Appendix A

Real Analysis

$L^1(\mathbb{R})$ and the L^1-norm $\|\ldots\|_{L^1(\mathbb{R})}$ were introduced in *Definition 1.1.1*, and are a basic space and norm in real analysis, i.e., advanced calculus and real variables. This appendix lists results from real analysis, e.g., [AB66], [Apo57], [Ben76], [Rud66]. There are many excellent texts.

A.1 Definition. CONVERGENCE OF SEQUENCES

a. $\mathbb{N}$ is the set positive integers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers.

b. A sequence of $\{c_n\} \subseteq \mathbb{C}$ *converges* to $c \in \mathbb{C}$ as $n \to \infty$, written $\lim_{n\to\infty} c_n = c$ if

$$\forall \epsilon > 0, \quad \exists N = N(\epsilon), \quad \text{such that} \quad \forall n \geq N, \qquad |c_n - c| < \epsilon. \tag{A.1}$$

The "statement" (A.1) is read as follows: for every $\epsilon > 0$, there is $N \in \mathbb{N}$ depending on ϵ, such that for any $n \geq \mathbb{N}$ the inequality, $|c_n - c| < \epsilon$, is valid.

c. If $\{c_n\} \subseteq \mathbb{C}$ is a sequence and if a subsequence $\{c_{n_k}\} \subseteq \{c_n\}$ converges to $c \in \mathbb{C}$, then c is a *limit point* of $\{c_n\}$.

Let $\{c_n\} \subseteq \mathbb{R}$, and suppose there is $s \in \mathbb{R}$ with the properties that

$$\forall \epsilon > 0, \quad \exists N = N(\epsilon), \quad \text{such that} \quad \forall n > N, \quad c_n < s + \epsilon$$

and

$$\forall \epsilon > 0 \text{ and } \forall N, \quad \exists n > N \quad \text{such that} \quad c_n > s - \epsilon.$$

Then s is the *limit superior* of $\{c_n\}$, and we write

$$s = \overline{\lim_{n\to\infty}}\, c_n.$$

The definition of limit superior means that for each $\epsilon > 0$, eventually (depending on ϵ) all terms of $\{c_n\}$ are less than $s+\epsilon$ and infinitely many terms of $\{c_n\}$ are greater than $s-\epsilon$. Intuitively, s is the "greatest" limit point of $\{c_n\}$. The *limit inferior* of $\{c_n\} \subseteq \mathbb{R}$ is defined as

$$\underline{\lim_{n\to\infty}}\, c_n = -\overline{\lim_{n\to\infty}}\,(-c_n).$$

d. If $\{a_n\}, \{b_n\} \subseteq \mathbb{R}$ are sequences then it is not difficult to prove that

$$\overline{\lim}\,(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n.$$

A.2 Definition. Convergence of Functions

Let $\{f_n\}$ be a sequence of functions $f_n : X \to \mathbb{C}$, where $X \subseteq \mathbb{R}$.

a. $\{f_n\}$ *converges pointwise* to a function $f : X \to \mathbb{R}$ as $n \to \infty$, written

$$\forall t \in X, \quad \lim_{n\to\infty} f_n(t) = f(t),$$

if

$$\forall t \in X \text{ and } \forall \epsilon > 0, \quad \exists N = N(t, \epsilon) \in \mathbb{N} \quad \text{such that}$$

$$\forall n \geq N, \quad |f_n(t) - f(t)| < \epsilon.$$

b. $\{f_n\}$ *converges uniformly* on X to a function $f : X \to \mathbb{R}$ as $n \to \infty$, written

$$\lim_{n\to\infty} f_n = f \quad \text{uniformly on} \quad X,$$

if

$$\forall \epsilon > 0, \ \exists N = N(\epsilon), \quad \text{such that} \quad \forall t \in X \text{ and } \forall n \geq N,$$

$$|f_n(t) - f(t)| < \epsilon.$$

In this case, N is independent of the variable t. This allows us to *switch operations* legitimately. An example is the following result where the operations are integration and taking limits, cf., *Theorem A.5* and [Ben76, Chapter 6] (where a characterization for the validity of such switching is given).

A.3 Theorem. $\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt$

Let $\{f_n\}$ be a sequence of continuous functions $f_n : [a,b] \to \mathbb{C}$, which converges uniformly on $[a,b]$ to a function $f : [a,b] \to \mathbb{C}$. Then f is continuous on $[a,b]$ and

$$\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b \left(\lim_{n\to\infty} f_n(t)\right)\,dt = \int_a^b f(t)\,dt.$$

A.4 Definition. Lebesgue Measure Zero

The *Lebesgue measure* of an interval with endpoints a and b, $b > a$, is defined as its length $b-a$. If $X = \bigcup_{j=1}^{\infty}(a_j, b_j)$, where $b_{j+1} \le a_j$, then the *Lebesgue measure* of X, denoted by $|X|$, is defined as $|X| = \sum_{j=1}^{\infty}(b_j - a_j)$. Unless there is possible confusion, we shall usually refer to the *measure* of X instead of the *Lebesgue measure* of X. This notion of measure extends naturally to many other sets $X \subseteq \mathbb{R}$, and we write $|X|$ to denote their measure.

A set $X \subseteq \mathbb{R}$ is a set of *measure* 0 if for each $\epsilon > 0$ there is a countable set $\{(a_j, b_j) \subseteq \mathbb{R} : j = 1, \dots\}$ of intervals such that

$$X \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j) \quad \text{and} \quad \Sigma(b_j - a_j) < \epsilon.$$

A property is valid *almost everywhere*, written *a.e.*, if it is true for all $t \in \mathbb{R}\backslash X$, where X is a set of measure 0. (Technically, an element of $f \in L^1(\mathbb{R})$ is a set of functions g which are equal a.e., and for which $\int |g(t)|\,dt < \infty$. In this book we shall think of elements in $L^1(\mathbb{R})$ as functions, and no problems will arise.)

The next result gives insight into the structure of $L^1(\mathbb{R})$. Recall the definition of $C_c^{\infty}(\mathbb{R})$ from *Definition 2.2.1*.

A.5 Theorem. FUNDAMENTAL PROPERTIES OF $L^1(\mathbb{R})$

Let $f \in L^1(\mathbb{R})$ and let $\epsilon > 0$.

a. There is a function $g_\epsilon = \Sigma c_j \mathbf{1}_{[a_j,b_j)}$, $b_j \leq a_{j+1}$, *a finite sum, such that*

$$\|f - g_\epsilon\|_{L^1(\mathbb{R})} < \epsilon.$$

b. There is a function $g_\epsilon \in C_c^\infty(\mathbb{R})$, *such that*

$$\|f - g_\epsilon\|_{L^1(\mathbb{R})} < \epsilon.$$

c. There is $\delta = \delta(\epsilon)$ such that if $|u| < \delta$ then

$$\|f - \tau_u f\|_{L^1(\mathbb{R})} < \epsilon.$$

(Recall that $(\tau_u f)(t) = f(t-u)$ as a function of t.)

d. If $\|f\|_{L^1(\mathbb{R})} = 0$, then f is the 0-element of $L^1(\mathbb{R})$, i.e., $f = 0$ a.e.

A.6 Theorem. FATOU LEMMA

Let $\{f_n\} \subseteq L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ be real-valued functions which have the property that for each n, $g \leq f_n$ a.e. If f is a function for which $\lim_{n\to\infty} f_n = f$ a.e., then $f \in L^1(\mathbb{R})$ and

$$\int f(t)\,dt \leq \underline{\lim}_{n\to\infty} \int f_n(t)\,dt.$$

(Recall that $\int f(t)\,dt$ indicates integration over $\mathbb{R}$.)

A.7 Remark. PERSPECTIVE ON THE FATOU LEMMA

Fatou proved *Theorem A.6* in 1906, and used it to prove the Parseval Theorem for $F \in L^2(\mathbb{T}_{2\Omega})$, e.g., [Haw70, pages 168-172]. Earlier, Lebesgue had proved the Parseval Theorem for $L^\infty(\mathbb{T}_{2\Omega}) \subseteq L^2(\mathbb{T}_{2\Omega})$. The Fatou Lemma can be used to prove the Beppo Levi Theorem, which was originally proved in 1906, as well as the general form of the Lebesgue Dominated Convergence Theorem (LDC), which Lebesgue published in 1908. Statements of these latter two results now follow.

A.8 Theorem. Beppo Levi Theorem

Let $\{f_n\} \subseteq L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$ have the property that for each n, $g \leq f_n$ a.e. If $f_n \leq f_{n+1}$ a.e. for each n and $\lim_{n\to\infty} \int f_n(t)\,dt$ is finite, then $\lim_{n\to\infty} f_n \equiv f \in L^1(\mathbb{R})$ and

$$\int f(t)\,dt = \lim_{n\to\infty} \int f_n(t)\,dt,$$

cf., [Ben76, Theorem 3.7] for a relaxation of the monotonicity of $\{f_n\}$.

We shall use the notation *LDC* in referring to the following result.

A.9 Theorem. Lebesgue Dominated Convergence Theorem

Let $\{f_n\} \subseteq L^1(\mathbb{R})$, and let f be a function for which $\lim_{n\to\infty} f_n = f$ a.e. If $g \in L^1(\mathbb{R})$ has the property that

$$\forall n, \quad |f_n(t)| \leq g(t) \quad a.e. \quad on \quad \mathbb{R},$$

then $f \in L^1(\mathbb{R})$ and

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(\mathbb{R})} = 0.$$

A.10 Definition. Measurable Functions and L^p-spaces

a. A complex-valued function f defined a.e. on $\mathbb{R}$ is *(Lebesgue) measurable* if there is a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ for which $\lim_{n\to\infty} f_n = f$ a.e. The elements of $L^1(\mathbb{R})$ are measurable; whereas the function f, defined by $f(t) = 1/t$ a.e. on $\mathbb{R}$, is measurable but is not only not in $L^1(\mathbb{R})$ but is not in $L^1_{\text{loc}}(\mathbb{R})$. *All of our functions in the statements of all of our theorems are assumed to be measurable.* However, we are not interested in measurable functions per se; and a real knowledge of this concept is not required in this book even though it is fundamental in any systematic treatment of real analysis.

b. Let $p \in [1, \infty)$. $L^p(\mathbb{R})$ is the space of functions $f : \mathbb{R} \to \mathbb{C}$, for which the *$L^p$-norm* of f,

$$\text{(A.2)} \qquad \|f\|_{L^p(\mathbb{R})} = \left(\int |f(t)|^p\,dt\right)^{1/p},$$

is finite. There are subtleties in this definition related to the meaning of Lebesgue integration and the parenthetical remark in *Definition A.4*. We shall not deal with such issues, and, as with measurable functions, they do not play a role in understanding the material in this book.

c. If $p = \infty$ and the aforementioned subtleties are once again acknowledged and forgotten, then $L^\infty(\mathbb{R})$ is defined as the space of all functions which are bounded except possibly on a subset of measure zero. The L^∞-*norm* $\|f\|_{L^\infty(\mathbb{R})}$ of $f \in L^\infty(\mathbb{R})$ is intuitively the supremum of $|f|$ on $\mathbb{R}$. Technically,

$$\|f\|_{L^\infty(\mathbb{R})} = \inf\{M : |\{t \in \mathbb{R} : |f(t)| > M\}| = 0\}. \tag{A.3}$$

This complicated array of symbols reduces to

$$\|f\|_{L^\infty(\mathbb{R})} = \sup\{|f(t)| : t \in \mathbb{R}\}$$

in the case $f \in C_c^\infty(\mathbb{R})$.

d. The L^p-spaces can be defined on subintervals of $\mathbb{R}$ as well as other sets X. In such cases, if $p < \infty$ then the integration over $\mathbb{R}$ in (A.2) is replaced by integration over X, with a similar adjustment in (A.3).

