Chapter 3

Fourier Series

3.1 Fourier series – definitions and convergence

3.1.1 Definition. FOURIER SERIES

a. Let $\Omega > 0$ and let $F : \widehat{\mathbb{R}} \to \mathbb{C}$ be a function. F is 2Ω-*periodic* with period 2Ω if $F(\gamma + 2\Omega) = F(\gamma)$ for all $\gamma \in \widehat{\mathbb{R}}$. For example, $F(\gamma) = \sin \gamma$ is 2π-periodic. If F is defined a.e. then F is 2Ω-*periodic* if $F(\gamma + 2\Omega) = F(\gamma)$ a.e.

b. Let $F \in L^1_{\mathrm{loc}}(\widehat{\mathbb{R}})$ be 2Ω-periodic. The *Fourier series* of F is the series,

$$(3.1.1) \qquad S(F)(\gamma) = \sum f[n] e^{-\pi i n \gamma / \Omega},$$

where

$$(3.1.2) \qquad \forall n \in \mathbb{Z}, \quad f[n] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{\pi i n \gamma / \Omega} \, d\gamma.$$

The numbers $f[n]$ are the *Fourier coefficients* of F. The symbol "$\sum$" denotes summation over all of $\mathbb{Z}$, i.e., "$\sum_{n=-\infty}^{\infty}$".

c. Formally, the right side of (3.1.1) can be thought of as defining the *Fourier transform* $\widehat{f}$ or F of the sequence $f = \{f[n]\}$, cf., *Remark 3.1.3d*. In fact, a sequence $f = \{f[n]\}$ is a function

$$\begin{aligned} f : \mathbb{Z} &\longrightarrow \mathbb{C} \\ n &\longmapsto f[n]. \end{aligned}$$

Letting $\ell^1(\mathbb{Z})$ be the space of all sequences $f = \{f[n]\}$ for which $\|f\|_{\ell^1(\mathbb{Z})} \equiv \sum |f[n]| < \infty$, the right side of (3.1.1) is well-defined for $f \in \ell^1(\mathbb{Z})$. In this context, we shall write

remarks c.
$\overset{\vee}{F}$ or $F^\vee$?

(3.1.3) $f \longleftrightarrow F, \quad \widehat{f} = F, \quad f = \overset{\vee}{F},$

just as we did in *Definition 1.1.2* for the case of Fourier transforms. Thus, in the case of sequences we write $\overset{\vee}{F}[n] = f[n]$.

The notation (3.1.3) is based on the presumption that $S(F)$ should equal F, e.g., *Remark 3.1.2* and *Theorem 3.1.6*; and that if the right side of (3.1.1) defines the Fourier transform of the sequence f then (3.1.2) is the Fourier inversion formula on $\mathbb{Z}$ corresponding to the Fourier inversion formula (1.1.1) on $\mathbb{R}$.

In fact, $S(F)$ often does equal F in the sense that the partial sums of the series $S(F)$ will converge in some way to F. With this in mind, if $F \in L^1_{\text{loc}}(\widehat{\mathbb{R}})$ is 2Ω-periodic we shall write

$$S_{M,N}(F)(\gamma) = \sum_{n=-M}^{N} f[n] e^{-\pi i n \gamma / \Omega},$$

where f is defined by (3.1.2). $S_N(F) \equiv S_{N,N}(F)$ is the Nth *partial sum* of $S(F)$.

d. The venerable subject of Fourier series has its share of venerable treatises, which include [Bary64], [Car30], [Edw67], [HR56], [Kah70], [KS63], [Kat76], [Kör88], [Rog59], and [Zyg59] (Zygmund's first edition is great, too).

e. We are using the notation "$f[n]$" for a Fourier coefficient of F to distinguish it from the notation "$f(n)$", which usually designates the value at $n \in \mathbb{Z}$ of a function f defined on $\mathbb{R}$. Similarly, we have chosen "$f[n]$" instead of "f_n", since "f_n" often indicates an element of a sequence of functions defined on $\mathbb{R}$. Also, we could use "c_n" instead of "$f[n]$", but then we lose contact with the letter "F".

We can and shall consider Fourier series of periodic functions on $\mathbb{R}$ instead of $\widehat{\mathbb{R}}$. Our choice of (3.1.1) to define Fourier series is based on the first part of c, the typical setting of spectral frequency information

(in terms of Greek letters such as "γ") associated with digital signals (sequences), *and* whim!

Finally, we could have defined Fourier series for 1-periodic or 2π-periodic functions; and then have developed the theory of Fourier series, unburdened by lots of Ωs. We have chosen the setting of 2Ω-periodic functions to give us the flexibility of dealing with different values of Ω which might arise in specific problems or applications. In more theoretical developments, we shall usually let $2\Omega = 1$.

3.1.2 Remark. FORMAL CALCULATION AND ELEMENTARY EXAMPLES

a. The reason we deal with (3.1.1) and (3.1.2) as a *pair* is that the decomposition of F into it fundamental parts, viz., the formula $S(F) = F$, is only effective if there is quantitative knowledge of the coefficients $f[n]$ in (3.1.3). In the case $S(F) = F$ the following *formal calculation* allows us to obtain (3.1.2) from (3.1.1):

$$\int_{-\Omega}^{\Omega} F(\gamma)e^{\pi im\gamma/\Omega}\,d\gamma = \int_{-\Omega}^{\Omega}\left(\sum f[n]e^{-\pi in\gamma/\Omega}\right)e^{\pi im\gamma/\Omega}\,d\gamma$$

$$= \sum_{n=-\infty}^{\infty} f[n]\int_{-\Omega}^{\Omega} e^{\pi i(m-n)\gamma/\Omega}\,d\gamma = 2\Omega f[m].$$

b. Let $\Omega > 0$ and let $\alpha \in (0,\Omega)$. Define F as $F = \mathbf{1}_{[-\alpha,\alpha)}$ on $[-\Omega,\Omega)$, extended 2Ω-periodically on $\widehat{\mathbb{R}}$, i.e., $F(\gamma + 2n\Omega) = F(\gamma)$ for all $\gamma \in \widehat{\mathbb{R}}$ and all $n \in \mathbb{Z}$. The Fourier series of F is

$$S(F)(\gamma) = \sum d_{(\alpha)}[n]e^{-\pi in\gamma/\Omega}$$

where $d_{(\alpha)}[0] = \frac{\alpha}{\Omega}$ and

$$\forall n \in \mathbb{Z}\backslash\{0\}, \quad d_{(\alpha)}[n] = \frac{\alpha}{\Omega}\frac{\sin\left(\frac{\pi n\alpha}{\Omega}\right)}{\left(\frac{\pi n\alpha}{\Omega}\right)},$$

cf., the Dirichlet function in *Example 1.3.1*.

