Preconditioning of finite frames

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Outline

Definition

Definition

$\Phi = \{\varphi_k\}_{k=1}^M \subseteq \mathbb{R}^N$ is a *frame* for $\mathbb{R}^N$ if $\exists\, A, B > 0$ such that $\forall x \in \mathbb{R}^N$,

$$A\|x\|^2 \leq \sum_{k=1}^{M} |\langle x, \varphi_k\rangle|^2 \leq B\|x\|^2.$$

If, in addition, $\|\varphi_k\| = 1$ for each k, we say that Φ is a *unit-norm frame*. The set of frames for $\mathbb{R}^N$ with M elements will be denoted by $\mathcal{F}$. In addition, we let $\mathcal{F}_u$ the the subset of unit-norm frames.

Analysis and Synthesis with frame

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$.

1. The *analysis operator*, is defined by

$$\mathbb{R}^N \ni x \mapsto \Phi^T x = \{\langle x, \varphi_k \rangle\}_{k=1}^M \in \mathbb{R}^M.$$

2. The *synthesis operator* is defined by

$$\mathbb{R}^M \ni c = (c_k)_{k=1}^M \mapsto \Phi c = \sum_{k=1}^M c_k \varphi_k \in \mathbb{R}^N.$$

3. The *frame operator* $S = \Phi\Phi^T$ is given by

$$\mathbb{R}^N \ni x \mapsto Sx = \sum_{k=1}^M \langle x, \varphi_k \rangle \varphi_k \in \mathbb{R}^N.$$

4. The *Gramian (operator)* $G = \Phi^T\Phi$ of the frame is the $M \times M$ matrix whose $(i,j)^{th}$ entry is $\langle \varphi_j, \varphi_i \rangle$.

Resolution of the identity with frame

If $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ is a frame,

$$\{\tilde{\varphi}_k\}_{k=1}^M = \{S^{-1}\varphi_k\}_{k=1}^M$$

is *the canonical dual frame*, and, for each $x \in \mathbb{R}^N$, we have

$$x = SS^{-1}(x) = S^{-1}S(x)$$

$$x = \sum_{k=1}^M \langle x, \varphi_k\rangle \tilde{\varphi}_k = \sum_{k=1}^M \langle x, \tilde{\varphi}_k\rangle \varphi_k. \tag{1}$$

Tight frames and FUNTFs

1. A frame Φ is a *tight frame* if we can choose $A = B$.
2. If $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^M$ is a frame then

$$\{\varphi_k^\dagger\}_{k=1}^M = \{S^{-1/2}\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$$

is a tight frame, and for every $x \in \mathbb{R}^N$

$$x = SS^{-1}x = SS^{-1/2}S^{-1/2}x = \sum_{k=1}^M \langle x, \varphi_k^\dagger\rangle \varphi_k^\dagger. \qquad (2)$$

3. If Φ is a tight frame of unit-norm vectors, we say that Φ is a *finite unit-norm tight frame (FUNTF)*. In this case, the reconstruction formula (1) reduces to

$$\forall x \in \mathbb{R}^N, \quad x = \frac{N}{M}\sum_{k=1}^M \langle x, \varphi_k\rangle \varphi_k. \qquad (3)$$

Example of FUNTFs

Example

Let $\omega = e^{2\pi i/M}$

$$\frac{1}{\sqrt{M}}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \dots & \omega^{(M-1)^2} \end{bmatrix}$$

Any (normalized) N rows from the $M \times M$ DFT matrix is a tight frame for $\mathbb{C}^N$.
Every tight frame of M vectors in $\mathbb{K}^N$ is obtained from an orthogonal projection of an ONB in $\mathbb{K}^M$ onto $\mathbb{K}^N$.

Examples of frames

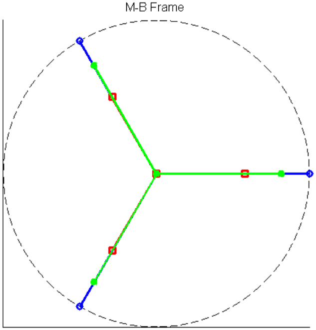

Figure : The MB-Frame

The frame potential

Theorem (Benedetto and Fickus, 2003)

For each $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$, *such that* $\|\varphi_k\| = 1$ *for each* k, *we have*

$$FP(\Phi) = \sum_{j=1}^{M} \sum_{k=1}^{M} |\langle \varphi_j, \varphi_k \rangle|^2 \geq \frac{M}{N} \max(M, N). \tag{4}$$

Furthermore,

- *If* $M \leq N$, *the minimum of FP is* M *and is achieved by orthonormal systems for* $\mathbb{R}^N$ *with* M *elements.*
- *If* $M \geq N$, *the minimum of FP is* $\frac{M^2}{N}$ *and is achieved by FUNTFs.*

$FP(\Phi)$ is the frame potential.

Proof

Proof.

$$FP(\{\varphi_k\}_{k=1}^M) = M + \sum_{k\neq\ell=1}^{M} |\langle\varphi_k,\varphi_\ell\rangle|^2 \geq M.$$

• If $M \leq N$ the minimizers are exactly orthonormal systems and the minimum is M.

• Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Observe that

$$Tr(G^2) = \sum_{k=1}^{M}\langle G^2 e_k, e_k\rangle = \sum_{k=1}^{M}\langle Ge_k, Ge_k\rangle = \sum_{k=1}^{M}\|Ge_k\|^2.$$

But

$$\|Ge_k\|^2 = \sum_{\ell=1}^{M}|G(\ell,k)|^2 = \sum_{\ell=1}^{M}|\langle\varphi_\ell,\varphi_k\rangle|^2.$$

Proof

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• Now assume $M \geq N$ and let $G = \Phi^*\Phi$. Observe that

$$Tr(G^2) = \sum_{k=1}^{M}\langle G^2 e_k, e_k\rangle = \sum_{k=1}^{M}\langle Ge_k, Ge_k\rangle = \sum_{k=1}^{M} \|Ge_k\|^2.$$

But

$$\|Ge_k\|^2 = \sum_{\ell=1}^{M} |G(\ell, k)|^2 = \sum_{\ell=1}^{M} |\langle\varphi_\ell, \varphi_k\rangle|^2.$$

□

Proof (continued)

Proof.

Consequently,

$$FP(\{\varphi_k\}_{k=1}^M) = Tr(G^2) = \sum_{k=1}^N \lambda_k^2$$

and, $trace(G) = \sum_{k=1}^N \lambda_k = M$.
Minimizing $FP(\{\varphi_k\}_{k=1}^M)$ is equivalent to minimizing

$$\sum_{k=1}^N \lambda_k^2 \quad such\, that \quad \sum_{k=1}^N \lambda_k = M.$$

□

Proof (continued)

Proof.

