone C is blunt and vice versa, as long as $C^{\circ\circ} = C$, see [13, Theorem 2.10.7]. But in our case $C(F(\Phi_I))^{\circ\circ} = C(F(\Phi_I))$, see [13, Lemma 2.7.9].

(iii) \Rightarrow (iv). If $C(F(\Phi_I))^{\circ}$ is not blunt, then it is contained in a proper hyperplane of \mathbb{R}^d whose interior is empty. Hence, also the interior of $C(F(\Phi_I))^{\circ}$ must be empty.

(iv) \Rightarrow (iii). We use a contra positive argument. Assume that $C(F(\Phi))^{\circ}$ is blunt. This is equivalent to $\text{span } C(F(\Phi))^{\circ} = \mathbb{R}^d$. But for the nonempty cone $C(F(\Phi))^{\circ}$ we have $\text{aff}(C(F(\Phi))^{\circ}) = \text{span } C(F(\Phi))^{\circ}$. Hence, $\text{aff}(C(F(\Phi))^{\circ}) = \mathbb{R}^d$, and so the relative interior of $C(F(\Phi))^{\circ}$ is equal to its interior, which therefore is nonempty. \square

The main idea of the previous results is the characterization of (m -)scalability of Φ in terms of properties of the convex polytopes $\text{co}(F(\Phi_I))$. However, it seems more “natural” to seek assumptions on the convex polytopes $\text{co}(\Phi_I)$ that will ensure that $\text{co}(F(\Phi))$ satisfy the conditions in Theorem 3.9 hold. Proposition 3.13, which gives a condition on Φ that precludes it to be scalable, is a step in this direction.

Nonetheless, we address the related question of whether $F(\Phi)$ is a frame for \mathbb{R}^d whenever Φ is a scalable frame for \mathbb{R}^N . This depends clearly on the redundancy of Φ as well as on the map F . In particular, we finish this section by giving a condition which ensures that $F(\Phi)$ is always a frame for \mathbb{R}^d when $M \geq d + 1$. In order to prove this result, we need a few preliminary facts.

For $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ and $h = (h_k)_{k=1}^d \in \mathbb{R}^d$, we have that

$$(3.5) \quad \langle F(x), h \rangle = \sum_{\ell=2}^N h_{\ell-1}(x_1^2 - x_{\ell}^2) + \sum_{k=1}^{N-1} \sum_{\ell=k+1}^N h_{k(N-1-(k-1)/2)+\ell-1} x_k x_{\ell}.$$

The right hand side of (3.5) is obviously a homogeneous polynomial of degree 2 in x_1, x_2, \dots, x_N . We shall denote the set of all polynomials of this form by \mathbf{P}_2^N . It is easily seen that \mathbf{P}_2^N is isomorphic to the subspace of real symmetric $N \times N$ matrices whose trace is 0. Indeed, for each $N \geq 2$, and each $p \in \mathbf{P}_2^N$,

$$p(x) = \sum_{\ell=2}^N a_{\ell-1}(x_1^2 - x_{\ell}^2) + \sum_{k=1}^{N-1} \sum_{\ell=k+1}^N a_{k(N-(k+1)/2)+\ell-1} x_k x_{\ell},$$

we have $p(x) = \langle Q_p x, x \rangle$, where Q_p is the symmetric $N \times N$ -matrix with entries

$$Q_p(1, 1) = \sum_{k=1}^{N-1} a_k, \quad Q_p(\ell, \ell) = -a_{\ell-1} \quad \text{for } \ell = 2, 3, \dots, N$$

and

$$Q_p(k, \ell) = \frac{1}{2} a_{k(N-(k+1)/2)+\ell-1} \quad \text{for } k = 1, \dots, N-1, \ell = k+1, \dots, N.$$

In particular, the dimension of \mathbf{P}_2^N is $d = (N+2)(N-1)/2$.