Next define F by $F(\gamma) = \max(1 - \frac{|\gamma|}{\alpha}, 0)$ on $[-\Omega,\Omega)$, extended 2Ω-periodically on $\widehat{\mathbb{R}}$. A straightforward calculation, similar to that in *Example 1.3.4*, shows that the Fourier series of F is

$$S(F)(\gamma) = \sum w_{(\alpha)}[n]e^{-\pi in\gamma/\Omega},$$

where $w_{(\alpha)}[0] = \frac{\alpha}{2\Omega}$ and

$$\forall n \in \mathbb{Z}\backslash\{0\}, \quad w_{(\alpha)}[n] = \frac{\alpha}{2\Omega}\left(\frac{\sin\left(\frac{\pi n \alpha}{2\Omega}\right)}{\frac{\pi n \alpha}{2\Omega}}\right)^2,$$

cf., the Fejér function in *Example 1.3.4*.

The Fourier coefficients in this example define the *Dirichlet* and *Fejér kernel* on $\mathbb{Z}$, cf., *Example 3.4.5*.

3.1.3 Remark. NOTATION AND SETTING

a. If $\Omega > 0$ and $F \in L^1_{\text{loc}}(\widehat{\mathbb{R}})$ is 2Ω-periodic, then we write $F \in L^1(\mathbb{T}_{2\Omega})$.

Mathematically, $\mathbb{T}_{2\Omega} = \widehat{\mathbb{R}}/(2\Omega\mathbb{Z})$ is a special quotient group referred to as the *circle group* depending on Ω. An engineering student need not be concerned with this terminology for the time being. The point is that, because of the periodicity of F, $F \in L^1(\mathbb{T}_{2\Omega})$ can be thought of as being defined on *any* fixed interval $I \subseteq \widehat{\mathbb{R}}$ of length 2Ω; and that this periodicity, combined with knowledge of F on any such interval, completely determines F on $\widehat{\mathbb{R}}$.

Similarly, $\lambda_0 \in \mathbb{T}_{2\Omega}$ indicates any element of the set $\{\lambda + 2n\Omega : n \in \mathbb{Z}\} \subseteq \widehat{\mathbb{R}}$ for some fixed $\lambda \in \widehat{\mathbb{R}}$. Further, $J \subseteq \mathbb{T}_{2\Omega}$ indicates a subset $J_I \subseteq I \subseteq \widehat{\mathbb{R}}$, where $I \subseteq \widehat{\mathbb{R}}$ is an interval of length 2Ω, or any one of the subsets

$$J_I + 2n\Omega = \{\gamma + 2n\Omega : \gamma \in J_I\}, \quad n \in \mathbb{Z}.$$

If $\Omega = 1/2$, we write $\mathbb{T} \equiv \mathbb{T}_1$.

This possibly cryptic exposition might be unraveled at this time by performing some of the calculations in *Exercise 3.1*, cf., part *c*.

In any case, if $F \in L^1(\mathbb{T}_{2\Omega})$, then $\int_I F(\gamma)\,d\gamma = \int_{-\Omega}^{\Omega} F(\gamma)\,d\gamma$ for any interval $I \subseteq \widehat{\mathbb{R}}$ of length 2Ω. As such we introduce the notation,

$$\int_{\mathbb{T}_{2\Omega}} F(\gamma)\,d\gamma = \frac{1}{2\Omega}\int_I F(\gamma)\,d\gamma.$$

The factor "$\frac{1}{2\Omega}$" is a normalization factor in the sense that if $F = 1$ on $\widehat{\mathbb{R}}$ then

$$F = 1 \in L^1(\mathbb{T}_{2\Omega}) \text{ and } \int_{\mathbb{T}_{2\Omega}} 1\,d\gamma = 1.$$

The *norm* of $F \in L^1(\mathbb{T}_{2\Omega})$ is

$$\|F\|_{L^1(\mathbb{T}_{2\Omega})} = \int_{\mathbb{T}_{2\Omega}} |F(\gamma)| \, d\gamma = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)| \, d\gamma.$$

b. If $\Omega > 0$, F is 2Ω-periodic, and $F^2 \in L^1_{\text{loc}}(\widehat{\mathbb{R}})$ is 2Ω-periodic, then we write $F \in L^2(\mathbb{T}_{2\Omega})$. The *norm* of $F \in L^2(\mathbb{T}_{2\Omega})$ is

$$\|F\|_{L^2(\mathbb{T}_{2\Omega})} = \left(\int_{\mathbb{T}_{2\Omega}} |F(\gamma)|^2 \, d\gamma \right)^{1/2} = \left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)|^2 \, d\gamma \right)^{1/2}.$$

By Hölder's Inequality,

(3.1.4)
$$\int_{\alpha}^{\beta} |F(\gamma)G(\gamma)| \, d\gamma$$
$$\leq \left(\int_{\alpha}^{\beta} |F(\gamma)|^2 \, d\gamma \right)^{\frac{1}{2}} \left(\int_{\alpha}^{\beta} |G(\gamma)|^2 \, d\gamma \right)^{\frac{1}{2}},$$

e.g., *Theorem A.15*; and so we have the inclusion $L^2(\mathbb{T}_{2\Omega}) \subseteq L^1(\mathbb{T}_{2\Omega})$ and the inequality

(3.1.5) $\qquad \forall F \in L^2(\mathbb{T}_{2\Omega}), \quad \|F\|_{L^1(\mathbb{T}_{2\Omega})} \leq \|F\|_{L^2(\mathbb{T}_{2\Omega})}.$

Recall the analogous (sic) situation on $\mathbb{R}$, i.e., *Exercise 1.35*. The inclusion and inequality (3.1.5) follow since, by taking $G = 1$ in (3.1.4), we obtain

$$\|F\|_{L^1(\mathbb{T}_{2\Omega})} = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)| \, d\gamma$$

$$\leq \frac{1}{2\Omega} \left(\int_{-\Omega}^{\Omega} |F(\gamma)|^2 \, d\gamma \right)^{\frac{1}{2}} (2\Omega)^{\frac{1}{2}} = \|F\|_{L^2(\mathbb{T}_{2\Omega})}.$$

The inclusion is also proper, e.g., *Exercise 3.6*.

c. Let $\Omega = \pi$ and let $F(\gamma) = \sin \gamma + \cos 2\gamma$ on $\widehat{\mathbb{R}}$. Then

$$\int_{\mathbb{T}_{2\Omega}} F(\gamma) \, d\gamma = \frac{1}{2\pi} \int_0^{2\pi} F(\gamma) \, d\gamma$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\gamma) \, d\gamma = \frac{1}{2\pi} \int_{\alpha}^{\alpha+2\pi} F(\gamma) \, d\gamma = 0,$$

for any fixed $\alpha \in \widehat{\mathbb{R}}$. Further, if $J_I = (\frac{\pi}{4}, \frac{\pi}{2}] \subseteq I = (0, 2\pi]$ then

$$\forall n \in \mathbb{Z}, \quad \frac{1}{2\pi} \int_{J_I} F(\gamma) \, d\gamma = \frac{1}{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\gamma) \, d\gamma = \frac{1}{2\pi} \int_{\frac{\pi}{4}+2\pi n}^{\frac{\pi}{2}+2\pi n} F(\gamma) \, d\gamma.$$