Solution: $\lambda_k = M/N$ for all k.
Hence $S = \frac{M}{N} I_N$ where I_N is the identity matrix. The corresponding minimizers $\{\varphi_k\}_{k=1}^M$ are FUNTFs

$$x = \frac{N}{M} \sum_{k=1}^{M} \langle x, \varphi_k \rangle \varphi_k \quad \forall x \in \mathbb{K}^N.$$

□

Construction of FUNTFs

Fact

- *Numerical schemes such as gradient descent can be used to find minimizers of the frame potential and thus find FUNTFs.*
- *The spectral tetris method was proposed by Casazza, Fickus, Mixon, Wang, and Zhou (2011) to construct all FUNTFs. Further contributions by Krahmer, Kutyniok, Lemvig, (2012); Lemvig, Miller, Okoudjou (2012).*
- *Other methods (algebraic geometry) have been proposed by Cahill, Fickus, Mixon, Strawn.*

Optimally conditioned frames

Remark

1. *FUNTFs can be considered "optimally conditioned" frames. In particular the condition number of the frame operator is* 1.
2. *There are many preconditioning methods to improve the condition number of a matrix, e.g., Matrix Scaling.*
3. *A matrix* A *is (row/column) scalable if there exit diagonal matrices* D_1, D_2 *with positive diagonal entries such that* D_1A, AD_2, *or* D_1AD_2 *have constant row/column sum.*

Main question

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one transform Φ into a tight frame? If yes can this be done algorithmically and can the class of all frames that allow such transformations be described?

Solution

1. *If Φ denotes again the $N \times M$ synthesis matrix, a solution to the above problem is the associated canonical tight frame*

$$\{S^{-1/2}\varphi_k\}_{k=1}^M.$$

   Involves the inverse frame operator.

2. *What "transformations" are allowed?*

Goals of this section

Remark

1. *How to transform a (non) tight frame into a tight one?*
2. *What "transformations" are allowed?*
3. *Give theoretical guarantees and algorithms.*
4. *For a given "transformation", what happens if a frame cannot be transformed exactly?*

Choosing a transformation

Question

Given a (non-tight) frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ can one find nonnegative numbers $\{c_k\}_{k=1}^M \subset [0, \infty)$ such that $\widetilde{\Phi} = \{c_k\varphi_k\}_{k=1}^M$ becomes a tight frame?

Remark

In matrix notation, one seeks a diagonal $(M \times M)$ matrix C with nonnegative entries such that ΦC is a tight frame.
More generally, one can ask when there exist (structured) matrices D such that ΦD is a tight frame.

Definition

Definition

A frame $\Phi = \{\varphi_k\}_{k=1}^M$ in $\mathbb{R}^N$ is *scalable*, if $\exists\, \{c_k\}_{k=1}^M \subset [0, \infty)$ such that $\{c_k\varphi_k\}_{k=1}^M$ is a tight frame for $\mathbb{R}^N$.
The set of scalable frames is denoted by $\mathcal{SC}(M, N)$.
In addition, if $\{c_k\}_{k=1}^M \subset (0, \infty)$, the frame is called *strictly scalable* and the set of strictly scalable frames is denoted by $\mathcal{SC}_+(M, N)$.

A more general definition

Definition

Given, $N \leq m \leq M$, a frame $\Phi = \{\varphi_k\}_{k=1}^M$ is said to be m-*scalable*, respectively, *strictly* $m-$*scalable*, if $\exists\, \Phi_I = \{\varphi_k\}_{k\in I}$ with $I \subseteq \{1, 2, \dots, M\}$, $\#I = m$, such that $\Phi_I = \{\varphi_k\}_{k\in I}$ is scalable, respectively, strictly scalable.
We denote the set of m-scalable frames, respectively, strictly m-scalable frames in $\mathcal{F}(M, N)$ by $\mathcal{SC}(M, N, m)$, respectively, $\mathcal{SC}_+(M, N, m)$.

Some basic examples

Example

1. When $M = N$, a frame $\Phi = \{\varphi_k\}_{k=1}^N \subset \mathbb{R}^N$ is scalable if and only if Φ is an orthogonal set.
2. When $M \geq N$, if Φ contains an orthogonal basis, then it is clearly $N-$scalable.
3. Thus, given $M \geq N$, the set $\mathcal{SC}(M, N, N)$ consists exactly of frames that contains an orthogonal basis for $\mathbb{R}^N$.

Useful remarks

Remark

We note that a frame $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *with* $\varphi_k \neq 0$ *for each* $k = 1, \ldots, M$ *is scalable if and only if* $\Phi' = \{\frac{\varphi_k}{\|\varphi_k\|}\}_{k=1}^M$ *is scalable.*

Useful remarks

Remark

Given a frame $\Phi \subset \mathbb{R}^N$, assume that $\Phi = \Phi_1 \cup \Phi_2$ where

$$\Phi_1 = \{\varphi_k^{(1)} \in \Phi : \varphi_k^{(1)}(N) \geq 0\}$$

and

$$\Phi_2 = \{\varphi_k^{(2)} \in \Phi : \varphi_k^{(2)}(N) < 0\}.$$

Let

$$\Phi' = \Phi_1 \cup (-\Phi_2).$$

Φ is scalable if and only if Φ' is scalable.
We shall assume that all the frame vectors are in the upper-half space, i.e., $\Phi \subset \mathbb{R}^{N-1} \times \mathbb{R}_{+,0}$ where $\mathbb{R}_{+,0} = [0, \infty)$.

Elementary properties of scalable frames

Proposition

Let $M \geq N$, and $m \geq 1$ be integers.

(i) *If $\Phi \in \mathcal{F}$ is m-scalable then $m \geq N$.*

(ii) *For any integers m, m' such that $N \leq m \leq m' \leq M$ we have that*

$$\mathcal{SC}(M, N, m) \subset \mathcal{SC}(M, N, m'),$$

and

$$\mathcal{SC}(M, N) = \bigcup_{m=N}^{M} \mathcal{SC}(M, N, m).$$

(iii) *$\Phi \in \mathcal{SC}(M, N)$ if and only if $T(\Phi) \in \mathcal{SC}(M, N)$ for one (and hence for all) orthogonal transformation(s) T on $\mathbb{R}^N$.*

(iv) *Let $\Phi = \{\varphi_k\}_{k=1}^{N+1} \in \mathcal{F}(N+1, N) \setminus \{0\}$ with $\varphi_k \neq \pm\varphi_\ell$ for $k \neq \ell$. If $\Phi \in \mathcal{SC}_+(N+1, N)$, then $\Phi \notin \mathcal{SC}_+(N+1, N+1)$.*

Scalable frames: When and How?