PROPOSITION 3.15. *Let $M \geq d + 1$ where $d = (N-1)(N+2)/2$, and $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{SC}_+(d+1) \setminus \mathcal{SC}(d)$. Then $F(\Phi)$ is a frame for \mathbb{R}^d .*

PROOF. Let $I \subset [M]$, $\#I = d + 1$, be an index set such that Φ_I is strictly scalable. Assume that there exists $h \in \mathbb{R}^d$ such that $\langle F(\varphi_k), h \rangle = 0$ for each $k \in I$. By (3.5) we conclude that $p_h(\varphi_k) = 0$ for all $k \in I$, where p_h is the polynomial in

\mathbf{P}_2^N on the right hand side of (3.5). Hence $\langle Q_{p_h} \varphi_k, \varphi_k \rangle = 0$ for all $k \in I$. Now, we have

$$(3.6) \quad \langle \varphi_k \varphi_k^T, Q_{p_h} \rangle_{HS} = \text{Tr}(\varphi_k \varphi_k^T Q_{p_h}) = \langle Q_{p_h} \varphi_k, \varphi_k \rangle = 0 \quad \text{for all } k \in I.$$

But as Φ_I is not d -scalable (otherwise, $\Phi \in \mathcal{SC}(d)$) it is not m -scalable for every $m \leq d$. Thus, Proposition 3.5 yields that

$$\dim \text{span}\{\varphi_k \varphi_k^T : k \in I\} = d + 1.$$

Equivalently, $\{\varphi_k \varphi_k^T : k \in I\}$ is a basis of the $(d + 1)$ -dimensional space S_N . Therefore, from (3.6) we conclude that $Q_{p_h} = 0$ which implies $p_h = 0$ (since $p \mapsto Q_p$ is an isomorphism) and thus $h = 0$.

Now, it follows that $F(\Phi_I)$ spans \mathbb{R}^d which is equivalent to $F(\Phi_I)$ being a frame for \mathbb{R}^d . Hence, so is $F(\Phi)$. \square

4. Topology of the Set of Scalable Frames

In this section, we present some topological features of the set $\mathcal{SC}(M, N)$. Hereby, we identify frames in $\mathcal{F}(M, N)$ with real $N \times M$ -matrices as we already did before, see, e.g., (3.1) in subsection 3.3. Hence, we consider $\mathcal{F}(M, N)$ as the set of all matrices in $\mathbb{R}^{N \times M}$ of rank N . Note that under this identification the order of the vectors in a frame becomes important. However, it allows us to endow $\mathcal{F}(M, N)$ with the usual Euclidean topology of $\mathbb{R}^{N \times M}$.

In [8] it was proved that $\mathcal{SC}(M, N)$ is a closed set in $\mathcal{F}(M, N)$ (in the relative topology of $\mathcal{F}(M, N)$). The next proposition refines this fact.

PROPOSITION 4.1. *Let $M \geq m \geq N \geq 2$. Then $\mathcal{SC}(M, N, m)$ is closed in $\mathcal{F}(M, N)$.*

PROOF. We prove the assertion by establishing that the complement $\mathcal{F} \setminus \mathcal{SC}(m)$ is open, that is, if $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$ is a frame which is not m -scalable, we prove that there exists $\varepsilon > 0$ such that for any collection $\Psi = \{\psi_k\}_{k=1}^M$ of vectors in \mathbb{R}^N for which

$$\|\varphi_k - \psi_k\| < \varepsilon \quad \text{for all } k \in [M],$$

we have that Ψ is a frame which is not m -scalable. Thus assume that $\Phi = \{\varphi_k\}_{k=1}^M$ is a frame which is not m -scalable and define the finite set \mathcal{I} of subsets by