Thus, if we let J be any one of the sets $J_I + 2\pi n \subseteq \widehat{\mathbb{R}}$, then $J \subseteq \mathbb{T}_{2\Omega}$ and

$$\int_J F(\gamma) \, d\gamma = \frac{1}{2\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} F(\gamma) \, d\gamma.$$

d. In *Definition 3.1.1c* we defined the Fourier transform of a sequence $f = \{f[n]\} \in \ell^1(\mathbb{Z})$. We now define the *Fourier transform* of $F \in L^1(\mathbb{T}_{2\Omega})$ as the sequence $f = \{f[n]\}$, where

$$\forall n \in \mathbb{Z}, \quad f[n] = \int_{\mathbb{T}_{2\Omega}} F(\gamma) e^{-\pi i n \gamma / \Omega} \, d\gamma,$$

and where "$\int_{\mathbb{T}_{2\Omega}}$" is defined in part *a*. In this case, the formal inversion formula is

$$F(\gamma) = \sum f[n] e^{\pi i n \gamma / \Omega}.$$

e. In *Chapter 1*, for $f \in L^1(\mathbb{R})$, the Fourier transform of f was defined on $\widehat{\mathbb{R}}(= \mathbb{R})$. In this chapter, we have two "dual" settings. First, for $f \in \ell^1(\mathbb{Z})$, the Fourier transform of f is defined on $\mathbb{T}_{2\Omega}$; and, second, for $F \in L^1(\mathbb{T}_{2\Omega})$, the Fourier transform of F is defined on $\mathbb{Z}$. Mathematically, $\mathbb{R}$ and $\widehat{\mathbb{R}}$ are locally compact abelian groups (LCAGs) which are not compact and which are *dual*, in a technical sense, to each other; similarly, the discrete LCAG $\mathbb{Z}$ is the dual group of the compact LCAG $\mathbb{T}_{2\Omega}$, and vice-versa, e.g., [Rud62], [Edw67], [Ben75].

3.1.4 Example. FOURIER SERIES OF REAL-VALUED FUNCTIONS

Let $F \in L^1(\mathbb{T}_{2\Omega})$ be real-valued, and let $f = \{f[n]\}$ be the sequence of Fourier coefficients of F. Thus,

$$f[n] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) \cos\left(\frac{\pi n \gamma}{\Omega}\right) \, d\gamma + \frac{i}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) \sin\left(\frac{\pi n \gamma}{\Omega}\right) \, d\gamma.$$

Formally, since $S(F)$ should be equal to F and therefore be real-valued, we have

$$S(F)(\gamma) = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda)\, d\lambda$$

$$+ \sum{}' \left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \cos\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda + \frac{i}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \sin\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda \right) \times$$

$$\left(\cos\left(\frac{\pi n \gamma}{\Omega}\right) - i \sin\left(\frac{\pi n \gamma}{\Omega}\right) \right)$$

$$= \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda)\, d\lambda + \sum{}' \left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \cos\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda \right) \cos\left(\frac{\pi n \gamma}{\Omega}\right)$$

$$+ \sum{}' \left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \sin\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda \right) \sin\left(\frac{\pi n \gamma}{\Omega}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n \gamma}{\Omega}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n \gamma}{\Omega}\right),$$

where

$$a_0 = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda)\, d\lambda,$$

and

$$a_n = \frac{1}{\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \cos\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda,$$

and

$$b_n = \frac{1}{\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \sin\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda,$$

for $n > 0$. The coefficients a_n, b_n are obtained since, for example, if $n > 0$ then

$$\left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \sin\left(\frac{-\pi n \lambda}{\Omega}\right) d\lambda \right) \sin\left(\frac{-\pi n \gamma}{\Omega}\right)$$

$$+ \left(\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\lambda) \sin\left(\frac{\pi n \lambda}{\Omega}\right) d\lambda \right) \sin\left(\frac{\pi n \gamma}{\Omega}\right)$$

$$= b_n \sin\left(\frac{\pi n \gamma}{\Omega}\right).$$

The Riemann-Lebesgue Lemma for $L^1(\mathbb{R})$ (*Theorem 1.4.1c*) has an analogue for $L^1(\mathbb{T}_{2\Omega})$.

3.1.5 Theorem. RIEMANN-LEBESGUE LEMMA

If $F \in L^1(\mathbb{T}_{2\Omega})$ then $\lim_{|n| \to \infty} f[n] = 0$, where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F, i.e., $\hat{f} = F$.

Proof. a. Assume $F \in C^1(\widehat{\mathbb{R}})$. Then, $G = F' \in L^1(\mathbb{T}_{2\Omega})$ has the properties that $\int_{-\Omega}^{\Omega} G(\gamma)\, d\gamma = 0$ and

$$(3.1.6) \qquad \forall \gamma \in [-\Omega, \Omega), \quad F(\gamma) = \int_{-\Omega}^{\gamma} G(\lambda)\, d\lambda + F(-\Omega),$$

cf., the approach in *Theorem 1.4.1c*.

We compute (for $n \neq 0$)

$$f[n] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{\pi i n \gamma/\Omega}\, d\gamma$$

$$= \frac{1}{2\Omega} \left[\frac{\Omega}{\pi i n} e^{\pi i n \gamma/\Omega} F(\gamma) \Big|_{-\Omega}^{\Omega} - \frac{\Omega}{\pi i n} \int_{-\Omega}^{\Omega} G(\gamma) e^{\pi i n \gamma/\Omega}\, d\gamma \right]$$

$$= \frac{-1}{2\pi i n} \int_{-\Omega}^{\Omega} G(\gamma) e^{\pi i n \gamma/\Omega}\, d\gamma;$$

and, hence,

$$|f[n]| \leq \frac{\Omega}{\pi |n|} \|G\|_{L^1(\mathbb{T}_{2\Omega})}.$$

Consequently, $\lim_{|n| \to \infty} f[n] = 0$.

b. Let $F \in L^1(\mathbb{T}_{2\Omega})$ and $\epsilon > 0$. There is $F_\epsilon \in C^1(\widehat{\mathbb{R}})$ which is 2Ω-periodic and for which $\|F - F_\epsilon\|_{L^1(\mathbb{T}_{2\Omega})} < \epsilon$. Then (3.1.5) is valid with F_ϵ and $G_\epsilon = F'_\epsilon \in L^1(\mathbb{T}_{2\Omega})$ instead of F and G. (The existence of F_ϵ can be proven in many ways, including the convolution of F with an approximate identity, which we shall discuss in *Section 3.4*.)