A.11 Example. PERSPECTIVE ON LDC

a. In light of LDC, it is natural to ask if the hypothesis,

$$\lim_{n\to\infty} \|f_n\|_{L^1(\mathbb{R})} = 0,$$

allows us to conclude that $\lim_{n\to\infty} f_n = 0$ a.e., cf., [Ben76, Chapter 6]. The following example shows that the answer is "no". For each $n \in \mathbb{N}$, consider the unique representation $n = 2^j + k$, $0 \le k < 2^j$; and for each n consider the interval $I_n = [\frac{k}{2^j}, \frac{k+1}{2^j}) \subseteq [0, 1)$. We define

$$f_n = \mathbf{1}_{I_n};$$

and note that $\{I_n\}$ keeps "sweeping over" $[0,1)$ with finer and finer partitions, i.e., $I_1 = [0,1)$, $I_2 = [0, \frac{1}{2})$, $I_3 = [\frac{1}{2}, 1)$, $I_4 = [0, \frac{1}{4})$, $I_5 = [\frac{1}{4}, \frac{1}{2})$, etc. Clearly, then, $\lim_{n\to\infty} f_n(t)$ does *not* exist for any $t \in [0,1)$. On the other hand, if $p \in (0, \infty)$ then $\|f_n\|_{L^p(\mathbb{R})} = (1/2^j)^{1/p}$, and hence $\lim_{n\to\infty} \|f_n\|_{L^p(\mathbb{R})} = 0$ since $j = j(n) \to \infty$ as $n \to \infty$.

b. Generally, if $\lim_{n\to\infty}\|f_n\|_{L^\infty(\mathbb{R})}=0$ then $\lim_{n\to\infty} f_n = 0$ a.e.

c. There is a reasonable sort of converse to LDC which is due to F. Riesz: *if* $\{f_n\}$ *is a sequence of functions on* $\mathbb{R}$ *with the property that*

$$\text{(A.4)} \qquad \forall\epsilon > 0, \quad \lim_{n\to\infty} |\{t \in \mathbb{R} : |f_n(t)| \geq \epsilon\}| = 0,$$

then there is a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ *for which*

$$\lim_{k\to\infty} f_{n_k} = 0 \text{ a.e.}$$

Note that (A.4) holds if $\lim_{n\to\infty}\|f_n\|_{L^1(\mathbb{R})}=0$. If "$\lim f_n = 0$" in the sense of (A.4), we say that $\{f_n\}$ *converges in measure* to 0.

A.12 Theorem. FUBINI THEOREM (1907)

If $f(x,y)$ is integrable on the rectangle R defined by $a \leq x \leq b$, $c \leq y \leq d$, then the functions $x \mapsto f(x,y)$ and $y \mapsto f(x,y)$ are integrable for almost all values of y and x, respectively. Furthermore, the functions

$$y \longmapsto \int_a^b f(x,y)\,dx \quad \text{and} \quad x \longmapsto \int_c^d f(x,y)\,dy$$

are integrable and

$$\int_R f(x,y)\,dxdy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx,$$

where "dxdy" is so-called "product measure", e.g., [Rud66].

Tonelli's Theorem, which follows, can be viewed as a converse to Fubini's Theorem with the further hypothesis, $f \geq 0$. Equivalently, we can think of Tonelli's Theorem as giving conditions so that f is integrable, thereby allowing us to use Fubini's Theorem.

A.13 Theorem. TONELLI THEOREM (1909)

Let f be nonnegative on the rectangle R defined in Theorem A.12. Then

$$\int_R f(x,y)\,dxdy = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx.$$

These equations mean that if any one of the expressions is infinite then the other two are infinite, and if any one is finite then the other two are finite and all three are equal.

Fubini's and Tonelli's Theorems can be stated quite generally on spaces of the form $X \times Y$ instead of R and for measures other than Lebesgue measure. For convenience we shall refer to one or the other collectively as the *Fubini-Tonelli Theorem*; and we shall not discuss *product measure* or go into any of the subtleties relating measurable functions on $X \times Y$ with their restrictions to X or Y. For $X = Y = \mathbb{R}$, or for even more general sets X and Y, the Fubini-Tonelli Theorem is

A.14 Theorem. FUBINI-TONELLI THEOREM

a. (Tonelli). If $f : X \times Y \to \mathbb{C}$ and

$$\int_X \left(\int_Y |f(x,y)|\, dy \right) dx < \infty, \tag{A.5}$$

then f is integrable, i.e., $f \in L^1(X \times Y)$.

b. (Fubini). Let $f \in L^1(X \times Y)$, and define the function $f_x : Y \to \mathbb{C}$, resp., $f_y : X \to \mathbb{C}$, by

$$f_x(y) = f(x,y), \quad \textit{resp.}, \quad f_y(x) = f(x,y).$$

Then $f_x \in L^1(Y)$ a.e. in x, $f_y \in L^1(X)$ a.e. in y, the functions defined a.e. by

$$\int_Y f_x(y)\, dy \quad \textit{and} \quad \int_X f_y(x)\, dx$$

are in $L^1(X)$ and $L^1(Y)$, respectively, and

$$\begin{aligned} \int_{X \times Y} f(x,y)\, dx dy &= \int_Y \int_X f_y(x)\, dx dy \\ &= \int_X \int_Y f_x(y)\, dy dx. \end{aligned} \tag{A.6}$$

c. If $f : X \times Y \to \mathbb{C}$ and (A.5)is valid, then each of the integrals in (A.6) is finite, and (A.6) is valid.

A.15 Theorem. Hölder Inequality

If $1 \le p \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$, and if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$ and

$$\|fg\|_{L^1(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R})}\|g\|_{L^q(\mathbb{R})}. \tag{A.7}$$

If $1 < p < \infty$, there is equality in (A.7) if and only if there are constants $A, B \ge 0$, not both 0, such that $A|f|^p = B|g|^q$ a.e.

A.16 Theorem. Minkowski Inequality

Let $p \ge 1$ and let f be a complex-valued function defined on $\mathbb{R}\times\mathbb{R}$. Then

$$\left(\int \left|\int f(t,u)\,dt\right|^p du\right)^{1/p} \le \int \left(\int |f(t,u)|^p\,du\right)^{1/p} dt, \tag{A.8}$$

i.e., the L^p-norm of a "sum" is less than or equal to the "sum" of the L^p-norms.

A.17 Remark. Consequences of Hölder's and Minkowski's Inequalities

a. Minkowski's Inequality is true for many spaces X besides $\mathbb{R}$ and for measures, other than Lebesgue measure, on these spaces, cf., [HLP52, pages 146-150], [Zyg59, pages 18-19]. In particular, if

$$f_1, \ldots, f_n \in L^p(\mathbb{R})$$

then

$$\left\|\sum_{j=1}^{n} f_j\right\|_{L^p(\mathbb{R})} \le \sum_{j=1}^{n} \|f_j\|_{L^p(\mathbb{R})} \tag{A.9}$$

for $1 \le p < \infty$; and thus $L^p(\mathbb{R})$ is a vector space. It is easy to see that $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are vector spaces.

b. Because of Hölder's Inequality, if $1 \le p \le r \le \infty$ then

$$\forall X \subseteq \mathbb{R}, \text{ for which } |X| < \infty, \quad L^r(X) \subseteq L^p(X). \tag{A.10}$$

Using Fatou's Lemma and Minkowski's Inequality we can prove the following result for $1 \le p \le \infty$. The $p = \infty$ case is more elementary.

A.18 Theorem. COMPLETENESS OF L^p

Let $1 \leq p \leq \infty$. $L^p(\mathbb{R})$ is complete, i.e., every Cauchy sequence in $L^p(\mathbb{R})$ converges in L^p-norm to an element of $L^p(\mathbb{R})$. (Recall that $\{f_n\} \subseteq L^p(\mathbb{R})$ is a *Cauchy sequence* if

$$\forall \epsilon > 0\ \exists N, \quad \text{such that} \quad \forall m, n \geq N,\ \|f_m - f_n\|_{L^p(\mathbb{R})} < \epsilon.)$$

We defined bounded variation in *Definition 1.1.5*. The following notion is a special case of both continuity and bounded variation.

A.19 Definition. ABSOLUTE CONTINUITY

a. A function $F : [a, b] \to \mathbb{C}$ is *absolutely continuous* on $[a, b]$, written $F \in AC[a, b]$, if

$$\forall \epsilon > 0, \quad \exists \delta = \delta(\epsilon) \quad \text{such that} \quad \forall \{(x_j, y_j) \subseteq [a, b] : j = 1, \cdots, n\},$$

a finite disjoint family of intervals, we can conclude that

$$\sum_{j=1}^{n} (y_j - x_j) < \delta \quad \text{implies} \quad \sum_{j=1}^{n} |F(y_j) - F(x_j)| < \epsilon.$$

We define $AC(\mathbb{R})$ similarly.

b. A function $F : \mathbb{R} \to \mathbb{C}$ is *locally absolutely continuous*, written $F \in AC_{\text{loc}}(\mathbb{R})$, if $F \in AC[a, b]$ for each interval $[a, b] \subseteq \mathbb{R}$.

A.20 Theorem. FUNDAMENTAL THEOREM OF CALCULUS

a. Let $f \in L^1[a, b]$ and let $r \in \mathbb{R}$. Define

$$\forall t \in [a, b], \quad F(t) = r + \int_a^t f(u)\, du$$

so that $F(a) = r$. Then $F \in AC[a, b]$ and

$$F' = f \quad \text{a.e.}$$

b. A function $F : [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$ if and only if there is $f \in L^1[a, b]$ such that

$$\text{(A.11)} \qquad \forall t \in [a, b], \quad F(t) - F(a) = \int_a^t f(u)\, du.$$

A.21 Remark. PERSPECTIVE ON FTC

Theorem A.20 is denoted by FTC. *Theorem A.20a*, denoted by FTCI, asserts that *the derivative of the integral is the identity map. Theorem A.20b*, denoted by FTCII, shows the importance of absolute continuity and asserts that *the integral of the derivative is the identity map.*

If $F \in AC[a,b]$, then F' exists a.e., $F' \in L^1[a,b]$, and (A.11) asserts that

$$F(t) - F(a) = \int_a^t F'(u)\,du.$$

A.22 Theorem. INTEGRATION BY PARTS

Let $f, g \in L^1[a,b]$, let $r, s \in \mathbb{C}$, and define the (absolutely continuous) functions

$$F(t) = r + \int_a^t f(u)\,du \quad \text{and} \quad G(t) = s + \int_a^t g(u)\,du$$

on $[a,b]$. Then

$$\int_a^b f(t)G(t)\,dt + \int_a^b g(t)F(t)\,dt = F(b)G(b) - F(a)G(a).$$

Appendix B

Functional Analysis

This appendix lists results from functional analysis that are used in the book. There are many excellent texts and expositions including [Die81], [GG81], [Hor66], [RN55], [Rud73], and [Tay58].

B.1 Definition. COMPACT SET

A set $S \subseteq \mathbb{R}$ is compact if, whenever

$$S \subseteq \bigcup_{\alpha} N_\alpha$$

for a collection of open intervals N_α, there is a finite subcollection $\{N_{\alpha_1}, \ldots, N_{\alpha_n}\}$ for which

$$S \subseteq \bigcup_{j=1}^{n} N_{\alpha_j}.$$

This definition generalizes to topological spaces. In the case of $\mathbb{R}$, compact sets $S \subseteq \mathbb{R}$ are precisely the closed bounded subsets of $\mathbb{R}$, e.g., [Apo57, Chapter 3].

B.2 Definition. METRIC SPACE

a. A *metric space* is a nonempty set M and a function $\rho : M \times M \to [0, \infty)$ satisfying the following properties:

$$\begin{array}{ll} \forall x, y \in M, & \rho(x,y) \geq 0, \\ \forall x, y \in M, & \rho(x,y) = 0 \quad \text{if and only if} \quad x = y, \\ \forall x, y \in M, & \rho(x,y) = \rho(y,x), \\ \forall x, y, z \in M, & \rho(x,z) \leq \rho(x,y) + \rho(y,z). \end{array}$$

The last property is the *triangle inequality*, and the function ρ is a *metric*. An excellent reference for metric spaces is [Gle91].

b. A sequence $\{x_n : n = 1, \ldots\}$, contained in a metric space M (with metric ρ), is a *Cauchy sequence* if

$$\forall \epsilon > 0 \ \ \exists N \ \text{ such that } \ \forall m, n > N, \quad \rho(x_m, x_n) < \epsilon.$$

If M is a metric space in which every Cauchy sequence $\{x_n\}$ converges to some element $x \in M$, i.e., $\lim_{n\to\infty} \rho(x_n, x) = 0$, then M is a *complete metric space.*

c. Two metric spaces M_1 and M_2, with metrics ρ_1 and ρ_2, respectively, are *isometric* if there is a bijection $f : M_1 \to M_2$ such that

$$\forall x, y \in M_1, \quad \rho_1(x, y) = \rho_2(f(x), f(y)).$$

In this case, f is an *isometry*. (Recall that a function $f : X_1 \to X_2$ is *bijective* if it is *injective* and *surjective*, i.e., if it is one-to-one and onto.)

d. Let M be a metric space with metric ρ. A subset $V \subseteq M$ is *closed* if, whenever $\{x_n\} \subseteq V$ and $\lim_{n\to\infty} \rho(x_n, x) = 0$ for some $x \in M$, we can conclude that $x \in V$. The *closure* $\overline{X}$ of a subset $X \subseteq M$ is the set of all elements $x \in M$ for which there is a sequence $\{x_n\} \subseteq X$ such that $\lim_{n\to\infty} \rho(x_n, x) = 0$.