$$\mathcal{I} := \{I \subset [M] : \#I = m\}.$$

By Proposition 3.10, for each $I \in \mathcal{I}$ there exists $y_I \in \bigcap_{k \in I} H^+(F(\varphi_k))$, that is, $\min_{k \in I} \langle y_I, F(\varphi_k) \rangle > 0$. By the continuity of the map F , there exists $\varepsilon > 0$ such that for each $\{\psi_k\}_{k=1}^M \subset \mathbb{R}^N$ with $\|\psi_k - \varphi_k\| < \varepsilon$ for all $k \in [M]$ we still have $\min_{k \in I} \langle y_I, F(\psi_k) \rangle > 0$. We can choose $\varepsilon > 0$ sufficiently small to guarantee that $\Psi = \{\psi_k\}_{k=1}^M \in \mathcal{F}$. Again from Proposition 3.10 we conclude that Ψ is not m -scalable for any $N \leq m \leq M$. Hence, $\mathcal{SC}(m)$ is closed in \mathcal{F} . \square

The next theorem is the main result of this section. It shows that the set of scalable frames is nowhere dense in the set of frames unless the redundancy of the considered frames is disproportionately large.

THEOREM 4.2. *Assume that $2 \leq N \leq M < d + 1 = N(N + 1)/2$. Then $\mathcal{SC}(M, N)$ does not contain interior points. In other words, for the boundary of $\mathcal{SC}(M, N)$ we have*

$$\partial \mathcal{SC}(M, N) = \mathcal{SC}(M, N).$$

For the proof of Theorem 4.2 we shall need two lemmas. Recall that for a frame $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}$ we use the notation

$$X_\Phi = \{\varphi_i \varphi_i^T : i \in [M]\}.$$

LEMMA 4.3. *Let $\{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ be such that $\dim \text{span } X_\Phi < \frac{N(N+1)}{2}$. Then there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0 \varphi_0^T \notin \text{span } X_\Phi$.*

PROOF. Assume the contrary. Then each rank-one orthogonal projection is an element of $\text{span } X_\Phi$. But by the spectral decomposition theorem every symmetric matrix in $\mathbb{R}^{N \times N}$ is a linear combination of such projections. Hence, $\text{span } X_\Phi$ coincides with the linear space S_N of all symmetric matrices in $\mathbb{R}^{N \times N}$. Therefore,

$$\dim \text{span } X_\Phi = \frac{N(N+1)}{2},$$

which is a contradiction. \square

The following lemma shows that for a generic M -element set $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$ (or matrix in $\mathbb{R}^{N \times M}$, if the φ_i are considered as columns) the subspace $\text{span } X_\Phi$ has the largest possible dimension.

LEMMA 4.4. *Let $D := \min\{M, N(N+1)/2\}$. Then the set*

$$\{\Phi \in \mathbb{R}^{N \times M} : \dim \text{span } X_\Phi = D\}$$

is dense in $\mathbb{R}^{N \times M}$.

PROOF. Let $\Phi = \{\varphi_i\}_{i=1}^M \in \mathbb{R}^{N \times M}$ and $\varepsilon > 0$. We will show that there exists $\Psi = \{\psi_i\}_{i=1}^M \in \mathbb{R}^{N \times M}$ with $\|\Phi - \Psi\| < \varepsilon$ and $\dim \text{span } X_\Psi = D$. For this, set $\mathcal{W} := \text{span } X_\Phi$ and let k be the dimension of \mathcal{W} . If $k = D$, nothing is to prove. Hence, let $k < D$. Without loss of generality, assume that $\varphi_1 \varphi_1^T, \dots, \varphi_k \varphi_k^T$ are linearly independent. By Lemma 4.3 there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0 \varphi_0^T \notin \mathcal{W}$. For $\delta > 0$ define the symmetric matrix

$$S_\delta := \delta (\varphi_{k+1} \varphi_0^T + \varphi_0 \varphi_{k+1}^T) + \delta^2 \varphi_0 \varphi_0^T.$$