Question

1. *When is a frame* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *scalable?*
2. *If* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is scalable, how to find the coefficients?*
3. *If* Φ *is not scalable, how close to scalable is it?*
4. *What are the topological properties of* $\mathcal{SC}(M, N)$*?*

A reformulation

Fact

Φ is (m-) scalable $\iff \exists\{x_k\}_{k\in I} \subset [0,\infty)$ with $\#I = m \geq N$ such that $\widetilde{\Phi} = \Phi X$ satisfies

$$\widetilde{\Phi}\widetilde{\Phi}^T = \Phi X^2 \Phi^T = \tilde{A} I_N = \frac{\sum_{k\in I} x_k^2 \|\varphi_k\|^2}{N} I_N \tag{5}$$

where $X = \mathrm{diag}(x_k)$.
(5) is equivalent to solving

$$\Phi Y \Phi^T = I_N \tag{6}$$

for $Y = \frac{1}{\tilde{A}} X^2$.

Scalable frames in $\mathbb{R}^2$

Question

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^2$ *be a frame contained in the first quadrant:*

$$\Phi = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{M-1} \\ 0 & b_1 & b_2 & \dots & b_{M-1} \end{pmatrix}.$$

Is Φ *scalable?*

Solution

Φ *is scalable* $\iff$ $\exists : x = \{x_k\}_{k=1}^M \subset [0,\infty)$ *with* $\|x\|_0 \geq 2$: *the rows of the following matrix are orthogonal*

$$\begin{pmatrix} x_1 & x_2 a_1 & x_3 a_2 & \dots & x_M a_{M-1} \\ 0 & x_2 b_1 & x_3 b_2 & \dots & x_M b_{M-1} \end{pmatrix}$$

This happens when $\sum_{k=1}^{M-2} x_{k+1}^2 a_k b_k = 0$ *has a nontrivial solution. Which is not the case, so* Φ *is not scalable*

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Scalable frames in $\mathbb{R}^2$

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Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^2$ *be a frame with,* $\varphi_1 = e_1$, $\varphi_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ *with,* $a_1, b_1 > 0$, $\varphi_3 = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$, *with* $a_2 < 0 < b_2$

$$\Phi = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{M-1} \\ 0 & b_1 & b_2 & \dots & b_{M-1} \end{pmatrix}.$$

Is Φ *scalable?*

Solution

There exist $x_2, x_3 > 0$ *such that* $\{\varphi_1, x_2\varphi_2, x_3\varphi_3\}$ *forms a tight frames, since* $x_1^2 a_1 b_1 + x_2^2 a_2 b_2 = 0$ *has nontrivial nonnegative solutions.*
Φ *is scalable.*

Scalable frames in $\mathbb{R}^2$

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$$\Phi = \begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{M-1} \\ 0 & b_1 & b_2 & \dots & b_{M-1} \end{pmatrix}.$$

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Φ *is scalable.*

Scalable frame in $\mathbb{R}^2$

Question

More generally, when is $\Phi = \{\varphi_k\}_{k=1}^M \subset S^1$ is a scalable frame in $\mathbb{R}^2$?

Solution

Assume that $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R} \times \mathbb{R}_{+,0}$, $\|\varphi_k\| = 1$, and $\varphi_\ell \neq \varphi_k$ for $\ell \neq k$. Let $0 = \theta_1 < \theta_2 < \theta_3 < \ldots < \theta_M < \pi$, then

$$\varphi_k = \begin{pmatrix} \cos\theta_k \\ \sin\theta_k \end{pmatrix} \in S^1.$$

Let $Y = (y_k)_{k=1}^M \subset [0, \infty)$, then (6) becomes

$$\begin{pmatrix} \sum_{k=1}^M y_k \cos^2\theta_k & \sum_{k=1}^M y_k \sin\theta_k \cos\theta_k \\ \sum_{k=1}^M y_k \sin\theta_k \cos\theta_k & \sum_{k=1}^M y_k \sin^2\theta_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (7)$$

Scalable frame in $\mathbb{R}^2$

Solution

(7) *is equivalent to*

$$\begin{cases} \sum_{k=1}^{M} y_k \sin^2\theta_k & = & 1 \\ \sum_{k=1}^{M} y_k \cos 2\theta_k & = & 0 \\ \sum_{k=1}^{M} y_k \sin 2\theta_k & = & 0. \end{cases}$$

Consequently, for Φ to be scalable we must find a nonnegative vector $Y = (y_k)_{k=1}^{M}$ in the kernel of the matrix whose k^{th} column is $\begin{pmatrix} \cos 2\theta_k \\ \sin 2\theta_k \end{pmatrix}$.

Scalable frame in $\mathbb{R}^2$

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Scalable frame in $\mathbb{R}^2$

Solution

The problem is equivalent to finding non-trivial nonnegative vectors in the nullspace of

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \dots & \cos 2\theta_M \\ 0 & \sin 2\theta_2 & \dots & \sin 2\theta_M \end{pmatrix}. \tag{8}$$

Describing $SC(3,2)$

Example

We first consider the case $M = 3$. In this case, we have $0 = \theta_1 < \theta_2 < \theta_3 < \pi$, and the (8) becomes

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \cos 2\theta_3 \\ 0 & \sin 2\theta_2 & \sin 2\theta_3 \end{pmatrix}. \tag{9}$$

Describing $SC(3, 2)$

Example

If $\theta_{k_0} = \pi/2$ for $k_0 \in \{2, 3\}$, then the corresponding frame contains an ONB and, hence is scalable.

For example, when $k_0 = 2$, then $0 = \theta_1 < \theta_2 = \pi/2 < \theta_3 < \pi$. In this case, the fame is $2-$ scalable but not $3-$ scalable.

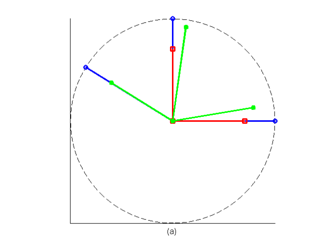

Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $\mathcal{SC}(3,2)$

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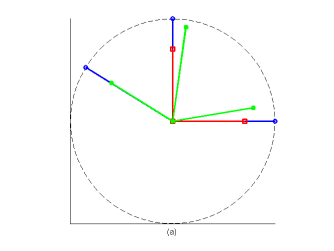

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Describing $\mathcal{SC}(3, 2)$

Example

Suppose $\theta_k \neq \pi/2$ for $k = 2, 3$. If $\theta_3 < \pi/2$, then the frame cannot be scalable. Indeed, $u = (z_1, z_2, z_3)$ belongs to the kernel of (9) if and only if

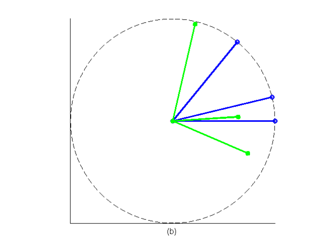
$$\begin{cases} z_1 &= \frac{\sin 2(\theta_3 - \theta_2)}{\sin 2\theta_2} z_3, \\ z_2 &= -\frac{\sin 2\theta_3}{\sin 2\theta_2} z_3, \end{cases} \tag{10}$$

where $z_3 \in \mathbb{R}$. The choice of the angles implies that $z_2 z_3 < 0$, unless $z_3 = 0$.