B.3 Definition. BANACH SPACE

a. A vector space $B \neq \{0\}$ over $\mathbb{C}$ is a *normed vector space* if there is a function $\|\ldots\| : B \longrightarrow [0, \infty)$ such that

$$\begin{aligned} &\forall x \in B, && \|x\| = 0 \quad \text{if and only if} \quad x = 0, \\ &\forall x, y \in B, && \|x + y\| \leq \|x\| + \|y\|, \\ &\forall c \in \mathbb{C} \ \text{ and } \ \forall x \in B, && \|cx\| = |c| \, \|x\|. \end{aligned}$$

The function $\|\ldots\|$ is a *norm.*

b. A normed vector space is a metric space with metric ρ defined by $\rho(x, y) = \|x - y\|$. A complete normed vector space is a *Banach space.*

Let $1 \le p \le \infty$. $L^p(\mathbb{R})$, with L^p-norm defined in *Definition A.10*, is a Banach space. This is a restatement of *Theorem A.18.*

c. Let B be a normed vector space and let $\{x_n : n = 1, \ldots\} \subseteq B$. $\sum x_n$ *converges* to $x \in B$ if $\lim_{n\to\infty} \|x - \sum_{j=1}^{n} x_j\| = 0$; and $\sum x_n$ is *absolutely convergent* if $\sum \|x_n\| < \infty$. It is straightforward to prove that *a normed vector space is a Banach space if and only if every absolutely convergent series is convergent*, e.g., [Ben76, page 232].

B.4 Definition. HILBERT SPACE

a. A *Hilbert space* $H \neq \{0\}$ is a Banach space for which there is a function $\langle \ldots, \ldots \rangle : H \times H \longrightarrow \mathbb{C}$ such that

$$\begin{aligned}
&\forall x, y \in H, && \overline{\langle x, y\rangle} &&= \langle y, x\rangle,\\
&\forall x, y, z \in H && \langle x + y, z\rangle &&= \langle x, z\rangle + \langle y, z\rangle,\\
&\forall c \in \mathbb{C} \text{ and } \forall x, y \in H, && \langle cx, y\rangle &&= c\langle x, y\rangle,\\
&\forall x \in H, && \|x\| &&= \langle x, x\rangle^{1/2},
\end{aligned}$$

where $\|\ldots\|$ is the Banach space norm. $\langle \ldots, \ldots \rangle$ is an *inner product.*

b. $L^2(\mathbb{R})$ is a Hilbert space with inner product defined by $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$, cf., [Ben76, Example I.2.4, page 234] for a structural converse of the form that every Hilbert space is some type of L^2 space. The integral is well-defined by Hölder's Inequality (*Theorem A.15*).

c. If H is a Hilbert space (actually, completeness is not required) then the properties of part *a* can be used to give *simple* proofs of the following facts:

$$|\langle x, y\rangle| \le \|x\|\,\|y\| \tag{B.1}$$

and

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{B.2}$$

(B.1) is the *Schwarz Inequality*, which in the case $H = L^2(\mathbb{R})$ is *Hölder's Inequality*. Of course, this does not mean there is a simple proof of Hölder's Inequality since Schwarz' Inequality *assumes* the existence of an inner product, and Hölder's Inequality *shows* the existence of an inner product for $H = L^2(\mathbb{R})$.

(B.2) is the *parallelogram law.* It can be shown that *a Banach space is a Hilbert space*, i.e., there is an inner product with the required properties, *if and only if the parallelogram law is valid.*

B.5 Theorem. MOORE-SMITH THEOREM

Let M be a complete metric space with metric ρ, and let $\{x_{m,n} : m, n \in \mathbb{N}\}$ be given. Assume there are sequences $\{y_m\}, \{z_n\} \subseteq M$ such that

roman 'uniformly'?

$$\lim_{n\to\infty} \rho(x_{m,n}, y_m) = 0 \text{ uniformly in } m$$

and

$$\forall n \in \mathbb{N}, \lim_{m\to\infty} \rho(x_{m,n}, z_n) = 0.$$

Then there is $x \in M$ such that the limits,

$$\lim_{m\to\infty}\lim_{n\to\infty} \rho(x_{m,n}, x), \ \lim_{n\to\infty}\lim_{m\to\infty} \rho(x_{m,n}, x), \ \textit{and} \ \lim_{m,n\to\infty} \rho(x_{m,n}, x),$$

all exist and are equal to 0*. The last limit signifies that*

$$\forall \epsilon > 0, \ \exists N > 0, \ \text{ such that } \ \forall m, n > N,$$
$$\rho(x_{m,n}, x) < \epsilon.$$

The Moore-Smith Theorem is elementary to prove, e.g., [Ben76, pages 235–237], and should be compared with LDC and its refinements, e.g., [Ben76, Chapter 6].

B.6 Definition. CONTINUOUS LINEAR FUNCTIONS

a. Let B_1 and B_2 be non-zero normed vector spaces with norms $\|\ldots\|_{B_1}$ and $\|\ldots\|_{B_2}$, respectively. The *norm* of a linear function $L : B_1 \to B_2$ is

$$\|L\| = \sup\{\|Lx\|_{B_2} : x \in B_1 \text{ and } \|x\|_{B_1} \le 1\}. \tag{B.3}$$

Thus, $\|L\|$ is the smallest constant $C \ge 0$ such that $\|Lx\|_{B_2} \le C\|x\|_{B_1}$ for all $x \in B_1$.

b. Clearly, *if* $\|L\| < \infty$ *then* L *is continuous.* The space of continuous linear functions $L : B_1 \to B_2$ is denoted by $\mathcal{L}(B_1, B_2)$. It is not difficult to show that *if* B_2 *is a Banach space then* $\mathcal{L}(B_1, B_2)$ *is a Banach space, where* $L = c_1L_1 + c_2L_2$ *is defined by* $Lx = c_1L_1(x) + c_2L_2(x)$, *and where* $\|L\|$ *is defined by* (B.3).

If $B = B_1 = B_2$ we denote $\mathcal{L}(B, B)$ by $\mathcal{L}(B)$. If $B = B_1$ and $B_2 = \mathbb{C}$ we denote $\mathcal{L}(B, \mathbb{C})$ by B'. B' is the space of *continuous linear functionals* on B and is called the *dual space* of B.

c. Let B_1 and B_2 be Banach spaces, with duals B_1' and B_2', and let $L \in \mathcal{L}(B_1, B_2)$. The *adjoint* of L is the linear function

$$\begin{aligned} L^* : B_2' &\longrightarrow B_1' \\ y &\longmapsto L^*y \end{aligned}$$

where $(L^*y)(x)$ is defined to be $y(Lx)$ for all $x \in B_1$. It is easy to see that $L^* \in \mathcal{L}(B_2', B_1')$.

In the case of Hilbert spaces H_1 and H_2, we write

$$\text{(B.4)} \qquad \forall x \in H_1 \text{ and } \forall y \in H_2, \quad \langle Lx, y\rangle_{H_2} = \langle x, L^*y\rangle_{H_1}$$

to define the adjoint. In (B.4), we have used the fact that Hilbert spaces H are *reflexive*, i.e., $H' = H$, cf., *Theorem B.14b.*

Let M_1 and M_2 be metric spaces with metrics ρ_1 and ρ_2, respectively. A function $f : M_1 \to M_2$ is *uniformly continuous* if

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ such that } \rho_1(x, y) < \delta \text{ implies } \rho_2(f(x), f(y)) < \epsilon.$$

B.7 Theorem. UNIFORMLY CONTINUOUS EXTENSIONS

a. Let M_1 *be a metric space, let* M_2 *be a complete metric space, and let* V *be a subset of* M_1. *Assume that* $f : V \to M_2$ *is uniformly continuous. Then* f *has a unique uniformly continuous extension to* $\overline{V} \subseteq M_1$.

b. Let B_1 *and* B_2 *be Banach spaces, and let* $L \in \mathcal{L}(B_1, B_2)$. *Then* L *is uniformly continuous.*

c. Let B_1 *and* B_2 *be Banach spaces, let* V *be a subspace of* B_1. *If* $L : V \to B_2$ *is a continuous linear function, then* L *has a unique continuous linear extension to* $\overline{V}$.

The following result is also called the *Banach-Steinhaus Theorem.*

B.8 Theorem. UNIFORM BOUNDEDNESS PRINCIPLE

a. Let M be a complete metric space with metric ρ, and let $\mathcal{C}$ be a set of continuous functions $f : M \to \mathbb{C}$. Assume

$$\forall x \in M, \quad \exists C_x > 0 \ \text{ such that } \ \forall f \in \mathcal{C}, \quad |f(x)| \leq C_x.$$

Then there is a constant $C > 0$ and a nonempty open ball U, i.e., $U = \{x \in M : \rho(x, x_0) < r\}$ for some $x_0 \in M$ and $r > 0$, such that

$$\forall x \in U \ \text{ and } \ \forall f \in \mathcal{C}, \quad |f(x)| \leq C.$$

b. Let B_1 be a Banach space, let B_2 be a normed vector space, and let $\mathcal{C} \subseteq \mathcal{L}(B_1, B_2)$. Then one of the following is true:

i. $\exists C$ such that $\forall L \in \mathcal{C}, \quad \|L\| \leq C$,

or

ii. There is a nonempty set $V \subseteq B_1$ such that $\overline{V} = B_1$ and

$$\forall x \in V, \quad \sup_{L \in \mathcal{C}} \|Lx\| = \infty.$$

(V is also the intersection of a countable family of open sets.)

c. Let B_1 be a Banach space, let B_2 be a normed vector space, and let $\{L_n\} \subseteq \mathcal{L}(B_1, B_2)$ have the property that

$$\forall x \in B_1, \quad \exists Lx \in B_2 \ \text{ such that } \ \lim_{n \to \infty} \|L_n x - Lx\| = 0.$$

Then $L \in \mathcal{L}(B_1, B_2)$.

B.9 Theorem. OPEN MAPPING THEOREM

Let B_1 and B_2 be Banach spaces and let $L \in \mathcal{L}(B_1, B_2)$ be bijective. Then L^{-1} exists and $L^{-1} \in \mathcal{L}(B_2, B_1)$.

B.10 Remark. BAIRE CATEGORY THEOREM

The Uniform Boundedness Principle and the Open Mapping Theorem both depend on completeness. In particular, they can be proved using the *Baire Category Theorem* which asserts that *if M is a complete metric space then the intersection of any countable family of dense open subsets of M is dense in M.*

Combining the Open Mapping Theorem with Alaoglu's Theorem [Ben76, pages 246–247], we have the following result.

B.11 Theorem. INJECTIVITY, SURJECTIVITY, AND CONTINUITY FOR L, L^*, L^{-1}, AND $(L^*)^{-1}$

Let B_1 and B_2 be normed vector spaces with dual Banach spaces B_1' and B_2' , and let $L \in \mathcal{L}(B_1, B_2)$.

a. $L^*(B_2') = B_1'$ if and only if L^{-1} exists and $L^{-1} \in \mathcal{L}(L(B_1), B_1)$.

b. Let B_1 and B_2 be Banach spaces. $L(B_1) = B_2$ if and only if $(L^*)^{-1}$ exists and $(L^*)^{-1} \in \mathcal{L}(L^*(B_2'), B_2')$. Further, if L^{-1} exists, then it is in $\mathcal{L}(L(B_1), B_1)$.

c. Let B_1 and B_2 be Banach spaces, let L be injective, and assume $\overline{L(B_1)} = B_2$. The following are equivalent:

i. $L(B_1) = B_2$;

ii. There is $C > 0$ such that for all $y \in B_2'$, $\|y\|_{B_2'} \leq C\|L^*y\|_{B_1'}$, i.e., L^* is an open mapping;

iii. $L^*(B_2') = B_1'$.