We compute (for $n \neq 0$)

$$|f[n]| \leq |f[n] - f_\epsilon[n]| + |f_\epsilon[n]|$$

$$\leq \|F - F_\epsilon\|_{L^1(\mathbb{T}_{2\Omega})} + \frac{\Omega}{\pi |n|} \|G_\epsilon\|_{L^1(\mathbb{T}_{2\Omega})},$$

where $f_\epsilon = \{f_\epsilon[n]\}$ is the sequence of Fourier coefficients of F_ϵ and where we have invoked part a in the second inequality.

We know that

$$\varlimsup_{|n|\to\infty} a_n \le \varlimsup_{|n|\to\infty} b_n + \varlimsup_{|n|\to\infty} c_n$$

in case $a_n \le b_n + c_n$ and $a_n, b_n, c_n \ge 0$, e.g., *Definition A.1.* Consequently,

$$\varlimsup_{|n|\to\infty} |f[n]| \le \varlimsup_{|n|\to\infty} \|F - F_\epsilon\|_{L^1(\mathbb{T}_{2\Omega})} < \epsilon.$$

Since the left side is non-negative and independent of ϵ we conclude that $\lim_{|n|\to\infty} |f[n]| = 0$. $\qquad\square$

We shall use the Riemann-Lebesgue Lemma to verify Dirichlet's fundamental theorem which provides sufficient conditions on a function $F \in L^1(\mathbb{T}_{2\Omega})$ so that $S(F)(\gamma_0) = F(\gamma_0)$ for a given point γ_0. The following ingenious proof is due to P. Chernoff [Che80], cf., [Lio86] and the classical proof as found in [Zyg59]. Dirichlet's theorem for Fourier series naturally preceded the analogous inversion theorem for Fourier transforms, e.g., *Sections 1.7 and 3.2.*

3.1.6 Theorem. DIRICHLET THEOREM

If $F \in L^1(\mathbb{T}_{2\Omega})$ and F is differentiable at γ_0 then $S(F)(\gamma_0) = F(\gamma_0)$ in the sense that

$$\lim_{M,N\to\infty} \sum_{n=-M}^{N} f[n] e^{-\pi i n\gamma_0/\Omega} = F(\gamma_0),$$

where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F, i.e., $\widehat{f} = F$.

Proof. a. Without loss of generality, assume $\gamma_0 = 0$ and $F(\gamma_0) = 0$. In fact, if $F(\gamma_0) \ne 0$ then consider the function $F - F(\gamma_0)$ (instead of F), which is also an element of $L^1(\mathbb{T}_{2\Omega})$, and then translate this function to the origin.

b. Since $F(0) = 0$ and $F'(0)$ exists we can verify that $G(\gamma) = F(\gamma)/(e^{-\pi i\gamma/\Omega} - 1)$ is bounded in some interval centered at the origin.

To see this, note that

$$(3.1.7) \quad G(\gamma) = \frac{F(\gamma)}{\gamma} \left(\frac{1}{(-\frac{\pi i}{\Omega}) + (-\frac{\pi i}{\Omega})^2 \frac{1}{2!}\gamma + (-\frac{\pi i}{\Omega})^3 \frac{1}{3!}\gamma^2 + \cdots} \right),$$

$-\Omega F'(0)/(\pi i)$

and, hence, $G(\gamma)$ is close to $-\Omega F'(0)/(\pi i)$ in a neighborhood of the origin, e.g., *Exercise 3.4.*

This boundedness near the origin, coupled with integrability of F on $\mathbb{T}_{2\Omega}$, yields the integrability of G on $\mathbb{T}_{2\Omega}$. Therefore, since $F(\gamma) = G(\gamma)(e^{-\pi i \gamma/\Omega} - 1)$ we compute $f[n] = g[n-1] - g[n]$, where $g = \{g[n]\}$ is the sequence of Fourier coefficients of G. Thus, the partial sum $S_{M,N}(F)(0)$ is the telescoping series

$$\sum_{n=-M}^{N} \Big(g[n-1] - g[n] \Big) = g[-M-1] - g[N].$$

Consequently, we can apply the Riemann-Lebesgue Lemma to obtain

$$\lim_{M,N \to \infty} S_{M,N}(F)(0) = 0. \qquad \Box$$

3.1.7 Remark. Fundamental Spaces and Elementary Convergence Results

a. Let $\Omega > 0$, and let $C^m(\widehat{\mathbb{R}})$, $0 \le m \le \infty$, be the space of m-times continuously differentiable functions on $\widehat{\mathbb{R}}$. It is convenient to define the following spaces:

$$C^m(\mathbb{T}_{2\Omega}) = \{F \in C^m(\widehat{\mathbb{R}}) : F \text{ is } 2\Omega\text{-periodic on } \widehat{\mathbb{R}}\}, \quad 0 \le m \le \infty,$$

$$AC(\mathbb{T}_{2\Omega}) = \{F \in AC_{\text{loc}}(\widehat{\mathbb{R}}) : F \text{ is } 2\Omega\text{-periodic on } \widehat{\mathbb{R}}\},$$

$$BV(\mathbb{T}_{2\Omega}) = \{F \in BV_{\text{loc}}(\widehat{\mathbb{R}}) : F \text{ is } 2\Omega\text{-periodic on } \widehat{\mathbb{R}}\},$$

$$C(\mathbb{T}_{2\Omega}) = C^0(\mathbb{T}_{2\Omega}) = \{F \in C^0(\widehat{\mathbb{R}}) : F \text{ is } 2\Omega\text{-periodic on } \widehat{\mathbb{R}}\}.$$

Clearly,

$$C^\infty(\mathbb{T}_{2\Omega}) \subseteq \ldots C^{m+1}(\mathbb{T}_{2\Omega}) \subseteq C^m(\mathbb{T}_{2\Omega}) \subseteq \ldots C^1(\mathbb{T}_{2\Omega})$$

$$\subseteq AC(\mathbb{T}_{2\Omega}) \subseteq C(\mathbb{T}_{2\Omega}) \cap BV(\mathbb{T}_{2\Omega}),$$

cf., *Remark 1.4.2.*

b. The hypothesis in *Theorem 3.1.6* that F is differentiable at γ_0 is strong; and the proof of *Theorem 3.1.6* is still valid if the hypothesis on $F \in L^1(\mathbb{T}_{2\Omega})$ is weakened to the condition that $\frac{F(\gamma)-F(\gamma_0)}{\gamma-\gamma_0}$ be integrable on some interval centered at γ_0, cf., *Exercise 3.42.* In particular, if $F \in BV(\mathbb{T}_{2\Omega})$ then

$$\forall \gamma \in \mathbb{T}_{2\Omega}, \quad \lim_{N \to \infty} S_N(F)(\gamma) = \frac{F(\gamma+) + F(\gamma-)}{2}.$$