Describing $SC(3, 2)$

Example

This is illustrated by



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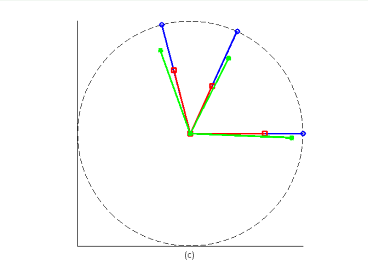
Describing $\mathcal{SC}(3,2)$

Example

Suppose that $0 = \theta_1 < \theta_2 < \pi/2 < \theta_3 < \pi$. From (10) $z_2 > 0$ for all $z_3 > 0$ and $z_1 > 0$ for all $z_3 > 0$ if and only if $\theta_3 - \theta_2 < \pi/2$. Consequently, when $0 = \theta_1 < \theta_2 < \pi/2 < \theta_3 < \pi$ the frame $\Phi \in \mathcal{SC}_+(3,2,3)$ if and only if $0 < \theta_3 - \theta_2 < \pi/2$.

Describing $SC(3, 2)$

Example



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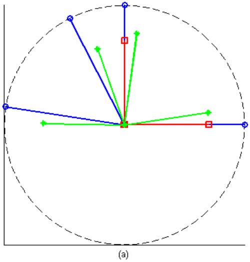
Describing $SC(4, 2)$

Example

When $M = 4$ we are lead to seek nonnegative non-trivial vectors in the null space of

$$\begin{pmatrix} 1 & \cos 2\theta_2 & \cos 2\theta_3 & \cos 2\theta_4 \\ 0 & \sin 2\theta_2 & \sin 2\theta_3 & \sin 2\theta_4 \end{pmatrix}.$$

Describing $SC(4, 2)$



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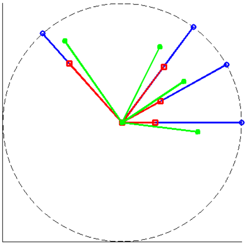
Describing $SC(4,2)$



(b)

Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

Describing $SC(4,2)$

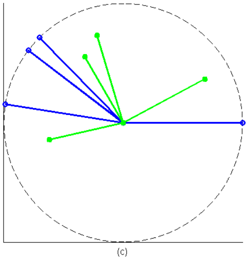


Figure : Blue=original frame; Red=the frames obtained by scaling; Green=associated canonical tight frame.

A more general reformulation

Setting

Recall that Φ *is* (m-) *scalable* $\iff \exists \{x_k\}_{k\in I} \subset [0,\infty)$ *such that* $\widetilde{\Phi} = \Phi X$ *satisfies*

$$\widetilde{\Phi}\widetilde{\Phi}^T = \Phi X^2 \Phi^T = I_N$$

where $X = \mathrm{diag}(x_k)$.
This is equivalent to

$$\begin{cases} \sum_{k=1}^{M} \varphi_k(j)^2 y_k = 1 & \text{for } j = 1,\ldots,N, \\ \sum_{k=1}^{M} \varphi_k(\ell)\varphi_k(j) y_k = 0 & \text{for } \ell, j = 1,\ldots,N,\ k > \ell. \end{cases} \tag{11}$$

A more general reformulation

Setting

(11) *leads to*

$$\begin{cases} \sum_{k=1}^{M} \big(\varphi_k(1)^2 - \varphi_k(j)^2\big) y_k = 0 & \text{for } j = 2, \ldots, N, \\ \sum_{k=1}^{M} \varphi_k(\ell)\varphi_k(j) y_k = 0 & \text{for } \ell, j = 1, \ldots, N,\ k > \ell. \end{cases} \tag{12}$$

When is a frame scalable: A generic solution

Question

When is $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ scalable?

Proposition

A frame Φ for $\mathbb{R}^N$ is m-scalable, respectively, strictly m-scalable, if and only if there exists a nonnegative $u \in \ker F(\Phi) \setminus \{0\}$ with $\|u\|_0 \leq m$, respectively, $\|u\|_0 = m$, and where $F(\Phi)$ is the $d \times M$ matrix whose k^{th} column is $F(\varphi_k)$.

A more general reformulation

Setting

Let $F : \mathbb{R}^N \to \mathbb{R}^d$, $d := (N-1)(N+2)/2$, *defined by*

$$F(x) = \begin{pmatrix} F_0(x) \\ F_1(x) \\ \vdots \\ F_{N-1}(x) \end{pmatrix}$$

$$F_0(x) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1^2 - x_3^2 \\ \vdots \\ x_1^2 - x_N^2 \end{pmatrix}, \ldots, F_k(x) = \begin{pmatrix} x_k x_{k+1} \\ x_k x_{k+2} \\ \vdots \\ x_k x_N \end{pmatrix}$$

and $F_0(x) \in \mathbb{R}^{N-1}$, $F_k(x) \in \mathbb{R}^{N-k}$, $k = 1, 2, \ldots, N-1$.

The map F when $N = 2$

Example

When $N = 2$ the map F reduces to

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ xy \end{pmatrix}.$$

Note that in the examples given above we consider

$$\widetilde{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}.$$

Some convex geometry notions

Fact

Let $X = \{x_i\}_{k=1}^M \subset \mathbb{R}^N$.