Let B_1 and B_2 be vector spaces, let $V \subseteq B_1$ be a subspace, and let $K : V \to B_2$ be a linear function. Using the Axiom of Choice, it can be shown that there is a linear function $L : B_1 \to B_2$ for which $Lx = Kx$ for all $x \in V$, e.g., [Tay58, pages 40–41]. If B_1 is a normed vector space, $B_2 = \mathbb{C}$, and K is continuous, then the following theorem provides a continuous norm preserving extension $L : B_1 \to \mathbb{C}$ of K, e.g., [Ben76, pages 244–246]. Completeness is not required as it is for *Theorems B.8* and *B.9*.

B.12 Theorem. HAHN-BANACH THEOREM

Let $V \subseteq B$ be a subspace of the normed vector space B. If $K : V \to \mathbb{C}$ is linear and continuous, then there is $L \in B'$ such that $L = K$ on V and $\|L\| = \|K\|$.

B.13 Corollary. EXISTENCE OF CONTINUOUS LINEAR FUNCTIONALS

Let B be a normed vector space.

a. If $x, y \in B$ and $x \neq y$, then there is $L \in B'$ such that $Lx \neq Ly$.

b. Let V be a closed subspace of B and let $y \in B \backslash V$. There is $L \in B'$ such that $Ly = 1$ and $Lx = 0$ for all $x \in V$.

Proof. Let V be the subspace of B generated by $x - y$, and define a linear function K on V by the rule $K(c(x-y)) \equiv c\|x-y\|$ for $c \in \mathbb{C}$. Note that $\|K\| = 1$ so that K is continuous. Apply *Theorem B.12* to obtain an extension $L \in B'$. Then $0 < \|x-y\| = L(x-y) = L(x)-L(y)$. This completes part *a*.

b. Let $d > 0$ have the property that $\|x-y\| \geq d$ for all $x \in V$, and let $V_y = \{x + cy : x \in V \text{ and } c \in \mathbb{C}\}$. Define the linear function K on V_y by the rule $K(x+cy) = c$. Thus, $K = 0$ on V and $K(y) = 1$. Also, K is continuous on V_y since $|K(x+cy)| = d^{-1}|c|d \leq d^{-1}|c|\|c^{-1}x+y\| = d^{-1}\|x+cy\|$. The result follows from *Theorem B.12* . □

B.14 Theorem. L^p-DUALITY THEOREM

a. $L^1(\mathbb{R})' = L^\infty(\mathbb{R})$*, where* $g : L^1(\mathbb{R}) \to \mathbb{C}$ *is well-defined by* $g(f) = \int g(t)f(t)\,dt$ *for all* $f \in L^1(\mathbb{R})$.

b. Let $1 < p < \infty$ *and define* p' *by* $\frac{1}{p}+\frac{1}{p'} = 1$*. Then* $L^p(\mathbb{R})' = L^{p'}(\mathbb{R})$*, where* $g : L^p(\mathbb{R}) \to \mathbb{C}$ *is well-defined by* $g(f) = \int g(t)f(t)dt$ *for all* $f \in L^p(\mathbb{R})$*. In particular, the Hilbert space* $H = L^2(\mathbb{R})$ *has the property that* $L^2(\mathbb{R})' = L^2(\mathbb{R})$.

c. $L^\infty(\mathbb{R})'$ *is the space of finitely additive bounded measures on the Borel algebra* $\mathcal{B}(\mathbb{R})$*, e.g.,* Chapter IV of N. Dunford and J. T. Schwartz, Linear Operators, Part I, John Wiley and Sons, New York, 1957. ($\mathcal{B}(\mathbb{R})$ was defined in *Remark 2.7.4*.)

italics ?

B.15 Remark. RIESZ, HELLY, AND HAHN-BANACH THEOREMS

a. F. Riesz introduced the L^p spaces, $p \neq 2$, in 1910, and proved the L^p-Duality Theorem (*Theorem B.14b*). Using a complicated argument with Lagrange multipliers, he then proved the following theorem. *Let* $p > 1$*, and let* $\{f_n\} \subseteq L^p(\mathbb{R})$ *and* $\{c_n\} \subseteq \mathbb{C}$ *be given sequences; then*

$$\text{(B.5)} \qquad \exists f \in L^{p'}(\mathbb{R}) \text{ such that } \forall n, \quad c_n = \int f(t)f_n(t)dt$$

if and only if there is $C > 0$ *such that for all finite sequences* $\{\lambda_n\} \subseteq \mathbb{C}$,

$$\text{(B.6)} \qquad |\sum \lambda_n c_n| \leq C\|\sum \lambda_n f_n\|_{L^p(\mathbb{R})}.$$

The proof that (B.5) implies (B.6) is clear. If $p = 2$ and $\{f_n\}$ is orthonormal then Riesz' Theorem is equivalent to the Riesz-Fischer Theorem (*Remark 3.4.11c*).

b. In 1912, the Austrian mathematician E. Helly (1884–1943) (who spent his last years in the USA as an actuary) gave an elementary proof that (B.6) is a sufficient condition for the validity of (B.5). In so doing, he proved a result which is *equivalent* to the Hahn–Banach Theorem [Die81].

c. The proof of (B.5) from (B.6) is a consequence of the Hahn-Banach Theorem as follows. Assume (B.6) and let $V = \text{span}\{f_n\} \subseteq L^p(\mathbb{R})$. (The *span* of a set X contained in a vector space B is the set of all finite linear combinations $x \equiv \sum \lambda_n x_n$, where $\lambda_n \in \mathbb{C}$ and $x_n \in X$.) Define $K : V \to \mathbb{C}$ by the rule that $K(\sum \lambda_n f_n) = \sum \lambda_n c_n$ for any finite sequence $\{\lambda_n\} \subseteq \mathbb{C}$. Clearly, K is linear, and it is continuous on V by (B.6). Thus, by *Theorem B.12*, there is $L \in L^{p'}(\mathbb{R})$ for which $L = K$ on V. Hence, $c_n = f(f_n) = \int f(t) f_n(t)dt$ for each n, where we have denoted L by f.

B.16 Remark. EMBEDDINGS OF DUAL SPACES

a. Let $B_1 \subseteq B_2$, where B_1 and B_2 are normed vector spaces and let $I : B_1 \to B_2$ be the identity mapping with adjoint $I^* : B_2' \to B_1'$. By definition,

$$\forall x \in B_1 \text{ and } \forall y \in B_2', \quad (I^* y)(x) = y(x),$$

i.e., $I^* y = y$ on $B_1 \subseteq B_2$.

b. Assume I, and hence I^*, are continuous. Note that *if $y \in B_2'$ then $y|_{B_1}$, the restriction of y to B_1, is an element of B_1'.* To see this first note that since $B_1 \subseteq B_2$ and $y \in B_2'$, then $y|_{B_1}$ is linear on B_1. $y|_{B_1}$ is also continuous on B_1 because of the continuity of I. In fact, since y is continuous on B_1 with the induced topology from B_2, then it is continuous on B_1 with its given norm convergence because this latter topology is stronger (finer) than the B_2 criterion. (Continuity of a function for a given topology on its domain implies continuity for any stronger topology on that domain.)

c. To say that B_2' is *embedded* in B_1', in which case we write $B_2' \subseteq B_1'$, we mean that I^* is a continuous injection. This means that whenever $I^* y = 0 \in B_1'$ then $y = 0$, i.e., $y(x) = 0$ for all $x \in B_2$.

d. In the setting of parts *a* and *b*, we further assume $\overline{B}_1 = B_2$. Let $y \in B_2'$ have the property that $I^* y = 0 \in B_1'$. Suppose $x \in B_2$ and

$\lim_{n\to\infty} \|x_n - x\|_{B_2} = 0$, where $\{x_n\} \subseteq B_1$. Then $\lim_{n\to\infty} y(x_n) = y(x)$, and $y(x_n) = (I^*y)(x_n) = 0$. Thus, $y(x) = 0$, and so $y \in B_2'$ is the 0-element. Hence, I^* is a continuous injection. I^* is also the identity function, i.e., for all $y \in B_2'$, $I^*y = y$ on a dense subspace of B_2.

e. We can summarize parts *a–d* by the following *Embedding Theorem: let* B_1 *and* B_2 *be normed vector spaces; if* $B_1 \subseteq B_2$ *in the sense that the identity map* $I : B_1 \to B_2$ *is continuous, and if* $\overline{B}_1 = B_2$, *then* $B_2' \subseteq B_1'$. Several embeddings are listed in *Definition 2.4.5c* and *Example 2.4.6b,d,f*. Some of these involve topological structures more general than Banach spaces. This generality does not give rise to any problems, since the Embedding Theorem is true quite generally.

B.17 Example. $C_b(\mathbb{R}), M_b(\mathbb{R})$, AND DUALITY

a. Let $C_b(\mathbb{R})$ be the Banach space of continuous bounded functions on $\mathbb{R}$ taken with the L^∞-norm $\|\ldots\|_{L^\infty(\mathbb{R})}$; and let $C_0(\mathbb{R})$ be the closed subspace of $C_b(\mathbb{R})$ whose elements f satisfy the condition that $\lim_{|t|\to\infty} f(t) = 0$. Recall from *Theorem 2.7.3* (RRT) that $C_0(\mathbb{R})' = M_b(\mathbb{R})$. $C_b(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$, and $C_b(\mathbb{R})'$ is the space of finitely additive bounded regular measures on the Borel algebra $\mathcal{B}(\mathbb{R})$, cf., *Theorem B.14c* and the reference given there.

b. Since the continuous inclusion map $I : C_0(\mathbb{R}) \to C_b(\mathbb{R})$ is not dense, we can *not* conclude that $C_b(\mathbb{R})' \subseteq M_b(\mathbb{R})$, as is apparent from the characterizations of $C_0(\mathbb{R})'$ and $C_b(\mathbb{R})'$ in part *a*, cf., *Remark B.16.*

c. The characterizations of $C_0(\mathbb{R})'$ and $C_b(\mathbb{R})'$ in part *a do* imply that

$$M_b(\mathbb{R}) \subseteq C_b(\mathbb{R})'. \tag{B.7}$$

In this regard, if $\mu \in C_0(\mathbb{R})'$ then μ extends to an element $\mu_e \in C_b(\mathbb{R})'$ by the Hahn-Banach Theorem. Of course, there is no apriori guarantee of a unique extension.

On the other hand, and without invoking the characterization of $C_b(\mathbb{R})'$ stated in part *a*, we can see the validity of (B.7) in the following way.

If $f \in C_b(\mathbb{R})$ we can choose $\{f_n\} \subseteq C_0(\mathbb{R})$ for which $\lim_{n\to\infty} f_n = f$ pointwise on $\mathbb{R}$ and $\sup_n \|f_n\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})} < \infty$. Then we apply a form of LDC for $L^1_\mu(\mathbb{R})$ which allows us to assert that $f \in L^1_{|\mu|}(\mathbb{R})$ and $\lim_{n\to\infty} \|f_n - f\|_{L^1_{|\mu|}(\mathbb{R})} = 0$, e.g., [Ben76, Chapter 3]. The integral $\mu(f)$ is well-defined, i.e., it is independent of the sequence $\{f_n\} \subseteq C_0(\mathbb{R})$. Further, $\mu : C_b(\mathbb{R}) \to \mathbb{C}$ is linear. To prove the continuity of μ, let $f \in C_b(\mathbb{R})$, let $\epsilon > 0$, and choose $\{f_n\}$ as above. Then

$$\exists N > 0 \text{ such that } \forall n \geq N, \quad |\mu(f - f_n)| < \epsilon;$$

and so, for such n,

$$|\mu(f)| \leq \epsilon + |\mu(f_n)| \leq \epsilon + \|\mu\| \|f\|_{L^\infty(\mathbb{R})}.$$

This is true for all $\epsilon > 0$, and so $\mu \in C_b(\mathbb{R})'$. We designate μ so defined on $C_b(\mathbb{R})$ by μ^*.