Further, *Exercises 3.26* and *3.49* deal with properties of Fourier coefficients and rates of convergence of $\{S_N(F)\}$ for functions F belonging to the spaces defined in part *a*.

c. With regard to *Theorem 3.1.6*, we can further assert that *if $F \in BV(\mathbb{T}_{2\Omega})$ and if F is also continuous on a closed subinterval $I \subseteq \mathbb{T}_{2\Omega}$, then $\{S_N(F)\}$ converges uniformly to F on I,* cf., [Zyg59, Volume I, pages 57-58]. The *Dirichlet Theorem* and this version of it for intervals of continuity are often referred to as the *Dirichlet-Jordan Test*, cf., *Section 3.2.3.*

3.1.8 Definition. RELATIONS BETWEEN FUNCTIONS DEFINED ON $\mathbb{Z}$ AND $\mathbb{T}_{2\Omega}$

a. If $f \in \ell^1(\mathbb{Z})$ and $\Omega > 0$ then $F = \hat{f}$ is an *absolutely convergent Fourier series*, and the space of such series is denoted by $A(\mathbb{T}_{2\Omega})$. By definition, the norm of $F = \hat{f} \in A(\mathbb{T}_{2\Omega})$ is

$$\|F\|_{A(\mathbb{T}_{2\Omega})} = \|f\|_{\ell^1(\mathbb{Z})} = \sum |f[n]|,$$

cf., *Definition 1.1.2* and *Example 2.4.6f*. We define

$$L^\infty(\mathbb{T}_{2\Omega}) = \{F \in L^\infty(\widehat{\mathbb{R}}) : F \text{ is } 2\Omega\text{-periodic on } \widehat{\mathbb{R}}\},$$

and the *norm* of $F \in L^\infty(\mathbb{T}_{2\Omega})$ is $\|F\|_{L^\infty(\mathbb{T}_{2\Omega})} = \|F\|_{L^\infty(\widehat{\mathbb{R}})}$, e.g., *Definition A.10*. $C(\mathbb{T}_{2\Omega})$ is a closed subspace of $L^\infty(\mathbb{T}_{2\Omega})$ if $C(\mathbb{T}_{2\Omega})$ is taken with the $\| \ldots \|_{L^\infty(\mathbb{T}_{2\Omega})}$ norm.

b. We have the proper inclusions,

$$(3.1.8) \quad A(\mathbb{T}_{2\Omega}) \subseteq C(\mathbb{T}_{2\Omega}) \subseteq L^\infty(\mathbb{T}_{2\Omega}) \subseteq L^2(\mathbb{T}_{2\Omega}) \subseteq L^1(\mathbb{T}_{2\Omega});$$

and the identity map corresponding to any of these inclusions is continuous, e.g., *Exercise 3.44*. In fact,

$$(3.1.9) \quad \|F\|_{L^1(\mathbb{T}_{2\Omega})} \leq \|F\|_{L^2(\mathbb{T}_{2\Omega})} \leq \|F\|_{L^\infty(\mathbb{T}_{2\Omega})} \leq \|F\|_{A(\mathbb{T}_{2\Omega})}.$$

We shall show in *Example 3.5.3* that $C^1(\mathbb{T}_{2\Omega}) \subseteq A(\mathbb{T}_{2\Omega})$.

    c. Let $\ell^2(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \|f\|_{\ell^2(\mathbb{Z})} = (\sum |f[n]|^2)^{1/2} < \infty\}$. Because of part *b*, we have the proper inclusions

$$(3.1.10) \qquad \ell^1(\mathbb{Z}) \subseteq X(\mathbb{Z}) \subseteq A'(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z}) \subseteq A(\mathbb{Z}),$$

where the notation $X(\mathbb{Z}), A'(\mathbb{Z}), A(\mathbb{Z})$ is defined as follows:

$$
\begin{aligned}
X(\mathbb{Z}) &= \{f : \mathbb{Z} \to \mathbb{C} \ : \ \widehat{f} \in C(\mathbb{T}_{2\Omega})\}, \\
A'(\mathbb{Z}) &= \{f : \mathbb{Z} \to \mathbb{C} \ : \ \widehat{f} \in L^\infty(\mathbb{T}_{2\Omega})\}, \\
A(\mathbb{Z}) &= \{f : \mathbb{Z} \to \mathbb{C} \ : \ \widehat{f} \in L^1(\mathbb{T}_{2\Omega})\},
\end{aligned}
$$

e.g., *Exercise 3.44*. If we *define* $\|f\|_{X(\mathbb{Z})} = \|\widehat{f}\|_{L^\infty(\mathbb{T}_{2\Omega})}$ for $f \in X(\mathbb{Z})$, resp., $\|f\|_{A'(\mathbb{Z})} = \|\widehat{f}\|_{L^\infty(\mathbb{T}_{2\Omega})}$ for $f \in A'(\mathbb{Z})$ and $\|f\|_{A(\mathbb{Z})} = \|\widehat{f}\|_{L^1(\mathbb{T}_{2\Omega})}$ for $f \in A(\mathbb{Z})$, then the identity map corresponding to any of the inclusions in (3.1.10) is continuous. In fact,

$$(3.1.11) \quad \|f\|_{A(\mathbb{Z})} \leq \|f\|_{\ell^2(\mathbb{Z})} \leq \|f\|_{A'(\mathbb{Z})} \leq \|f\|_{X(\mathbb{Z})} \leq \|f\|_{\ell^1(\mathbb{Z})}.$$

From *Theorem 3.1.5*, we know that

$$A(\mathbb{Z}) \subseteq c_0(\mathbb{Z}),$$

where $c_0(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{C} : \lim_{|n| \to \infty} f[n] = 0\}$, cf., *Example 3.3.4a*.

    d. $A'(\mathbb{Z})$ is the space of *pseudo-measures* on $\mathbb{Z}$, cf., *Example 2.4.6f*. In light of RRT, it is natural to define the "bounded Radon measures $M_b(\mathbb{Z})$" on $\mathbb{Z}$ as the dual space $c_0(\mathbb{Z})'$ of $c_0(\mathbb{Z})$, where the norm of $f \in c_0(\mathbb{Z})$ is defined as $\|f\|_{\ell^\infty(\mathbb{Z})}$, e.g., *Appendix B*. This handwaving in terms of plausible analogy fails in this case since $c_0(\mathbb{Z})' = \ell^1(\mathbb{Z})$, i.e., $M_b(\mathbb{Z}) = \ell^1(\mathbb{Z})$!