1. *The polytope generated by* X *is the convex hull of* X*, denoted by* P_X *(or* $\mathrm{co}(X)$*).*
2. *The affine hull generated by* X *is denoted by* $\mathrm{aff}(X)$.
3. *The relative interior of the polytope* $\mathrm{co}(X)$ *denoted by* $ri\,\mathrm{co}(X)$*, is the interior of* $\mathrm{co}(X)$ *in the topology induced by* $\mathrm{aff}(X)$.
4. *It is true that* $ri\,\mathrm{co}(X) \neq \emptyset$ *whenever* $\#X \geq 2$*, and*

$$ri\,\mathrm{co}(X) = \left\{ \sum_{k=1}^M \alpha_k x_k : \alpha_k > 0, \sum_{k=1}^M \alpha_k = 1 \right\},$$

A key tool: The Farkas Lemma

Lemma (Farkas's lemma)

For every real $N \times M$-matrix A exactly one of the following cases occurs:

(i) *The system of linear equations $Ax = 0$ has a nontrivial nonnegative solution $x \in \mathbb{R}^M$, i.e., all components of x are nonnegative and at least one of them is strictly positive.*

(ii) *There exists $y \in \mathbb{R}^N$ such that $y^T A$ is a vector with all entries strictly positive.*

Farkas lemma with $N = 2$, $M = 4$

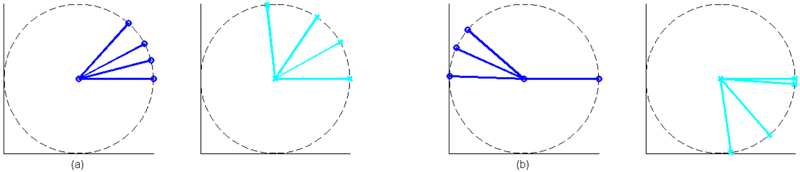


Figure : Bleu=original frame; Green=image by the map F. Both of these examples result in non scalable frames.

Farkas lemma with $N = 2$, $M = 4$

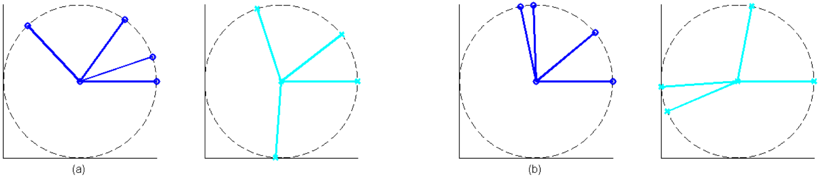


Figure : Bleu=original frame; Green=image by the map F. Both of these examples result in scalable frames.

Proof of Farkas Lemma

Proof.

Let $S = \{a_k\}_{k=1}^M \subset \mathbb{R}^N$ where a_k is the k^{th} column vector of A. The two alternatives correspond to $0 \in \text{co}(S)$ and $0 \notin \text{co}(S)$. □

Scalable frames and Farkas's lemma

Theorem

Let $M \geq N \geq 2$, and let m be such that $N \leq m \leq M$. Assume that $\Phi = \{\varphi_k\}_{k=1}^M \in \mathcal{F}^*(M,N)$ is such that $\varphi_k \neq \pm\varphi_\ell$ when $k \neq \ell$. Then the following statements are equivalent:

(i) *Φ is $m-$scalable, respectively, strictly $m-$scalable,*

(ii) *There exists a subset $I \subset \{1,2,\ldots,M\}$ with $\#I = m$ such that $0 \in \operatorname{co}(F(\Phi_I))$, respectively, $0 \in ri\operatorname{co}(F(\Phi_I))$.*

(iii) *There exists a subset $I \subset \{1,2,\ldots,M\}$ with $\#I = m$ for which there is no $h \in \mathbb{R}^d$ with $\langle F(\varphi_k), h\rangle > 0$ for all $k \in I$, respectively, with $\langle F(\varphi_k), h\rangle \geq 0$ for all $k \in I$, with at least one of the inequalities being strict.*

Sketch of the proof

Proof.

(i) $\Longleftrightarrow$ (ii). This equivalence follows directly if we can show the following equivalences for $\Psi \subset \Phi$:

$$0 \in \operatorname{co}(F(\Psi)) \quad \Longleftrightarrow \quad \ker F(\Psi) \setminus \{0\} \text{ contains a } \geq 0 \text{ vector and}$$
$$0 \in ri\,\operatorname{co}(F(\Psi)) \quad \Longleftrightarrow \quad \ker F(\Psi) \text{ contains a } > 0 \text{ vector.}$$

$\Rightarrow$: easy.
$\Leftarrow$ Case 1: Let $I \subset [M]$ be such that $\Psi = \Phi_I$, $I = \{i_1, \ldots, i_m\}$, and let $u = (c_1, \ldots, c_m)^T \in \ker F(\Psi)$ be a non-zero nonnegative vector. Set $A := \sum_{k=1}^m c_k > 0$ and with $\lambda_k := c_k/A$, we see that $0 \in \operatorname{co}(F(\Psi))$.
Case 2: each $c_k > 0$ leading to $\lambda_k > 0$.
(ii) $\Longleftrightarrow$ (iii) In the first case this follows from Farkas's lemma. □

A useful property of F

For $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ and $h = (h_k)_{k=1}^d \in \mathbb{R}^d$, we have that

$$\langle F(x), h\rangle = \sum_{\ell=2}^{N} h_{\ell-1}(x_1^2 - x_\ell^2) + \sum_{k=1}^{N-1}\sum_{\ell=k+1}^{N} h_{k(N-1-(k-1)/2)+\ell-1}x_k x_\ell. \tag{13}$$

Consequently, fixing $h \in \mathbb{R}^d$, $\langle F(x), h\rangle$ is a homogeneous polynomial of degree 2 in $x_1, x_2, \ldots, x_N$. The set of all polynomials of this form can be identified with the subspace of real symmetric $N \times N$ matrices whose trace is 0.

A useful property of F

Remark

$\langle F(x), h\rangle = \langle Q_h x, x\rangle = 0$ defines a quadratic surface in $\mathbb{R}^N$, and condition (iii) in the last Theorem stipulates that for Φ to be scalable, one cannot find such a quadratic surface such that the frame vectors (with index in I) all lie on (only) "one side" of this surface.

A geometric characterization of scalable frames

Theorem

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N \setminus \{0\}$ be a frame for $\mathbb{R}^N$. Then the following statements are equivalent.

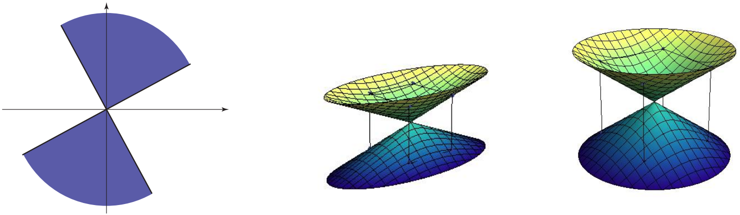
(i) Φ *is not scalable.*

(ii) *There exists a symmetric $M \times M$ matrix Y with $trace(Y) < 0$ such that $\langle \varphi_j, Y\varphi_j \rangle \geq 0$ for all $j = 1, \ldots, M$.*

(iii) *There exists a symmetric $M \times M$ matrix Y with $trace(Y) = 0$ such that $\langle \varphi_j, Y\varphi_j \rangle > 0$ for all $j = 1, \ldots, M$.*

Scalable frames in $\mathbb{R}^2$ and $\mathbb{R}^3$

Figures show sample regions of vectors of a non-scalable frame in $\mathbb{R}^2$ and $\mathbb{R}^3$.