The inclusion (B.7) is accomplished by the mapping $\mu \mapsto \mu^*$. The fact that many extensions μ_e of μ exist does not contradict (B.7). In fact, $\nu_e \equiv \mu_e - \mu^* \in C_b(\mathbb{R})'$ vanishes on $C_0(\mathbb{R})$; and if ν_e is not identically 0 on $C_b(\mathbb{R})$, then μ_e is not countably additive on $\mathcal{B}(\mathbb{R})$ and so it does not correspond to an element of $M_b(\mathbb{R})$.

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Appendix C

Fourier Analysis Formulas

$$f \longleftrightarrow F,$$

$$F(\gamma) = \widehat{f}(\gamma) = \int f(t)e^{-2\pi i t\gamma}\,dt, \quad f(t) = \int F(\gamma)e^{2\pi i t\gamma}\,d\gamma.$$

$$f_\lambda(t) = \lambda f(\lambda t), \quad f_\lambda(t) \longleftrightarrow \frac{\lambda}{|\lambda|}F\left(\frac{\gamma}{\lambda}\right), \quad \lambda \in \mathbb{R}\backslash\{0\}.$$

$$f \longleftrightarrow F \text{ implies } F(t) \longleftrightarrow f(-\gamma).$$

$$\tau_u f(t) = F(t-u), \quad (\tau_u f) \longleftrightarrow e^{-2\pi i u\gamma}\,F(\gamma), \quad e^{2\pi i t\gamma_0}f(t) \longleftrightarrow F(\gamma-\gamma_0).$$

$$f_1 * f_2 \longleftrightarrow F_1F_2, \quad f_1f_2 \longleftrightarrow F_1 * F_2.$$

$$\int f_1(t)f_2(t)\,dt = \int F_1(-\gamma)F_2(\gamma)\,d\gamma, \quad \int f_1(t)\overline{f_2(t)}\,dt = \int F_1(\gamma)\overline{F_2(\gamma)}\,d\gamma,$$

$$\|f\|_{L^2(\mathbb{R})} = \|F\|_{L^2(\widehat{\mathbb{R}})}.$$

$$f'(t) \longleftrightarrow 2\pi i\gamma F(\gamma), \quad f^{(n)}(t) \longleftrightarrow (2\pi i\gamma)^n F(\gamma),$$

$$-2\pi i t f(t) \longleftrightarrow F'(\gamma), \quad (-2\pi i t)^n f(t) \longleftrightarrow F^{(n)}(\gamma).$$

$$\int f(at-b)e^{-2\pi i t\gamma}\,dt = \tfrac{1}{a}e^{-2\pi i(b/a)\gamma}F\left(\tfrac{\gamma}{a}\right), \quad a>0.$$

Poisson Summation Formula

a. $\lambda\sum f(\lambda j) = \sum F(n/\lambda), \quad \lambda>0,$

b. $\left(\sum \delta_{nT}\right)^\wedge = (1/T)\sum \delta_{n/T}, \quad T>0.$

Dirichlet $d(t) = \dfrac{\sin t}{\pi t}$, **Fejér** $w(t) = \dfrac{1}{2\pi}\left(\dfrac{\sin(t/2)}{t/2}\right)^2$,

Gauss $g(t) = \dfrac{1}{\sqrt{\pi}}e^{-t^2}$, **Poisson** $p(t) = \dfrac{1}{\pi(1+t^2)}$,

Triangle $\Delta(t) = \max(1-|t|, 0)$.

$$\int d_\lambda(t)\,dt = \int w_\lambda(t)\,dt = \int g_\lambda(t)\,dt = \int p_\lambda(t)\,dt = 1,$$

$$w_\lambda(t) = \frac{2\pi}{\lambda}d_{\lambda/2}(t)^2.$$

$$\mathbf{1}_{[-T,T)} \longleftrightarrow d_{2\pi T}, \qquad d_\lambda \longleftrightarrow \mathbf{1}_{[-\lambda/(2\pi),\lambda/(2\pi))},$$

$$T\Delta_{1/T} \longleftrightarrow w_{2\pi T}, \qquad w_\lambda \longleftrightarrow \max\left(1-\tfrac{|2\pi\gamma|}{\lambda}, 0\right),$$

$$e^{-2\pi r|t|} \longleftrightarrow p_{1/r}, \qquad p_\lambda \longleftrightarrow e^{-2\pi|\gamma|/\lambda},$$

$$e^{-\pi r t^2} \longleftrightarrow g_{\sqrt{\pi/r}}, \qquad g_\lambda \longleftrightarrow e^{-(\pi\gamma/\lambda)^2},$$

$$T, \lambda, r > 0.$$

Heaviside Function H:

a. $H(t) \longleftrightarrow \frac{1}{2\pi i} pv\left(\frac{1}{\gamma}\right) + \frac{1}{2}\delta(\gamma),$

b. $\dfrac{1}{2}\delta(t) - \dfrac{1}{2\pi i} pv\left(\frac{1}{t}\right) \longleftrightarrow H(\gamma),$

c. $H(t) - \frac{1}{2} \longleftrightarrow \dfrac{1}{2\pi i} pv\left(\frac{1}{\gamma}\right).$

Appendix D

Contributors to Fourier Analysis

Abel, Niels Henrik	1802–1829	Norway
Akhiezer (Achieser), Naum Ilich	1901–1980	Ukraine
Alembert, Jean Le Rond D'	1717–1783	France
Archimedes	c. 287–212B.C.	Greece (Syracusa, Sicilia, Italy)
Arzelá, Cesare	1847–1912	Italy
Ascoli, Giulio	1843–1896	Italy

Obviously this is an incomplete list of past contributors to Fourier analysis.

With the complexity of history's chaotic paths through time, it is impossible to draw any serious conclusions from nationalistic labelling. Our listing of countries should be read in this spirit, viz., as a superficial organizational device. Further, many contributors made significant contributions in several countries, e.g., Erdélyi, Euler, Pólya, and Reiter. Others have a labyrinthine heritage, e.g., Cantor. Dirichlet, one of the protagonists of this book, succeeded Gauss at Göttingen; and was, indeed, German born (at Düren between Aachen and Köln). On the other hand, his grandfather lived in Verviers, Belgium; and several generations of the Dirichlet family lived in the Province of Liège. In fact, the family name originated from the Belgian village of Richelette, not so far from Maastricht in The Netherlands.

Babenko, Konstantin Ivanovich	1919–1987	Russia
Banach, Stefan	1892–1945	Poland
Bary, Nina Karlovna	1901–1962	Russia
Bernoulli, Daniel	1700–1782	Switzerland
Bernstein, Sergei Natanovich	1880–1968	Russia
Besicovitch, Abram Samoilovitch	1891–1970	Berdjansk, Russia, England
Bessel, Friedrich Wilhelm	1784–1846	Germany
Beurling, Arne Karl-August	1905–1986	Sweden
Boas, Ralph Philip	1912–1992	USA
Bôcher, Maxime	1867–1918	USA
Bochner, Salomon	1899–1982	USA
Bois-Reymond, Paul David Gustav du	1831–1889	Germany
Bohr, Harald	1887–1951	Denmark
Bolzano, Bernhard	1781–1848	Czechoslovakia (Bohemia)
Bosanquet, Lancelot Stephen	1903–1984	England
Bromwich, Thomas John I'Anson	1875–1929	England
Burkill, John Charles		England
Cantor, Georg Ferdinand Ludwig Philip	1845–1918	Germany
Carleman, Torsten	1892–1949	Sweden
Carlini, Francesco	1783–1862	Italy
Carlson, Fritz David	1888-1952	Sweden
Carslaw, Horatio Scott	1870–1954	Scotland, Australia
Cauchy, Augustin Louis	1789–1857	France
Cesàro, Ernesto	1859–1906	Italy
Cooper, Jacob Lionel Bakst	1915–1979	South Africa England
Cornu, Marie Alfred	1841–1902	France
Cramér, Harald	1893–1985	Sweden
De Moivre, Abraham	1667–1754	France, England
De Morgan, Augustus	1806–1871	India (Madura), England
Darboux, Jean Gaston	1842–1917	France
Dedekind, Julius Wilhelm Richard	1831–1916	Germany
Denjoy, Arnaud	1884–	France

66 Franklin, Philip -1965 USA

Dini, Ulisse	1845–1918	Italy
Dirichlet, Johann Peter Gustav Lejeune (Lejeune-Dirichlet)	1805–1859	Germany
Ditkin, V.		
Doetsch, Gustav	1892–1977	Germany
Doss, Raouff	–1988	USA — one f
Eberlein, William F.	1917–1986	USA
Erdélyi, Arthur	1908–1977	Hungary, Scotland, USA, Scotland
Euler, Leonhard	1707-1783	Switzerland, Russia
Fatou, Pierre Joseph Louis	1878–1929	France
Fejér, Lipót (b. Weisz. In Hungarian, "white" is "fehér")	1880–1959	Hungary
Fermat, Pierre de	1601–1665	France
Fischer, Ernst	1875–1954	Germany
Flett, Thomas Muirhead	1923–1976	England
Fourier, Jean Baptiste Joseph	1768–1830	France
Fréchet, Maurice René	1878–1973	France
Fredholm, Erik Ivar	1866–1927	Sweden
Fresnel, Augustin	1788–1827	France
Fubini, Guido	1879–1943	Italy
Gabor, Dennis	1900-1979	Hungary, England
Gauss, Carl Friedrich	1777–1855	Germany
Gibbs, Josiah Willard	1839–1903	USA
Glicksberg, Irving Leonard	1925–	USA
Green, George F.	1793–1841	England
Hankel, Hermann	1839–1873	Germany
Hardy, Godfrey Harold	1877–1947	England
Hausdorff, Felix	1868–1942	Germany
Heaviside, Oliver	1850–1925	England
Heine, Heinrich Eduard	1821–1881	Germany
Heisenberg, Werner	1901–1976	Germany
Herglotz, Gustav	1881–1953	Germany
Hermite, Charles	1822–1905	France
Herrero, Domingo A.	–1991	USA

Karamata, Jean -1967 Switzerland

Herz, Carl Samuel	1930–1995	USA, Canad
Hilbert, David	1862–1943	Germany
Hille, Einar	1894–1980	Sweden, US
Hirschman, Jr., Isadore Isaac	1922–1990	USA
Hobson, Ernest William	1856–1933	England
Hölder, (Ludwig) Otto	1859–1937	Germany
Ingham, Albert Edward	1900-1967	England
d? ~~Izumi, S.~~		~~Japan~~
Jackson, Dunham	1888–1946	USA
Jacobi, Karl Gustav Jacob	1804–1851	Germany
Jensen, Johan Ludvig William Valdemar	1859–1925	Denmark
Jordan, (Marie-Ennemond) Camille	1838–1922	France
Kac, Mark	1914–1984	Poland, USA
Lord Kelvin (William Thomson)	1824–1907	Ireland, Scot and
Kennedy, Patrick Brendan	1929–1966	Ireland
Kirchhoff, Gustav Robert	1824–1887	Germany
Kluvánek, Igor	1931–1993	Slovakia, Australia
Kolmogorov, Andrei Nikolaevich	1903–1987	Russia
Köthe, Gottfried	1905–1989	Austria, Germany
d? Kozlov, V. Ya		Russia
Krein, Mark Grigorivich	1907–1989	Odessa (Ukraine)
Kronecker, Leopold	1823–1891	Germany
d? Kuttner, B.		
Kuzmin, Rodion Osieviz	1891–1949	Russia
Lagrange, Joseph-Louis (Lagrangia)	1736–1813	Italy (Torino), France
Landau, Edmund	1877–1938	Germany
Laplace, Pierre Simon de	1749–1827	France
Legendre, Adrien Marie	1752–1833	France
Lebesgue, Henri-Léon	1875–1941	France
Leibenzon, Leonid Samuilovich	1879–1951	Russia
Levi, Beppo	1875–1928	Italy
Levinson, Norman	1912–1975	USA