3.1.9 Definition. MEASURES ON $\mathbb{T}$

a. A linear function $T : C(\mathbb{T}) \to \mathbb{C}$ is an element of the *dual space* $C(\mathbb{T})'$ (of the vector space $C(\mathbb{T})$) if $\lim_{n \to \infty} T(F_n) = 0$ for every sequence $\{F_n\} \subseteq C(\mathbb{T})$ for which $\lim_{n \to \infty} \|F_n\|_{L^\infty(\mathbb{T})} = 0$.

b. We denote $C(\mathbb{T})'$ by $M(\mathbb{T})$. $M(\mathbb{T})$ is the space of *Radon measures* on $\mathbb{T}$. The functionals $T \in M(\mathbb{T})$ are often denoted by μ, ν, etc., and in this case we have the usual notation,

$$T(F) = \mu(F) = \int_{\mathbb{T}} F(\gamma) \, d\mu(\gamma),$$

cf., *Section 2.7*. We also define $M_+(\mathbb{T}) = \{\mu \in M(\mathbb{T}) : \mu(F) \geq 0 \text{ if } F \geq 0\}$.

c. By the definition of $M(\mathbb{T})$, if $\mu \in M(\mathbb{T})$ then

$$\exists C_\mu > 0 \quad \text{such that} \quad \forall F \in C(\mathbb{T}), \quad |\mu(F)| \leq C_\mu \|F\|_{L^\infty(\mathbb{T})}.$$

$\|\mu\|_1$ denotes the infimum over all such constants C_μ. As in the case of measures on $\widehat{\mathbb{R}}$, $L^1(\mathbb{T})$ is naturally embedded in $M(\mathbb{T})$, and if the correspondence is denoted by $F \mapsto \mu_F$, then it is easy to show that $\|F\|_{L^1(\mathbb{T})} = \|\mu_F\|_1$.

d. The relationship between $M(\mathbb{T})$ and $M(\widehat{\mathbb{R}})$ is established by the fact that $M(\mathbb{T})$ can be identified with

$$\{\mu \in M(\widehat{\mathbb{R}}) : \tau_1 \mu = \mu\},$$

i.e., $M(\mathbb{T})$ can be considered as the subspace of 1-periodic elements of $M(\widehat{\mathbb{R}})$. Of course, $\mu \in M(\mathbb{T})$ is *bounded* in the sense of the norm inequality in part c; *but* if $\mu \in M(\mathbb{T}) \backslash \{0\}$, then the 1-periodic measure on $\widehat{\mathbb{R}}$ corresponding to μ is not in $M_b(\widehat{\mathbb{R}})$. For example, if $F \in L^1(\mathbb{T}) \backslash \{0\}$ then the 1-periodic function F_1 on $\widehat{\mathbb{R}}$, which equals F on $[0, 1)$, is not in $L^1(\widehat{\mathbb{R}})$. As another example, define the *Dirac measure* δ_γ at $\gamma \in \mathbb{T}$ by the formula $\delta_\gamma(F) = F(\gamma)$, where $F \in C(\mathbb{T})$. Then $\mu = \tau_\gamma(\sum \delta_n)$ is the 1-periodic measure on $\widehat{\mathbb{R}}$ corresponding to δ_γ, and $\mu \notin M_b(\widehat{\mathbb{R}})$. The verification of these assertions is left to *Exercise 3.50*.

3.1.10 Example. PERIODICITY: POTPOURRI AND TITILLATION

a. *Periodicity.* We defined 2Ω-periodicity on $\widehat{\mathbb{R}}$ in *Definition 3.1.1.* Let $p \in \mathbb{C}$ and let $F : \mathbb{C} \to \mathbb{C}$ be a function. F is *p-periodic* with *period* p if $F(z + p) = F(z)$ for all $z \in D$. If $D \subseteq \mathbb{C}$ is a domain then F is *doubly periodic* with *periods* $p_1, p_2 \in \mathbb{C}$ if $\text{Im}(p_2/p_1) > 0$ and

$$\forall z \in D, \quad F(z + p_1) = F(z) \text{ and } F(z + p_2) = F(z).$$

F is *quasi-periodic* if

$$\forall (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

$$F(x + 1, \omega) = e^{-2\pi i \omega} F(x, \omega) \text{ and } F(x, \omega + 1) = F(x, \omega).$$

b. *Jacobi theta function.* As examples, we first note that entire doubly periodic functions are constants. Also $F(z) = e^z$ is $2\pi i$-periodic on $\mathbb{C}$.

The *Jacobi theta function* ϑ_3 is a 1-periodic entire function, depending on a parameter $t \in \mathbb{C}$, and is defined as

$$\vartheta(z; t) = \vartheta(z) = \sum e^{-\pi n^2 t + 2\pi i n z},$$

where $\text{Re}\, t > 0$. $\vartheta(z)$ is 1-periodic on $\mathbb{C}$.

c. *Elliptic functions.* An *elliptic function* is a meromorphic function in the plane which is doubly periodic in its domain of definition. If $p_1, p_3 \in \mathbb{C}$, $\text{Im}(p_3/p_1) > 0$, and $p_{m,n} \equiv 2m p_1 + 2n p_3$ for $m, n \in \mathbb{Z}$, then the *Weierstrass $\mathcal{P}$-function* is defined as

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum \left\{ \frac{1}{(z - p_{m,n})^2} - \frac{1}{p_{m,n}^2} \right\},$$

where summation is over all $(m, n) \in \mathbb{Z} \times \mathbb{Z} \backslash \{(0, 0)\}$. $\mathcal{P}$ is an elliptic function.

d. *Zak transform.* The *Zak transform* Zf of $f : \mathbb{R} \to \mathbb{C}$ is formally defined as

(3.1.12)
$$\forall (x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}},$$

$$Zf(x, \omega) = a^{1/2} \sum_k f(xa + ka) e^{2\pi i k \omega},$$

where $a > 0$. Zf is a quasi-periodic function.

e. *Elliptic integrals.* Seventeenth and eighteenth century problems from astronomy for computing arc length of orbits, or from mechanics for computing the period of a simple pendulum, led to the problem of evaluating integrals of the form

$$(3.1.13) \qquad s(y) = \int_0^y \left(\frac{1 - k^2 x^2}{1 - x^2} \right)^{1/2} dx$$

or

$$(3.1.14) \qquad s(y) = \int_0^y \frac{d\varphi}{(1 - k^2 \sin^2 \varphi)^{1/2}},$$

respectively. These are examples of *elliptic integrals*; and Liouville (1833) proved that such integrals can not be evaluated in terms of algebraic, trigonometric, logarithmic, or exponential functions. The study of elliptic integrals was an important part of eighteenth and nineteenth century mathematics featuring the likes of Fagnano, Euler, and Legendre, and leading to the analyses of Gauss, Abel, and Jacobi. General elliptic integrals are of the form $\int R(x, \sqrt{P(x)}) \, dx$, where $P(x)$ is a third or fourth degree polynomial with distinct roots and $R(x, \omega)$ is a rational function of x and ω.

f. *Quintics.* Instead of studying the function $s(y)$ in (3.1.13) and (3.1.14), Abel (1802-1829) in 1826 and Jacobi in 1827 analyzed the inverse of elliptic integrals; and these are, in fact, the elliptic functions. The analogue in trigonometry is to study the sine function instead of the multiple-valued arcsine,

$$s(y) = \int_0^y \frac{dx}{\sqrt{1 - x^2}}.$$

Jacobi's theta functions are relatively elementary functions from which elliptic functions can be constructed. Later, in the early 1860s, Weierstrass introduced $\mathcal{P}(z)$ as the elliptic function inverting a specific elliptic integral; and then he proved that every elliptic function can be expressed in terms of $\mathcal{P}(z)$ and $\mathcal{P}'(z)$ [Hil74, page 141].