(a) (b) (c)

Figure : (a) shows a sample region of vectors of a non-scalable frame in $\mathbb{R}^2$. (b) and (c) show examples of sets in $\mathcal{C}_3$ which determine sample regions in $\mathbb{R}^3$.

Other necessary and sufficient conditions for scalability

Theorem

Let $\Phi \in \mathcal{F}(M,N)$ be a unit-norm frame. Then the following hold:

(a) (*A necessary condition for scalability*) *If Φ is scalable, then*

$$\min_{\|d\|_2=1} \max_i |\langle d, \varphi_i \rangle| \geq \frac{1}{\sqrt{N}}. \tag{14}$$

(b) (*A sufficient condition for scalability*) *If*

$$\min_{\|d\|_2=1} \max_i |\langle d, \varphi_i \rangle| \geq \sqrt{\frac{N-1}{N}}, \tag{15}$$

then Φ is scalable.

Remark

For $N = 2$ the conditions in the last theorem are necessary and sufficient. But this fails for $N > 2$.

Other necessary and sufficient conditions for scalability

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then Φ is scalable.

Remark

For $N = 2$ the conditions in the last theorem are necessary and sufficient. But this fails for $N > 2$.

Topology of scalable frames

Proposition

Let $M \geq m \geq N \geq 2$.

(a) *Let $\Phi = \{\varphi_k\}_{k=1}^M \mathcal{F}(M,N) \setminus \mathcal{SC}(M,N)$. Then there exists $\epsilon > 0$ such that each set of vectors $\{\psi_k\}_{k=1}^M \subset \mathbb{R}^N$ with*

$$\|\varphi_k - \psi_k\| < \epsilon \quad \text{for all } k = 1, \ldots, M$$

is a frame for $\mathbb{R}^N$ which is not scalable.

(b) *$\mathcal{SC}(M,N,m)$ is closed in $\mathcal{F}(M,N)$.*

Topology of scalable frames

Theorem

Assume that $2 \leq N \leq M < d+1 = N(N+1)/2$. Then $\mathcal{SC}(M,N)$ does not contain interior points. In other words, for the boundary of $\mathcal{SC}(M,N)$ we have

$$\partial\mathcal{SC}(M,N) = \mathcal{SC}(M,N).$$

Remark

When $2 \leq N \leq M < d+1 = N(N+1)/2$, $\mathcal{SC}(M,N)$ is a "hollow set". It can be shown that in this regime, the probability of a unit norm frame whose vectors are drawn randomly from the unit ball according to the uniform distribution is 0.

Fritz John's Theorem

Theorem (F. John (1948))

Let $K \subset B = B(0,1)$ be a convex body with nonempty interior. There exits a unique ellipsoid $\mathcal{E}_{min}$ of minimal volume containing K.
Moreover, $\mathcal{E}_{min} = B$ if and only if there exist $\{\lambda_k\}_{k=1}^m \subset (0,\infty)$ and $\{u_k\}_{k=1}^m \subset \partial K \cap S^{N-1}$, $m \geq N+1$ such that

(i) $\sum_{k=1}^m \lambda_k u_k = 0$

(ii) $x = \sum_{k=1}^m \lambda_k \langle x, u_k \rangle u_k, \forall x \in \mathbb{R}^N$

where ∂K is the boundary of K and S^{N-1} is the unit sphere in $\mathbb{R}^N$.

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame for $\mathbb{R}^N$. We apply F. John's theorem to the convex body $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$. Let $\mathcal{E}_\Phi$ denote the ellipsoid of minimal volume containing P_Φ, and $V_\Phi =$ Vol$(\mathcal{E}_\Phi)/\omega_N$ where ω_N is the volume of the euclidean unit ball.

Theorem

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ is the euclidean unit ball B.

F. John's characterization of scalable frames

Setting

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame for $\mathbb{R}^N$. We apply F. John's theorem to the convex body $K = P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$. Let $\mathcal{E}_\Phi$ denote the ellipsoid of minimal volume containing P_Φ, and $V_\Phi = Vol(\mathcal{E}_\Phi)/\omega_N$ where ω_N is the volume of the euclidean unit ball.

Theorem

Let $\Phi = \{\varphi_k\}_{k=1}^M \subset S^{N-1}$ be a frame. Then Φ is scalable if and only if $V_\Phi = 1$. In this case, the ellipsoid $\mathcal{E}_\Phi$ of minimal volume containing $P_\Phi = conv(\{\pm\varphi_k\}_{k=1}^M)$ is the euclidean unit ball B.

A first measure of scalability: Volume of the frame's John's ellipsoid

Remark

Let $\Phi \subset S^{N-1}$ be a frame. Then V_Φ is a "measure of scalability": the closer it is to 1 the more scalable is the frame.

A quadratic programing approach to scalability

Setting

$$\Phi = \{\varphi_i\}_{i=1}^M \textit{ is scalable } \iff \exists \{c_i\}_{i=1}^M \subset [0,\infty) : \Phi C \Phi^T = I,$$

where $C = \textit{diag}(c_i)$.

$$C_\Phi = \{\Phi C \Phi^T = \sum_{i=1}^M c_i \varphi_i \varphi_i^T : c_i \geq 0\}$$

is the (closed) cone generated by $\{\varphi_i \varphi_i^T\}_{i=1}^M$.

$$\Phi = \{\varphi_i\}_{i=1}^M \textit{ is scalable } \iff I \in C_\Phi.$$

$$D_\Phi := \min_{C \geq 0 \textit{ diagonal}} \left\| \Phi C \Phi^T - I \right\|_F$$

Comparing D_Φ to the frame potential

Proposition

(a) Φ *is scalable if and only if* $D_\Phi = 0$.

(b) *If* $\Phi = \{\varphi_k\}_{k=1}^M \subset \mathbb{R}^N$ *is a unit norm frame we have*

$$D_\Phi^2 \leq N - \frac{M^2}{\mathrm{FP}(\Phi)},$$

where $\mathrm{FP}(\Phi)$ *is the frame potential of* Φ.

A second measure of scalability

Remark

Let $\Phi \subset S^{N-1}$ be a frame. Then D_Φ is a "measure of scalability": the closer it is to 0 the more scalable is the frame.

Distance to the set of scalable frames

Let $\Phi \in \mathcal{F}(M,N)$ be a unit norm frame and denote

$$d_\Phi := \inf_{\Psi \in \mathcal{SC}(M,N)} \|\Phi - \Psi\|_F.$$

Proposition

If $\Phi \in \mathcal{F}_u(M,N)$ such that $d_\Phi < 1$ then there exists $\hat{\Phi} \in \mathcal{SC}(M,N)$ such that $\|\Phi - \hat{\Phi}\|_F = d_\Phi$.