Leeuw, Karel de -1978 USA

Lévy, Paul P.	1886–1971	France
Lewy, Hans	1904–1988	Germany, USA
Lindelöf, Ernst	1870–1946	Finland
Liouville, Joseph	1809–1882	France
Lipschitz, Rudolf Otto Sigismund	1832–1903	Germany
Littlewood, John Edensor	1885–1977	England
Loomis, Lynn Harold	1915–1994	USA
Lowdenslager, David B.	–1963	USA
Leeuw, Karel de [ON PREVIOUS PAGE]		
Lozinskii, Sergei Mikhailovich	1914–1985	Russia
Lusin, Nikolai Nikolaevich	1883–1950	Russia
Maclaurin, Colin	1698–1746	Scotland
Mandelbrojt, Szolem	1899–1983	Poland, France
Marcinkiewicz, Josef	1910–1940	Poland
Mellin, Hjalnar	1854–1933	Finland
Mengoli, Pietro	1626–1686	Italy
Mensov, Dimitrii Egven'evich	1892–1988	Russia
Mertens, Franz	1840–1927	Austria
Michelson, Albert Abraham	1852–1931	USA
Minkowski, Hermann	1864–1909	Russia, Switzerland, Germany
Osgood, William Fogg	1864–1943	USA
Paley, Raymond Edward Alan Christopher	1907–1933	England
Parseval (des Chênes), Marc-Antoine	1755-1836	France
Peano, Giuseppe	1858–1932	Italy
Pichorides, Stylianos K.	1940–1992	Greece
Pitt, H. R.		
Plancherel, Michel	1885-1967	Switzerland
Plessner, Abram Iezekiilovich	1900–	Russia
Poisson, Siméon Denis	1781–1840	France
Pollard, Harry	1919–1985	USA (Boston)
Pollard, Samuel	1894–1945	China, England
Pólya, George (György)	1887–1985	Hungary, Switzerland, USA

Porcelli, Pasquale -1972 USA

Pringsheim, Alfred	1850–1941	Germany, Switzerland
Pythagoras	c. 570–500B.C.	Greece
Rademacher, Hans	1892–1969	Germany, USA
Radon, Johann	1887–1956	Czechoslovakia
Rajchman, Aleksander	–1940	Poland
Lord Rayleigh (Strutt, John William)	1842–1919	England
Reiter, Hans Jakob	1921–1992	Austria, USA, England, The Netherlands, Austria
Riemann, Georg Friedrich Bernhard	1826–1866	Germany
Riesz, Frigyes	1880–1956	Hungary
Riesz, Marcel	1886–1969	Hungary, Sweden
Rivière, Nestor M.	1940–1978	Argentina, USA
Rogosinski, Werner W.	–1964	, England
Rubio de Francia, José Luis	1949–1988	Spain
Salem, Raphaël	1898–1963	Greece, France
Schaeffer, A. C.	–1957	USA
Schmidt, Erhard	1876–1959	Germany
Schmidt, Robert		Germany (Kiel)
Schoenberg, Isaac J.	1903–1990	Romania, USA
Schrödinger, Edwin	1887–1961	Austria
Schuster, Sir Arthur R.	1851–1934	Germany, England
Seliverstov		
Silov, G. E.		
Smith, Henry John Stephen	1826–1883	England
Stechkin, S. B.		Russia
Steinhaus, Hugo		
Stieltjes, Thomas Jan	1856–1894	The Netherlands
Stirling, James	1692–1770	Scotland
Stromberg, Karl Robert	1931– 1994	USA
~~Sunouchi, G.~~		~~Japan~~
Szász, Otto	1884–1952	Hungary, USA
Szegö, Gábor	1895–1985	Hungary, USA
Szidon, S.		

Tamarkin, Jacob David	1888–1945	Russia, USA
Taylor, Brook	1685–1731	England
Thomson, William (Lord Kelvin)	1824–1907	Ireland, Scotland
Titchmarsh, Edward Charles	1899–1963	England
Tonelli, Leonida	1885–1946	Italy
d? Ulyanov, P. L.		
Vallée-Poussin, Charles Jean de la	1866–1962	Belgium
van der Corput, Jan G.	1890–1975	The Netherlands
van Vleck, Edward Burr	1863–1943	USA
Verblunsky		
Viète, François	1540–1603	France
Vinogradov, Ivan Matveevich	1891–1983	Russia
Vitali, Giuseppe	1875–1932	Italy
Volterra, Vito	1860–1940	Italy
von Neumann, John (János Neumann)	1903–1957	Hungary, USA
Walsh, Joseph Leonard	1895–1973	USA
Weierstrass, Karl Theodor Wilhelm	1815–1897	Germany
Weiss, Mary	1930–1966	USA
Weyl, Hermann	1885–1955	Germany, USA
Widder, David Vernon	1899–1990	USA
Wiener, Norbert	1894–1964	USA
Wigner, Eugene Paul(Jenö Pál)	1902–1995	Hungary, USA
Wilbraham		
Williamson, John	1901–	England
Wintner, Aurel	1903–1958	Hungary, USA
Young, Grace Chisholm	1868–1944	England
Young, William Henry	1863–1942	England
Zygmund, Antoni	1900–1992	Poland, USA
Zuckerman, H.S.	-1970	USA

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All 3 lists have to be done carefully with Techsetters in conjunction with my "corrections" to these lists.

? Ordering ?
Symbol w words
Caps (Roman)
l.c. (")
Caps (Script)
l.c. (")
Greek

L^p-norm, 341
L^∞-norm, 342

~~Index~~ Notation

~~1_X, viii~~ delete with $\mathbb{1}_{[-T,T)}$
2Ω-periodic function, 185
H-sets, 211
OK ~~L^1-cross correlation, 77~~ OK
OK ~~L^2-autocorrelation, 137, 147, 154, 169~~ OK
U-set, 207, 215, 325
U-sets, 302
ℓ^2-autocorrelation, 258
λ-dilation, 10
p-periodic, 198
z-transform, 319
z-transforms, 253
S'- convolution, 119
$(\frac{p}{q})$, 286
$A'(\mathbb{Z})$, 196
$A(\mathbb{R})$, 109
$A(\mathbb{T}_{2\Omega})$, 195
$A(\mathbb{Z})$, 196
$A(\widehat{\mathbb{R}})$, 2
$AC(\mathbb{T}_{2\Omega})$, 194
A_*, 250
B', 353
$BQM(\mathbb{R})$, 150
$BV(I)$, 4
$BV(\mathbb{T}_{2\Omega})$, 194
$BV_{\text{loc}}(\mathbb{R})$, 4
B_*, 254
$B(\widehat{\mathbb{R}})$, 139
~~C, 104, 173, 176~~ delete
$C(\mathbb{T}_{2\Omega})$, 194
$C^1(\mathbb{R})$, 95
$C^1(\mathbb{R}\setminus\{0\})$, 97
$C^m(\widehat{\mathbb{R}})$, 194
$C^0(\mathbb{T}_{2\Omega})$, 194
$C^\infty(\mathbb{R})$, 89
$C_c^\infty(\mathbb{R})$, 90
$C^m(\mathbb{T}_{2\Omega})$, 194
$C_0(\mathbb{R})$, 358
$C_0(\mathbb{R})'$, 132
$C_b(\mathbb{R})$, 358
$C_c(\mathbb{R})$, 59, 87
$C_c(\mathbb{R})'$, 132
$C_0(\widehat{\mathbb{R}})$, 24
$C_b(\mathbb{R})$, 86
$D'(\mathbb{R})$, 92
$D'(\mathbb{T})$, 301
DFT, 73, 272, 276
D_N, 226
Dg, 96, 97
$E(\alpha)$, 212
$E\{X\}$, 142
FM, 158
$F^\vee$, 2, 62, 186, 272
F_X, 141
GHA, 150
H, 39, 70
A^T, 178
AR, 261
ARMA, 261
FFT, 288
FTC, 347
AC, 346

$H(x)$, 258
H_n, 113
$I_{\text{loc}}(\gamma)$, 241
$L(\mathbb{Z}_N)$, 273
LTI, 128
~~L^1-cross-correlation, 77~~
$L^1(\mathbb{R})$, 1
$L^1_{\text{loc}}(\mathbb{R})$, 1
$L^1_\mu(\mathbb{R})$, 270
~~L^2-autocorrelation, 137, 147, 154, 169~~
$L^2(\mathbb{R})$, 59
$L^2_W(\mathbb{T})$, 257
$L^2_p(\mathbb{R})$, 142
$L^1(\mathbb{T}_{2\Omega})$, 188
$L^2(\mathbb{T}_{2\Omega})$, 189
$L^\infty(\mathbb{T}_{2\Omega})$, 195
$M(\mathbb{R})$, 98
$MATLAB$, 73, 266, 313, 318
$M_+(\mathbb{R})$, 99
$M_+(\mathbb{T})$, 197
$M_b(\mathbb{R})$, 98
$M_d(\mathbb{R})$, 135
$M_{b+}(\mathbb{R})$, 99
$P >> 0$, 136
PW_Ω, 66
RRT, 132, 134, 196
ST, 118
$S * T$, 114
$S_N(F)$, 186
$S_{M,N}(F)$, 186
S_{MEM}, 247
S_Ω, 53
T', 96
T'_g, 97
T_g, 94
U-set, 215
U^n_N, 277
V_g, 156
$W(\mathbb{R})$, 150
$WSSP$, 144
W_N, 226, 272
X^c, viii
Zf, 198
~~$[0, 1)$, 327~~
$\mathbb{Z}_N$, 271
$\mathbb{C}$, viii
Δu, 49
Δ, 17
$\Delta_\epsilon s(\gamma)$, 152
$\text{Im}\, c$, viii
$\mathbb{N}$, viii
$\Omega{-}BL$, 12
$\mathbb{Q}$, viii
$\mathbb{R}$, v, viii
$\text{Re}\, c$, viii
$\mathbb{T}$, 188
$\mathbb{T}_{2\Omega}$, 188
$\mathbb{Z}$, v, viii
1_X, viii
$\mathbf{1}_{[-T,T)}$, 11
$\mathcal{L}(B)$, 353
$\mathcal{L}(B_1, B_2)$, 353
$\mathcal{D}_N$, 276
$\mathcal{F}$, 59, 62, 236
$\mathcal{F}_N$, 275
~~$\mathcal{F}f$, 62~~
$\mathcal{G}^-_N$, 277
$\mathcal{G}^\pm_N$, 275
$\mathcal{L}$, 29, 180
δ, 86

MEM, 247
M($\mathbb{T}$), 197
RMS, 66
ONB, 231, 275
LCAG, 190
MA
$L^p(\mathbb{R})$, 341
[illegible]
LDC, 341
S(F), 185

$\delta \circ g$, 88
δ_r, 87
δ_γ, 197
$\ell^1(\mathbb{Z})$, 186
~~ℓ^2-autocorrelation, 258~~
$\ell^2(\mathbb{Z})$, 196
$\varnothing$, viii, 207
$\equiv$, viii
$\int f(t) d\mu(t)$, 99
$\int g(t)\, dt$, 1
$\int_{\mathbb{T}_{2\Omega}} F(\gamma)\, d\gamma$, 188
$\int_{\mathbb{T}} F(\gamma)\, d\mu(\gamma)$, 197
$\mu(F)$, 197
$\mu^\vee$, 301
μ_C, 104, 135, 176
~~[illegible]~~
$\rho(f,g)$, 107
sgn t, viii, 71
$\sigma(\mathcal{H})$, 122
σ_X^2, 142
σ_X, 142
sinc, 14
~~supp T, 95~~
supp [illegible], 89, 95 (64)
$\mathring{f}_N$, 299
$\mathring{f}_T$, 305
~~$\check{F}$, 2, 62, 186, 272~~ (see earlier)
$\sum$, 185
$\tau_t T$, 116
$\tau_u f$, 9
$\tilde{f}(t)$, 59, 77 ← where $\tilde{g}(t)$ is
$\vartheta(z)$, 200
$\vartheta(z;t)$, 198
$\widehat{T}$, 106, 110
$\widehat{\mathbb{Z}}_N$, 273
$\delta(m,n)$, 231
$\sum'$, 217
$\widehat{\mathbb{R}}$, 2
$\widehat{\mathbb{R}}/(2\Omega\mathbb{Z})$, 188
$\hat{f}$, 2, 62, 186, 272
$\tilde{T}$, 119
~~$\tilde{g}(t)$, 77~~
~~$\{r_n\}$, 327~~ (n), 327
~~$\{(nr) : n \in \mathbb{N}\}$, 327~~
$c_0(\mathbb{Z})$, 196
$d_{(\alpha)}$, 187
e_γ, 9
~~[illegible]~~, 2,
$f[n]$, 185
$f * g$, 25
$f \circledast g(t)$, 77
$f \longleftrightarrow F$, 186, 272
~~[illegible]~~ F, 62,
f_C, 104
f_N, 300
f_s, 301
f_λ, 10
fft, 317
g', 97
h_n, 113
m_X, 142
$pv \int g(t)\, dt$, 1
$pv(\frac{1}{t})$, 101
~~$r \in \mathbb{R} \setminus \mathbb{Q}$, 327~~
$w_{(\alpha)}$, 188
$\mathcal{B}(\mathbb{R})$, 134
$\mathcal{E}'(\mathbb{R})$, 108
$\mathcal{F}$, 113
$\mathcal{H}$, 122, 180
$\mathcal{M}_+(\mathbb{R})$, 134
$\mathcal{O}_C$, 179
$\mathcal{P}(z)$, 198
d, 14
$\|f\|_{L^p(\mathbb{R})}$, 341
span, 257, 357
pdf, 141
$\overline{\lim}$, 338
$\underline{\lim}$, 338
$|X|$, 339