Then there exists *at most one* $\delta > 0$ such that $S_\delta \in \mathcal{W}$ (regardless of whether $\varphi_{k+1} \varphi_0^T + \varphi_0 \varphi_{k+1}^T$ and $\varphi_0 \varphi_0^T$ are linearly independent or not). Therefore, we find $\delta > 0$ such that $\delta < \varepsilon/M$ and $S_\delta \notin \mathcal{W}$. Now, for $i \in [M]$ put

$$\psi_i := \begin{cases} \varphi_i & \text{if } i \neq k+1 \\ \varphi_{k+1} + \delta \varphi_0 & \text{if } i = k+1 \end{cases}$$

and $\Psi := \{\psi_i\}_{i=1}^M$. Let $\lambda_1, \dots, \lambda_{k+1} \in \mathbb{R}$ such that

$$\sum_{i=1}^{k+1} \lambda_i \psi_i \psi_i^T = 0.$$

Then, since $\psi_{k+1} \psi_{k+1}^T = \varphi_{k+1} \varphi_{k+1}^T + S_\delta$, we have that

$$\lambda_{k+1} S_\delta = - \sum_{i=1}^{k+1} \lambda_i \varphi_i \varphi_i^T \in \mathcal{W},$$

which implies $\lambda_{k+1} = 0$ and therefore also $\lambda_1 = \dots = \lambda_k = 0$. Hence, we have $\dim \text{span } X_\Psi = k+1$ and $\|\Phi - \Psi\| < \varepsilon/M$. If $k = D-1$, we are finished. Otherwise, repeat the above construction at most $D-k-1$ times. \square

REMARK 4.5. For the case $M \geq N(N + 1)/2$, Lemma 4.4 has been proved in [2, Theorem 2.1]. In the proof, the authors note that X_Φ spans S_N if and only if the frame operator of X_Φ (considered as a system in S_N) is invertible. But the determinant of this operator is a polynomial in the entries of φ_i , and the complement of the set of roots of such polynomials is known to be dense.

PROOF OF THEOREM 4.2. Assume the contrary. Then, by Lemma 4.4, there even exists an interior point $\Phi = \{\varphi_i\}_{i=1}^M \in \mathcal{SC}(M, N)$ of $\mathcal{SC}(M, N)$ for which the linear space $\mathcal{W} := \text{span } X_\Phi$ has dimension M . Since Φ is scalable, there exist $c_1, \dots, c_M \geq 0$ such that

$$\sum_{i=1}^M c_i \varphi_i \varphi_i^T = \text{Id}.$$

Without loss of generality we may assume that $c_1 > 0$.

By Lemma 4.3 there exists $\varphi_0 \in \mathbb{R}^N$ with $\|\varphi_0\| = 1$ such that $\varphi_0 \varphi_0^T \notin \mathcal{W}$. As in the proof of Lemma 4.4, we set

$$S_\delta := \delta (\varphi_1 \varphi_0^T + \varphi_0 \varphi_1^T) + \delta^2 \varphi_0 \varphi_0^T.$$

Then, for $\delta > 0$ sufficiently small, $S_\delta \notin \mathcal{W}$ and $\Psi := \{\varphi_1 + \delta \varphi_0, \varphi_2, \dots, \varphi_M\} \in \mathcal{SC}(M, N)$. Hence, there exist $c'_1, \dots, c'_M \geq 0$ such that

$$\sum_{i=1}^M c_i \varphi_i \varphi_i^T = \text{Id} = c'_1 (\varphi_1 + \delta \varphi_0) (\varphi_1 + \delta \varphi_0)^T + \sum_{i=2}^M c'_i \varphi_i \varphi_i^T = \sum_{i=1}^M c'_i \varphi_i \varphi_i^T + c'_1 S_\delta.$$

This implies $c'_1 S_\delta \in \mathcal{W}$, and thus $c'_1 = 0$. But then we have

$$c_1 \varphi_1 \varphi_1^T + \sum_{i=2}^M (c_i - c'_i) \varphi_i \varphi_i^T = 0,$$

which yields $c_1 = 0$ as the matrices $\varphi_1 \varphi_1^T, \dots, \varphi_M \varphi_M^T$ are linearly independent. A contradiction. \square

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