Ruffini and Abel proved that quintic polynomial equations can not necessarily be solved by algebraic operations. In 1858, Hermite used elliptic functions to obtain solutions of such equations, cf., [Kle56].

g. *Shape of the sun.* We can integrate the Newtonian equations of motion of a secondary body in the equatorial plane of a rotationally symmetric central body; the solution is in terms of the Weierstrass $\mathcal{P}$-function [SB65]. There are important applications of this technique. In particular, the motion of equatorial artificial earth satellites is characterized, and orbital apsidal line shifts of the secondary body can be computed. This latter point is interesting because of the apsidal line shift of mercury's orbit about the sun. This shift can be accounted for by Newtonian methods if the sun is sufficiently "flat", as an oblate spheroid. Robert Dicke and others, e.g., Hill and Stebbins in 1975, have provided an experimental tour de force, and shown that the sun is too spherical, by an order of magnitude, to account for even ten percent of mercury's apsidal line shift by Newtonian methods. Einstein's theory of general relativity does explain the shift.

h. *Coherent states.* A *coherent state* is a family of functions of the form

$$(3.1.15) \qquad \varphi(t) = g(t-a)e^{2\pi itb}e^{2\pi icxw},$$

parameterized by $(a, b, c) \in \mathbb{R}^3$. In the quantum physics literature, g is often taken to be the Gaussian. There is a natural relation between coherent states and the Heisenberg group, e.g., our *Gabor representations and wavelets*, AMS Contemporary Math., 91(1989), 9–27. Closure problems for coherent states have a history going back to von Neumann's classic from the early 1930s [vN55, page 407]. Zak's role in the evolution of the Zak transform and its use in quantum mechanics has been documented in [Jan88]. The Zak transform and knowledge of its zero set are relevant for solving a variety of closure problems for coherent states, e.g., [BF94, Chapter 3].

As we have seen, $\vartheta(z)$ is a 1-periodic entire (and therefore non-elliptic) function used in the construction of meromorphic doubly periodic elliptic functions. It turns out that $\vartheta(z)$ also plays a role in the non-analytic quasi-periodic Zak transform of the Gaussian. In fact, it

is easy to check that the Zak transform of a Gaussian is a product of $\vartheta(z;t)$, for certain t, and a Gaussian. Thus, the zeros of the Zak transforms of the Gaussian are determined by the zeros of ϑ, cf., [BF94, computation after Theorem 7.8].

3.2   History of Fourier series

George Sarton, who founded the journal *Isis* in 1912, wrote that the "main duty of the historian of mathematics... is to explain the humanity of mathematics, to illustrate its greatness, beauty and dignity...". Alas, we can neither achieve such a noble goal with its accompanying deep scholarship, nor even present a lapidary exposition of the history of Fourier series. Fortunately such expositions abound, e.g., the historical commentaries of Riemann [Rie1873], Gibson [Gib1893], Carslaw [Car30], Hobson [Hob26], Plancherel [Pla25], cf., [Zyg59, Preface], the masterful entries on *Fourier series* in the Encyclopedia Britannicas of the last fifty years, and the relevant biographical entries in the *Dictionary of Scientific Biography*. There are also important historical contributions by Burkhardt, Plessner, and Tonelli referenced in these works.

Our treatment in this section is selective and perhaps idiosyncratic. We shall not discuss the history of Fourier series vis a vis its major applications to heat and light and celestial mechanics by Fourier and Fresnel and Hill, respectively. (Of course, there are brilliant, but perhaps curmudgeonly, thermodynamicists who assert that Fourier's theory of heat [Fou1822] did not really treat heat.) We shall mostly deal with the relation of Fourier series with real analysis, and to some extent with number theory, cf., [Ben76] and [Mon94], respectively. We shall not discuss its relation with functional analysis, that begins with the profound work of Beurling [Beu89], or with complex analysis, that begins with the work of F. and M. Riesz, Lusin and Privalov, and Hardy and Littlewood .

3.2.1 d'Alembert (1717-1783), Euler (1707-1783), D. Bernoulli (1700-1782), and ~~Lagra~~ Lagrange (1736-1813).

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In *Section 1.8* we discussed some partial differential equations from mathematical physics, and it turns out that Fourier series originated in dealing with such equations.

In 1747, d'Alembert solved the *vibrating string problem*. This problem is to solve the equation,

$$(3.2.1) \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for a function $u(x,t)$, where $x \in [0,L]$, $c \in \mathbb{R}$, $t \geq 0$, $u(0,t) = u(L,t) = 0$ for all $t \geq 0$, and $u(x,0) = f(x)$ is given on $[0,L]$. d'Alembert's solution u is in terms of f, and so f must be twice differentiable in this case. On the other hand, equation (3.2.1) was derived, after *significant* assumptions, to represent the motion of a taut string, such as a violin string, after it is released from a given initial position f on $[0,L]$. One can imagine an initial position f to have corners, so that f' need not exist everywhere.

In 1748, Euler made an important observation about the vibrating string problem. He noted that the motion of the string is completely determined for $x \in [0,L]$ and time $t \geq 0$ if the form of the string and its velocity at $t = 0$ are given. In particular, Euler was able to find the solution of (3.2.1) for a given initial position f and initial velocity g of the string. Euler's solution allowed for the initial positions f to have discontinuous derivatives. This led to a disagreement with d'Alembert on an issue which ultimately comes down to defining the notion of *function* e.g., [Bir73, pages 16ff.] which is taken from [Rie1873], cf., [G-V92]. In any case, Euler felt he had solved the vibrating string problem for very general initial positions f.

Daniel Bernoulli entered the discussion in 1753 in the midst of the d'Alembert-Euler disagreement. Daniel Bernoulli had developed hydrodynamics from the principle of conservation of energy, and was a professor of anatomy and botany at Basel, before becoming a professor of physics. He approached the vibrating string problem with Brook Taylor's observation (1715) that if

$$(3.2.2) \qquad u_n(x,t) = \sin\left(\frac{\pi n x}{L}\right) \cos\left(\frac{\pi n c t}{L}\right), \quad n \in \mathbb{Z},$$

then (3.2.1) is satisfied for $u = u_n$, and $u_n(0,t) = u_n(L,t) = 0$ for all $t \geq 0$.