$\mathcal{S}'(\mathbb{R})$, 107
$\mathcal{S}(\mathbb{R})$, 106
AR, 261
ARMA, 261
$B(\widehat{\mathbb{R}})$, 139
DFT, 272, 276
FFT, 288
LCAG, 190
L^2 autocorrelation, 147
MA, 261
[illegible]
ONB, 231
RRT, 196
Riemann ζ-function, 221
$\delta(m,n)$, 231
pdf, 141
1/3-Cantor set, 211

~~Index~~ Names

Abel, 199, 200, 232, 315, 323
Adelson, 52
Airy, 163
Archimedes, 132, 149
Artin, 288
Atal, 262
Auslander, 278
Bôcher, 55
Bary, 210–212
Bass, 157
Beltrami, 208
Bernoulli, D., 202, 203, [redacted]
Bernstein, 209
Berry, 140
Bertrand, 208
Bertrandi[redacted] ...as, 57, 157
Betti, 208
Beurling, 109, 147, 269–271, 302
Bochner, 138, 140
Bombieri, 183, 332
Bonnet, 206
Boole, 124
Borel, 207
Braque, vi
Bremmer, 124
Brown, 211, 288
Burg, 247
Burkhardt, 201
Burrus, 289
Burt, 52
Butzer, 299
Cantor, 207, 209, 210
Carleman, 261, 268, 269
Carleson, 213, 293, 329
Carlini, 288
Carslaw, 201
Catherine de Medici, 78
Cauchy, 204
Chover, 246
Cicero, 149
Clairaut, 203
Cohen, 240, 311
Coifman, 298
Colombeau, 120
Cooley, 288
Cramér, 157
d'Alembert, 202, 203
deLeeuw, 211
du Bois-Reymond, 212
Darboux, 204
Dedekind, 288
Dicke, 200
Dieudonné, 224
Dini, 205, 207, 208
Dirac, 86, 87
Dirichlet, 43, 193, 204–206, 214, 215, 286

Bohl, P., 57
Chernoff, P., 193
Bernoulli, J., 222
Bertrandias

Doetsch, 124
Doss, 140, 246
Dunford, 356
Dym, 246
Ehrenpreis, 126
Eichler, 288
Einstein, 149, 150
Euler, 199, 202, 203, 280, 286, 322
~~F. Riesz, 45, 70, 134, 135, 201~~
Fagnano, 199
Fant, 262
Fantappié, 302
Fatou, 209, 212
Fefferman, C., 298
Fejér, 212, 227, 251, 252, 329
Fichte, 210
Fischer, 234
Fourier, vii, 201, 204
~~François Viète, 78~~
Fredholm, 233
Fresnel, 158, 201
Frobenuis, 288
Gabardo, 247
Gabor, vi
Gauss, 199, 209, 278, 280, 283, 286, 289
Gel'fand, 93
Gibbs, 55, 207
Gibson, 201
Gohberg, 174, 246
Goldberg, 246
Gregory, 221
Gris, vi
Grothendieck, 224
Hörmander, 93, 126
Hardy, 155, 201, 211, 212, 328
Hartman, 210
Hasse, 288
Heaviside, 124, 126
Hecke, 286
Hegel, 210
Heideman, 289
Heisenberg, 148
Helson, 311
Henry IV, 78
~~Henry Wilbraham, 55~~
Herbart, 210
Herglotz, 138
Hermite, 200
Herz, 147
Hewitt, 211
Higgins, 299
Hilbert, 233
Hill, 200, 201
Hirata, 119
Hobson, 201, 212
Horváth, 119
Hunt, 213
Itakura, 262
Ivašev-Musatov, 211
Iwaniec, 183
Jacobi, 199
Jakob, 221
Jerosch, 212
Johnson, 289
Jury, 262
Körner, 211
Kaashoek, 174
Kahane, 211, 237, 287, 329

Kant, 210
Katznelson, 211, 329
Khinchin, 157
Kolmogorov, 157, 213, 216, 258, 259, 262, 269
Krein, 140,
Krein, 246, 252, 270
Kronecke, 215
Kronecker, 215, 286
Kummer, 209
Lagrange, 124, 204, 288
Landau, 247, 278
Laplace, 56, 163
Lauwerier, 123
Lebesgue, 207, 328
Legendre, 286
Legendrei, 199
Leibniz, vi, 124, 221
Levinson, 262, 267, 269
Liouville, 199
Lipschitz, 205
Littlewood, 201, 211, 213, 328
Logan, 179
Lojasiewicz, 126
Lord Kelvin, 56, 81, 164
Love, 55
Lusin, 201, 209, 210, 212
M. Riesz, 201
Malgrange, 126
Mallat, 52
Malliavin, 215, 269
Martineau, 123
Masani, 149
Maxwell, 207
McClellan, 278
McGehee, 312
Mengoli, 221
Mensov, 210, 211
Meyer, 52, 298
Michelson, 8, 55, 56
Mikusiński, 119, 124
Minkowski, 288
Morgan, 288
Napoleon, 208
Neri, 122
Neumann, 232
Newton, 56, 148
Nietzche, 210
Noether, 288
Norbert Wiener, 68
Oberguggenberger, 119
Ogata, 119
Oldenburg, 221
Oliver Heaviside, 124
Orton, 123
P. Bohl, 57
P. Chernoff, 193
Paley, 213, 268, 269
Parks, 278
Parseval, 63
Phillips, 140 , R. S.,
Picasso, vi
Pigno, 312
Pincherle, 124
Plancherel, 201, 212
Plessner, 201, 213
Poincaré, 55
Poisson, 204
Pollard, 147 , H.,
Pringsheim, 4

under M

Privalov, 201
Pyatetskii-Shapiro, 211
Rader, 293
Radon, 135
Rajchman, 211, 212
~~Rieman, 208~~
Riemann, vi, 163, 201, 204, 206–208, 302
Riesz, F., 45, 70, 134, 135, 201, 251, 356
Rudin, 240, 246, 311
Ruffini, 200
Russell, 210
Saito, 262
Salem, 173, 211–213, 240
Sarton, 201
Schaar, 286
Schaeffer, 211
Schelling, 210
Schmidt, 233
Schoenberg, 140
Schopenhauer, 210
Schrödinger, 148
Schroeder, 262
Schur, 278
Schuster, 149, 150
Schwartz, 93, 119, 124, 298, 302, 356
Seliverstov, 213
Shanks, 280
Shilov, 93
Shiraishi, 119
Siegel, 286, 288
~~Sir Arthur Schuster, 149~~
~~Sir Geoffrey I. Taylor, 149~~
Sjölin, 298
Riesz, M., 201
Smith, 207, 312
Splettstösser, 299
Stebbins, 200
Stein, 73, 246
Steinhaus, 209
Stens, 299
Stokes, 163
Stone, 228
Stratton, 55, 56
Sullivan, 288
Szegö, 269
Taibleson, 321
Tate, S. R., 299
Tauber, 323
~~Taylor, 149, 150, 202, 214~~
Tevzadze, 298
Tillmann, 123
Tolimieri, 278
Tonelli, 201
Tukey, 288, 311
van der Pol, 124
van den Bos, 247
von Neumann, 200
Vallée-Poussin, 210
Varopoulos, 215
Vitali, 208
Volterra, 124, 207, 208, 232
Weierstrass, 199, 207, 208
Weil, 138
Weiss, 73
Weyl, 56, 57, 114, 212
Wickerhauser, 298
Wiener, 57, 113, 124, 129, 145, 148, 149, 153, 207, 211, 237, 241, 244, 262, 268,
Taylor, 214
Viète, 78
Taylor, B., 202
Taylor, G., 149

269
Wik, 244, 245
Wilbraham, 55
Wiles, 214
Williamson, 174
Winograd, 293
Wold, 157
Yaglom, 149
Yosida, 140
Young, 209, 212
Yule, 261
Zak, 200
Zygmund, 186, 211, 213, 234, 329

Index

~~Spectral Estimation Theorem, 253~~
absolutely continuous, 346
absolutely convergent Fourier series, 195
adjoint, 353
~~Alaoglu Theorem, 355~~
algebraic number, 209
aliasing, 58, 311
all-pole model, 261
all-pole model method of spectral estimation, 266
almost everywhere, 339
amplitude, 8
analytic functionals, 302
antenna array, 287
approximate identity, 28, 225
apsidal line shift, 200
arithmetic progressions, 323
asymptotically unbiased estimator, 146
Atlantic submarine cable problem, 124
autocorrelation, 258
autoregressive (AR) model, 261
autoregressive moving average(ARMA) model, 261
Axiom of Choice, 355
Banach-Alaoglu Theorem, 355
Baire Category Theorem, 354
Banach algebra, 177, 241
Banach space, 350
Banach-SteinhausTheorem, 354
bandlimited, 12
bandpass, 130
barometric variations, 289
Beppo Levi Theorem, 341
Bessel Inequality, 235
Betti number, 208
Beurling Theorem, 271
Beurling-Malliavin Theorem, 268
bijective, 350
bilateral Laplace transform, 110, 176
binary expansion, 297
Bit Reversal, 295, 296
Blaschke products, 213
Bonnet Theorem, 40
Borel algebra, 134
bounded set, 90
bounded measures, 98
bounded positive measures, 99
bounded quadratic means, 150
bounded Radon measures, 98
bounded variation, 4
~~Butterflies, 295~~
butterfly, 295

under S

lc b r

cable equation, 81
Calderón-Zygmund Theory, 331
Cantor function, 22, 80, 104
Cantor measure, 104, 176
Cantor set with ratio of dissection α, 212
Cantor-Lebesgue Lemma, 209
Carleson Theorem, 213, 298
~~Carleson's Theorem, 298~~
carrier frequency, 11
carrier wave, 11
Cauchy principal value, 1
Cauchy sequence, 350
~~causal, 129~~
causal signal, 32, 129, 180
Central Limit Theorem, 34
character, 273
charge, 100
chirp z-algorithm, 319
chirp signal, 158
chirp transform algorithm, 165
~~Cicero, 149~~
circle group, 188
~~classical~~, 148,
Classical Sampling Theorem, 222, 269, 308
~~classical Tauberian theorems, 155~~
Classical Uncertainty Principle Inequality, 246
closed, 90, 350
closed ideal, 177
closed translation invariant subspace, 156
closure problem, 269
closure theorems, 269
Cohen Factorization Theorem, 241
coherent state, 200
commutative algebra, 224, 237
commutative group, 273
compact, 349
complete, 234, 346, 350
~~complete metric space, 350~~
composition, 88, 178
continuous, 135
continuous linear functionals, 133
continuous nowhere differentiable functions, 208
convection, 47
~~converges, 351~~
converges in measure, 343
converges pointwise, 338
converges uniformly, 338
convolution, 25, 114, 115, 224, 237
Cornu spiral, 167
correlation ergodic process, 154
Coulomb law, 100
creative formulas, 105
crest factor, 287
cumulative distribution function, 141
Data compression, 104
Daubechies Theorem, 253
de la Vallée-Poussin function, 80
de la Vallée-Poussin kernel, 80
Decomposition of Measures, 135
deterministic autocorrelation, 151
deterministic cross-correlation, 183
DFT Matrix, 276
differential topology, 288