Comparison of D_Φ and V_Φ

Theorem

Let $\Phi = \{\varphi_i\}_{i=1}^M \in \mathcal{F}_u(M, N)$, *then*

$$\frac{N(1 - D_\Phi^2)}{N - D_\Phi^2} \leq V_\Phi^{4/N} \leq \frac{N(N - 1 - D_\Phi^2)}{(N-1)(N - D_\Phi^2)} \leq 1, \tag{16}$$

where the leftmost inequality requires $D_\Phi < 1$. *Consequently,* $V_\Phi \to 1$ *is equivalent to* $D_\Phi \to 0$.

Examples in $\mathbb{R}^4$

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

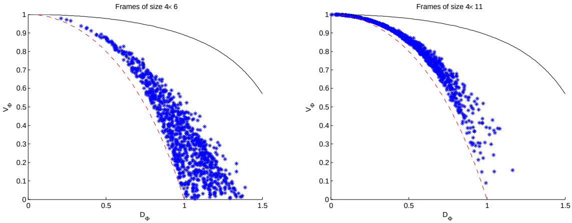

Figure : Relation between V_Φ and D_Φ with $M = 6, 11$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Comparing the measures of scalability

Values of V_Φ and D_Φ for randomly generated frames of M vectors in $\mathbb{R}^4$.

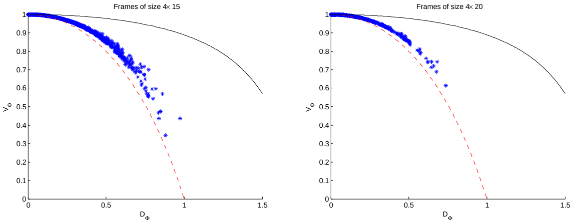

Figure : Relation between V_Φ and D_Φ with $M = 15, 20$. The black line indicates the upper bound in the last theorem, while the red dash line indicates the lower bound.

Comparison of the Measures D_Φ and V_Φ with d_Φ

Theorem

Let $\Phi \in \mathcal{F}_u(M, N)$ and assume that $d_\Phi < 1$. Then with $K := \min\{M, \frac{N(N+1)}{2}\}$ and $\omega := D_\Phi + \sqrt{K}$ we have

$$\frac{D_\Phi}{\omega + \sqrt{\omega^2 - D_\Phi^2}} \leq d_\Phi \leq \sqrt{KN\left(1 - V_\Phi^{2/N}\right)}. \tag{17}$$

Size of the set of scalable frames

Theorem

Given $\Phi = \{\varphi_i\}_{i=1}^M \subset \mathbb{R}^N$*, where each vector* φ_i *is drawn independently and uniformly from* $\mathbb{S}^{N-1}$*, let* $P_{M,N}$ *denote the probability that* Φ *is scalable. Then the following holds:*

(i) *When* $M < \frac{N(N+1)}{2}$, $P_{M,N} = 0$.

(ii) *When* $M \geq \frac{N(N+1)}{2}$, $P_{M,N} > 0$ *and*

$$C_N \left(1 - A_\alpha^{N-1}\right)^M \geq 1 - P_{M,N} \geq \left(1 - A_a^{N-1}\right)^{M-N},$$

where

$$\alpha = \frac{1}{2} \arccos \sqrt{\frac{N-1}{N}}, \quad a = \arccos \frac{1}{\sqrt{N}},$$

and where C_N *is the number of caps with angular radius* α *needed to cover* $\mathbb{S}^{N-1}$*. Consequently,* $\lim_{M\to\infty} P_{M,N} = 1$.

Laplacian pyramid based Laurent polynomial LP^2 matrix

Setting

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. $\mathcal{M}_{q,p}(z)$ *will denote the set of all* $q \times p$ *matrices whose entries are Laurent polynomials in* $z \in \mathbb{T}$ *with real coefficients, and* $\mathcal{M}_q(z) := \mathcal{M}_{q,1}(z)$ *will denote the set of all column vectors of length* q.
Consider a nonzero column vector with Laurent polynomial entries, denoted by

$$\mathtt{H}(z) := [H_0(z), H_1(z), \ldots, H_{q-1}(z)]^T \in \mathcal{M}_q(z).$$

To the (Laurent polynomial valued) vector $\mathtt{H}(z)$ *we associate the Laplacian pyramid based Laurent polynomial (LP*^{2}*) matrix* $\Phi_{\mathtt{H}}(z)$ *defined by*

$$\Phi_{\mathtt{H}}(z) := \left[\begin{array}{cc} \mathtt{H}(z) & \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z) \end{array}\right] \in \mathcal{M}_{q\times(q+1)}(z),$$

Laplacian pyramid based Laurent polynomial LP^2 matrix

Setting

Note that

$$\begin{aligned}\mathtt{H}^*(z) &:= \overline{\mathtt{H}(z)}^T \\ &= [\overline{H_0(z)}, \overline{H_1(z)}, \dots, \overline{H_{q-1}(z)}] \\ &= [H_0(z^{-1}), H_1(z^{-1}), \dots, H_{q-1}(z^{-1})].\end{aligned}$$

Laplacian pyramid based Laurent polynomial LP^2 matrix

Setting

It follows that

$$\Phi_{\mathtt{H}}(z) \begin{bmatrix} \mathtt{H}^*(z) \\ \mathtt{I} \end{bmatrix} = \mathtt{I}, \quad \forall z \in \mathbb{T}.$$

Consequently, $\operatorname{rank} \Phi_{\mathtt{H}}(z) = q$ *for all* $z \in \mathbb{T}$. *Hence, for each* $z \in \mathbb{T}$ *the columns of* $\Phi_{\mathtt{H}}(z)$ *form of frame for* $\mathbb{C}^q$.

Paraunitary LP^2 matrices

Setting

The LP^2 matrix $\Phi_{\mathrm{H}}(z)$ is said to be paraunitary, if

$$\Phi_{\mathrm{H}}(z)\Phi_{\mathrm{H}}^*(z) = \mathrm{I}.$$

In this case, the pair $(\Phi_{\mathrm{H}}(z), \Phi_{\mathrm{H}}(z)^*)$ can be used to construct a a tight filter bank.

Paraunitary LP^2 matrices

Setting

The existence of a tight filter bank from a paraunitary LP^2 matrix $\Phi_{\mathtt{H}}(z)$ *is equivalent to the existence of a column matrix* $\mathtt{H}(z)$ *such that* $\mathtt{H}^*(z)\mathtt{H}(z) = 1$, *that is,* $\sum_{k=0}^{q-1} |H_k(z)|^2 = 1$ *for all* $z \in \mathbb{T}$.

Question

Can a column vector $\mathtt{H}(z)$ *such that* $\mathtt{H}^*(z)\mathtt{H}(z) \neq 1$ *be modified into* $\widetilde{\mathtt{H}}(z)$ *for which* $\widetilde{\mathtt{H}}^*(z)\widetilde{\mathtt{H}}(z) = 1$ *leading to a paraunitary LP^2 matrix* $\Phi_{\widetilde{\mathtt{H}}}(z)$.

Definition

Definition

An LP^2 matrix $\Phi_{\mathtt{H}}(z)$ for which there exists a diagonal matrix $M(z)$ such that $\Phi_{\mathtt{H}}(z)M(z)$ is paraunitary, i.e.

$$[\Phi_{\mathtt{H}}(z)M(z)][M^*(z)\Phi^*_{\mathtt{H}}(z)] = \mathtt{I}.$$

is called a scalable LP^2 matrix.

Characterizing LP^2 matrix

Theorem

Let $\Phi_{\mathtt{H}}(z)$ be an LP^2 matrix associated with $\mathtt{H}(z) \in \mathcal{M}_q(z)$. Then we have

$$\Phi_{\mathtt{H}}(z)\mathrm{diag}([2 - \mathtt{H}^*(z)\mathtt{H}(z), 1, \ldots, 1])\Phi^*_{\mathtt{H}}(z) = \mathtt{I}.$$

Characterizing LP^2 matrix

Theorem

Let $\mathtt{H}(z) = [H_0(z), H_1(z), \ldots, H_{q-1}(z)]^T \in \mathcal{M}_q(z)$, *and let* $\Phi_{\mathtt{H}}(z)$ *be the associated* LP^2 *matrix. Suppose that* $B(z) \in \mathcal{M}_{(q+1)\times(q+1)}(z)$ *is diagonal satisfying* $\Phi_{\mathtt{H}}(z)B(z)\Phi_{\mathtt{H}}^*(z) = \mathtt{I}$. *Then* $B(z) = \mathrm{diag}([2 - \mathtt{H}^*(z)\mathtt{H}(z), 1, \ldots, 1])$ *for* $z \in \mathbb{T} \setminus S_{\mathtt{H}}$, *where the set* $S_{\mathtt{H}} \subset \mathbb{T}$ *is defined as*

$$S_{\mathtt{H}} := \{z \in \mathbb{T} : H_0(z)\overline{H_1(z)} = 0 \text{ or } 1 - |H_0(z)|^2 - |H_1(z)|^2 = 0\}$$

if $q = 2$, *and as*

$$S_{\mathtt{H}} := \{z \in \mathbb{T} : H_{k-1}(z)\overline{H_{i+k-1}(z)} = 0, \text{ for some } k = 1, \ldots, q-1,\ i = 1, \ldots, q$$

if $q \geq 3$.

Filters and wavelet

Setting

Let $\lambda \geq 2$. A filter $h : \mathbb{Z} \to \mathbb{R}$ is called lowpass if $\sum_{k\in\mathbb{Z}} h(k) = \sqrt{\lambda}$, and highpass if $\sum_{k\in\mathbb{Z}} h(k) = 0$.
The z-transform of a filter h is defined as $H(z) := \sum_{k\in\mathbb{Z}} h(k)z^{-k}$. A Laurent polynomial column vector $\mathtt{H}(z) \in \mathcal{M}_q(z)$ is called the (synthesis) polyphase representation of a filter h if

$$\mathtt{H}(z) = [H_{\nu_0}(z), H_{\nu_1}(z), \dots, H_{\nu_{q-1}}(z)]^T,$$

where $H_\nu(z)$ is the z-transform of the filter h_ν defined as $h_\nu(k) = h(\lambda k + \nu)$, $k \in \mathbb{Z}$.

Polyphase representation

Setting

Let h be a lowpass filter, and let $\mathtt{H}(z) \in \mathcal{M}_q(z)$ be its polyphase representation. Suppose that there exists a Laurent polynomial $m_{\mathtt{H}}(z)$ such that $2 - \mathtt{H}^*(z)\mathtt{H}(z) = |m_{\mathtt{H}}(z)|^2$. Then,

$$\Phi_{\mathtt{H}}(z)\mathrm{diag}([m_{\mathtt{H}}(z), 1, \ldots, 1]) = \left[\begin{array}{cc} m_{\mathtt{H}}(z)\mathtt{H}(z) & \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z) \end{array}\right]$$

is paraunitary, i.e. $\Phi_{\mathtt{H}}(z)$ is scalable.

Fejér-Riesz lemma

Question

The construction of tight wavelet frames hinges on the existence of a Laurent polynomial $m_{\mathtt{H}}(z)$ such that $2 - \mathtt{H}^*(z)\mathtt{H}(z) = |m_{\mathtt{H}}(z)|^2$. This is possible if and only if $2 - \mathtt{H}^*(z)\mathtt{H}(z) \geq 0$, for all $z \in \mathbb{T}$.

Tight wavelet filter banks

Theorem

Let h be a 1-D lowpass filter with positive accuracy and dilation $\lambda \geq 2$, and let $\mathtt{H}(z)$ be its polyphase representation. Suppose $2 - \mathtt{H}^*(z)\mathtt{H}(z) > 0$, $\forall z \in \mathbb{T}$. Then there is a polynomial $m_{\mathtt{H}}(z)$ such that $[m_{\mathtt{H}}(z)\mathtt{H}(z), \mathtt{I} - \mathtt{H}(z)\mathtt{H}^*(z)]$ gives rise to a tight wavelet filter bank whose lowpass filter $\widetilde{h}$ is associated with $m_{\mathtt{H}}(z)\mathtt{H}(z)$ and has the same accuracy as h. Furthermore, if the support of h is contained in $\{0, 1, \ldots, s\}$, then the support of $\widetilde{h}$ is contained in $\{0, 1, \ldots, 2s\}$.

Examples

Example

Let $h := [1/4 - a/2, 1/4, a, 1/4, 1/4 - a/2]$
Then, the associated z-transform $H(z)$ is,

$$H(z) = \sqrt{2}\left(\frac{1}{4} - \frac{a}{2}\right)\left(z^{-2} + z^2\right) + \frac{\sqrt{2}}{4}(z^{-1} + z) + \sqrt{2}a, \quad z \in \mathbb{T},$$

and the components of the polyphase representation
$\mathtt{H}(z) = [H_0(z), H_1(z)]^T$, $z \in \mathbb{T}$, are given as

$$H_0(z) = \sqrt{2}\left(\frac{1}{4} - \frac{a}{2}\right)\left(z^{-1} + z\right) + \sqrt{2}a, \quad H_1(z) = \frac{\sqrt{2}}{4}(1 + z).$$

Thank You!

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