Differentiation of Fourier Series, 219
diffusion equation, 47
dilation, 10
Dini test, 206
Dini-Lipschitz-Lebesgue test, 205
Diophantine Approximation, 57
Diophantine approximation, 214
Diophantine polynomial equations, 286
dipole, 99
dipole moment, 100
Dirac δ-function, 3, 86
Dirac measure, 86, 197
Dirichlet, 188
Dirichlet L-functions, 286
Dirichlet Box Principle, 206, 214
Dirichlet Computation, 283
Dirichlet function, 14, 226
Dirichlet kerne, 226
Dirichlet kernel, 30, 188
Dirichlet problem, 49, 232
Dirichlet Theorem, 43, 193
Dirichlet-Jordan Test, 195
discrete, 135
Discrete Classical Sampling Theorem, 304
Discrete Fourier Transform, 73, 272
distribution, 92
distribution function, 140
distributional derivative, 96
division problem, 125
double series theorem, 223
doubling condition, 245
doubly periodic, 198
du Bois-Reymond, 227
dual space, 132, 197, 353
duality formulas, 96
du Bois-Reymond example, 219

earth satellites, 200
eigenfunction, 331
eigenfunctions, 76, 114, 183
eigenvalue problem, 68, 278
eigenvalues, 68, 76, 113, 248, 278
elliptic function, 198
elliptic integrals, 199
Embedding, 357
Embedding Theorem, 358
equatorial, 200
equidistributed modulo, 327
equidistributed modulo 1, 327
equidistribution mod 1, 57
even function, 2
even part, 71
exchange formula, 114, 117, 118, 176
expansion theorem, 126
expectation, 142
extension problem, 246

Factorization, 240
fast convolution algorithm, 318
Fast Fourier Transform, 73, 288
fast Fourier transform, 213
Fatou Lemma, 340
Fejér function, 18, 226
Fejér kernel, 30, 188, 227
Fejér Theorem, 227

Fejér-Riesz Theorem, 246, 249, 251, 253, 330, 332
Fejér-Riesz Theorem, 268
Fermat Last Theorem, 209
Fermat's Last Theorem, 214
FFT Algorithm, 289
FFT algorithm, 291, 293
filter, 129
finite energy, 32, 59
Fourier coefficients, 185
Fourier series, 301
Fourier series of distributions, 301
Fourier transform, 2, 62, 106, 110, 185, 190, 272
Fourier transform inversion formula, 2
Fourier-Stieltjes transform, 139
frequency modulated, 158
frequency response, 129
Fresnel integral, 158
Fubini Theorem, 343
Fubini-Tonelli Theorem, 344
function, 202–205
fundamental solution, 126
Fundamental Theorem of Algebra, 321
Fundamental Theorem of Calculus, 346
fundamentals, 203

Gabor wavelet decompositions, 156
Gauss, 262
Gauss Computation, 281
Gauss elimination, 267
Gauss function, 17
Gauss kernel, 32
Gauss sum, 275
Gauss sums, 206, 286, 288
Gauss Theorem, 280
Gaussian, 17
general relativity, 200
generalized function, 92
geometric mean, 257
Gibbs phenomenon, 53

Hölder Inequality, 345
Haar wavelet, 30
Hahn-Banach Theorem, 355
Hamel bases, 171
Hardy and Littlewood Theorem, 223
Hardy space, 223
harmonic analyzer, 55
heat equation, 47, 56
Heaviside function, 39
Hecke rings, 214
Heisenberg group, 200
Helly, 357
Helson, 311
Herglotz Theorem, 252
Herglotz-Bochner Theorem, 138
Hermite functions, 68, 113
Hermite polynomials, 75, 113
Hermitian, 179
Hermitian matrix, 247
hidden periods, 149
Hilbert space, 351
Hilbert transform, 121, 180
Hilbert transforms, 223
homomorphism, 273
Huygens-Fresnel principle, 158

hyperdistributions, 302

~~ideal, 241~~
ideals, 209
idelic pseudo-measures, 267
idempotent problem, 65
image processing, 52
impulse response, 129
infinite frequencies, 166
injective, 350
inner product, 258, 273, 351
integers, 271
Integrability of Trigonometric Series, 218
integrable function, 1
Integration by Parts, 347
Integration of Fourier Series, 216, 222
Integration of Series, 215
interferometer, 8
inversion formula, 43
Inversion Formula for the DFT, 274
involution, 77
isometric isomorphism, 258
isometry, 350

Jacobi theta function, 198
Jordan Decomposition Theorem, 40, 328
Jordan Inequality, 161, 182, 332
Jordan pointwise inversion formula, 4, 39

Kahane and Katznelson Theorem, 213
Kolmogorov Theorem, 257, 260, 270
Krein Theorem, 252
Kronecker set, 215
Kronecker Theorem, 174, 206, 213, 215, 327

~~L^2 autocorrelation, 147~~
lacunary Fourier series, 208
Lagrange multipliers, 356
Lagrange series, 203
Lagrange Theorem, 287
Laplace asymptotic method, 163
Laplace equation, 49
Laplace transform, 29, 180
Law of Quadratic Reciprocity, 206, 283, 285
law of sines, 324
~~least squares, 262~~
least squares linear prediction, 247
Least Squares Method, 262
Lebedev-Milin Inequalities, 267
Lebesgue constants, 229
Lebesgue Dominated Convergence Theorem, 341
Lebesgue measure, 99, 339
Lebesgue set, 45, 328
Legendre Symbol, 286
Levinson Recursion Algorithm, 267
limit inferior, 338
limit point, 337
limit superior, 338
linear prediction, 267
~~Linear Prediction Model, 261~~

linear translation invariant system, 128
linear translation-invariant system, 261
Lipschitz conditions, 205
Littlewood Conjecture, 312
Littlewood Flatness Problem, 184, 206, 286, 333
Local Membership, 241
local membership, 244, 245
locally absolutely continuous, 346
locally compact abelian groups, 190
locally finite Borel measure, 134
locally integrable function, 1
lowpass, 130
LTI systems, 331
Lusin Conjecture, 211, 212, 329

m-file, 318
MATLAB, 266, 313
MATLAB program, 318
matrix, 179
maximal ideal space, 176
maximal ideals, 178
Maximum Entropy Method (MEM), 247
Maximum Entropy Theorem, 248
Maximum likelihood Method, 247
mean, 142, 143
measurable, 341
measure, 99
measure 0, 207, 339
measures, 98
MEM, 256, 267
method of stationary phase, 163
metric, 350
metric space, 349
Michelson interferometer, 147
minimum phase filter, 253
Minkowski Inequality, 345
modulated signal, 11
modulation, 10
modulo N, 271
moment problems, 137, 247
monopole moment, 100
Moore-Smith Theorem, 352
moving average (MA) model, 261
multiplication of distributions, 114
multiplicity problem, 278
multiplier, 123
multipoles, 99

neuron excitation, 81
norm, 189, 350, 352
normed vector space, 350

odd part, 71
Ogata, 119
ONB, 231, 275
open, 90
Open Mapping Theorem, 354
operational calculus, 124
orthonormal, 231
orthonormal basis, 231
oscillator equation, 76
oscillatory integral, 158, 163

Paley-Wiener Logarithmic Integral Theorem, 31, 180, 268
Paley-Wiener space, 66
parallelogram law, 352

Parseval Formula, 275, 287
Parseval formula, 63, 222, 234
Parseval-Plancherel formula, 153
Parseval-Plancherel Theorem, 63
partial summation formula, 315
period, 185
periodization, 299, 305
periodogram, 8
phase angle, 8
Pisot-Vigayraghavan (P-V) number, 212
Plancherel Theorem, 59
Poisson function, 15
Poisson integral formula, 51
Poisson kernel, 31
Poisson Summation Formula, 114, 183, 303, 306
pole-zero linear prediction model, 261
positive, 92, 99
positive definite, 136
positive definite function, 9
positive definite matrices, 264
positive definite matrix,, 248
Positive Distributions, 136
positive measures, 99
positive semidefinite, 178
positive semidefinite matrices, 332
positive semidefinite matrix, 247
potential energy, 75
power, 147, 152, 153
power spectrum, 8, 151, 258
predicting the future, 258
Prediction, 258
prediction, 259
prediction error, 265
Prime Number Theorem, 110
primes, 206
principal value distribution, 101
Pringsheim Theorem, 5
probability density function, 141
probability measure, 140
probability space, 140
product, 118
pseudo-measures, 109, 196, 323
Pythagorean Theorem, 105

quadratic form, 288
quadratic nonresidue, 286
quadratic residue, 286
quasi-analytic function, 268
quasi-periodic, 198

Radon measure, 135
Radon measures, 98
random variable, 141
rational approximation, 206, 214
rectangular pulse, 11
regular maximal ideal, 270
restriction theory, 246
Riemann ζ-function, 221, 286
Riemann Hypothesis, 109, 173
Riemann integral, 206
Riemann zeta function, 109, 173
Riemann-Lebesgue Lemma, 19, 192
Riemann-Stieltjes Radon measure, 133
Riesz product, 329
Riesz Representation Theorem, 134

Riesz' Theorem, 356
Riesz-Fischer Theorem, 234, 356
OK ~~RMS error, 66~~ OK
Root Mean Square (RMS), 66
~~roots, 266~~

Salem Theorem, 241
sampled signal, 301
sampled values, 310
sampling, 299
sampling function, 310
sampling period, 310
sampling rate, 310
~~sampling theorem, 269~~
Schrödinger equation, 75, 148
Schur Lemma, 325
Schwartz space, 106
Schwarz Inequality, 351
Second Mean Value Theorem, 40
selfsimilarity, 208
separation of variables method, 76
sets of uniqueness, 207
Shanks Finite Identity, 281
sieve theorem, 332
~~signals, 158~~
spectral analysis question, 270
~~Spectral Estimation, 265~~
spectral estimation, 135, 256, 265
spectral estimation problem, 246
spectral estimator, 247
spectral peak, 265, 266
~~Spectral synthesis, 109~~
spectral synthesis, 109, 215, 267
Spectral Theorem for Unitary Operators, 252

spectrogram, 8
spectrogram electrocorticogram data, 159
spline, 73, 177
square-integrable functions, 59
standard deviation, 142
stationary phase, 157, 163
stationary points, 163
stationary sequence, 258
statistical randomness, 207
steady state, 49
Stieltjes transform, 165, 180
Stone-Weierstrass Theorem, 228
stroboscopic effect, 311
summability method, 155
sun, 200
sun spots, 149, 262
sunlight, 148
support, 64, 89, 95, 207
surgery invariants, 288 l.c. s
surjective, 350
symbol of $\mathcal{H}$, 122
~~Szegö Factorization Theorem, 330~~
Szegö Factorization Theorem, 256, 330
Szegö Alternative, 257, 260
Szegö Factorization, 268 delete

Tauberian condition, 323
Tauberian Theorem, 155, 323
tautochrone equation, 232
tempered distribution, 107
test function, 90
~~The Maximum Entropy Theorem, 248~~ put under M
Theorema aureum, 286
tidal analysis, 56

Spectral Estimation Theorem, 253

time invariance, 128
Toeplitz matrix, 248, 267, 333
Tonelli Theorem, 343
total charge, 100
trace, 248, 277
transcendental, 327
transfer function, 129
translation, 9, 116
~~translation invariance~~, 128,
translation invariant 331, 334
tree of spaces, 297
triangle inequality, 350

uncertainty principle 65, 316
~~uncertainty principle inequal~~ity,
148,
Uniform Boundedness Principle,
354
uniformly continuous 353
~~uniformly distributed, 327~~
uniformly distributed modulo 1,
327
unimodular trigonometric polynomials, 286
uniqueness theorem, 35
unit, 237
unit impulse, 86
univalent function theory, 267

van der Corput Lemma, 182
variance, 142
Viète formula, 78, 174
vibrating string problem, 202
violin string, 202

W. H. Young Inequality, 79
wavelet theory, 52, 129, 169, 253, 316
weak convergence, 223
Weierstrass $\mathcal{P}$-function, 198
Weierstrass Approximation Theorem, 228
weight, 245
weighted norm inequality, 245
Weil distribution, 109
Weyl Equidistribution Theorem, 57, 327
Wiener s-function, 150
Wiener Inversion Theorem, 237, 241, 243
Wiener space, 150
Wiener Tauberian Theorem, 110, 152, 155, 241, 271, 323
Wiener-Hopf equation, 174
Wiener-Khinchin Theorem, 157
Wiener-Pitt Theorem, 174
Wiener-Plancherel formula, 149, 151, 153
Wiener-Wintner Theorem, 157
Wik Theorem, 245

Yule-Walker Equations, 267

Zak transform, 198