Using infinite sums of terms of the form (3.2.2), Bernoulli wrote down expressions which he asserted were the most general solutions of the vibrating string problem (3.2.1), i.e., solutions for the most general initial position f. His argument was both formal and in terms of the physics of sound, cf., [BS82], [Pie83] for beautiful treatments of the fundamentals and harmonics used by Bernoulli. Later in 1753, Euler noted that Bernoulli's claim of general solutions could only be correct *if "arbitrary curves" f defined on $[0, L]$ could be written as*, what were later called, *Fourier series*. Further, because of the periodicity of the individual terms in Bernoulli's series, Euler judged that Bernoulli was incorrect as far as generality of solution, cf., [Bra86, pages 462-464]. Once again, ill-defined terms such as "arbitrary curves", instead of a precise definition of *function*, were the root cause of these different opinions.

To add to the intellectual melee, Lagrange, at age 23, wrote in 1759 in support of Euler's solution being the most general. Amazingly, his "proof" used trigonometric series similar to Bernoulli's. For the case $L = 1$, Lagrange's solution was essentially of the form

$$u(x,t) = \int_0^1 \sum_{n=1}^{\infty} (\sin \pi n x \cos \pi n c t) f(y) \sin \pi n y \, dy$$

(3.2.3)

$$+ \frac{2}{c\pi} \int_0^1 \sum_{n=1}^{\infty} \frac{1}{n} (\sin \pi n x \sin \pi n c t) g(y) \sin \pi n y \, dy.$$

Note that if $t = 0$ and if there is an interchange of summation and integration, then (3.2.3) gives rise to the Fourier series expansion of f. *Lagrange series* were almost history! Lagrange seemed intent on verifying Euler's claims versus d'Alembert. In another bizarre twist, Euler did the formal calculation of *Remark 3.1.2* to compute Fourier coefficients in 1777 when Fourier was a 9-year old, cf., [Car30, page 4] for a similar contribution by Clairaut.

3.2.2 Fourier (1768-1830).

Fourier submitted his 234 page manuscript, "Sur la propagation de la chaleur", to the Institut de France in 1807. At that time, Fourier was almost 40, and had only three unrelated published papers. His work as Prefect of the Department of Isère in Grenoble dealt with drainage of marshlands, consultation on the achievements of Napoleon's Institut d'Egypte, modeled after the Institut de France, and with planning the first road between Grenoble and Torino, Lagrange's hometown.

The turbulent story of the evolution of this paper includes its critique by Lagrange, et al., a prize competition which Fourier won in 1812 along with Lagrange's reservations, the publication of *the* book [Fou1822] in 1822, and the disappearance of the original paper, e.g., [Grat72] along with some of the other references listed at the beginning of *Section 3.2*. Darboux rediscovered the paper at the École Nationale des Ponts et Chaussées in 1890.

As we saw in *Section 3.2.1*, the technology was already in place for Fourier series long before Fourier came on the scene. What did Fourier do? He never claimed discovery of the Fourier coefficients (3.1.2) that he used. However, he had a point of view which introduced a "new epoch", to use Riemann's phrase. In the 18th century, Fourier coefficients were an integral part (sic) of trigonometric series which had already been derived by other means. Fourier asserted that an arbitrary function could be expanded in a trigonometric series whose coefficients could be computed as in *Remark 3.1.2*. Such an assertion led to questions of convergence of series and integration of arbitrary functions (in the definition of Fourier coefficients) and, of course, to questions about the meaning of *function*. Fourier's examples and applications in [Fou1822] are extraordinary; and were influential in establishing the field of *Fourier's series*.

There are also related subsequent contributions by Cauchy and Poisson; but we shall go directly to Dirichlet.

3.2.3 Dirichlet (1805-1859).

Who gave the first correct definition of *function*? Scholars of good will and excellent credentials disagree, cf., [Monn72, especially pages 57-65]. As mentioned in *Remark 1.7.7b*, it seems to us that Dirichlet has a valid claim.

In 1829 [Dir1829, page 121], he wrote that a *continuous function* f on $[0, h]$ is defined by the property that it "has a finite and well determined value for each value of β between 0 and h, and moreover such that the difference $f(\beta + \epsilon) - f(\beta)$ decreases without limit (to 0) when ϵ becomes smaller and smaller".

In 1837 [Dir1837], he wrote the following, in which the parenthetical remark shows that continuity was not an intrinsic part of his definition. "Imagine a and b to be two fixed values and x a variable, which is supposed to assume one after the other all values between a and b. If to each x there corresponds a unique finite y in such a manner that while x runs continuously through the interval from a to b, $y = f(x)$ varies gradually also, then y is a continuous function of x for this interval. (Since in what follows, we shall only discuss continuous functions, this attribute can be omitted without loss.)" The lack of rigor in defining continuity in terms of the word "gradually" is compensated by his precision in 1829. "It is not at all necessary that y depends on x in this whole interval by the same law, and it is not even necessary to imagine a dependency expressible by mathematical operations. ...This definition does not prescribe a common law to the different parts of the curve; it can be thought of as being composed of parts of the most different kinds or completely without law." This last part addresses the confusion from the 18th century analysis, when a formula such as $f(x) = x^2$ on $[a, b]$ was often thought to characterize a function, instead of characterizations such as $f = 2$ on $[0, \frac{a+b}{2}]$ and $f(x) = x^2$ on $(\frac{a+b}{2}, b]$.

Of course, it was precisely in the papers [Dir1829] and [Dir1837], where Dirichlet defined the notion of function, that he also proved the fundamental *Theorem 3.1.6*. Dirichlet's theorem was generalized by Lipschitz in 1864, supposing so-called *Lipschitz conditions*; and generalized still further by Dini, who wrote an important book on Fourier series in 1880. In the spirit of Dini's analysis, there is the following *Dini-Lipschitz-Lebesgue test* for uniform convergence. *If $F \in L^2(\mathbb{T}_{2\Omega})$ and*

$$\lim_{\lambda \to 0} |F(\gamma + \lambda) - F(\gamma)| \log |\lambda| = 0$$

uniformly in an open interval I, then $S(F) = F$ on any closed subinterval $J \subseteq I$, and the convergence of the Fourier series $S(F)$ to F is

uniform on J, e.g., [HR56, Theorem 59], cf., [Zyg59, page 52] for the original *Dini test*.

For perspective, recall that the Jordan Theorem, *Theorem 1.7.6*, is the analogue for Fourier transforms of the Dirichlet Theorem, *Theorem 3.1.6*. The second mean value theorem (*Lemma 1.7.3*) was used to prove *Theorem 1.7.6*, and, in fact, Bonnet (Memoires des Savant Étrangers of the Belgian Academy, 23 (1948-1850) used *Lemma 1.7.3* directly to prove *Theorem 3.1.6*. Dirichlet's original proof in 1829 used an argument similar to that required to prove *Lemma 1.7.3*.

Dirichlet made major contributions to number theory. It is not difficult to prove that the sequence $\{4n - 1 : n \in \mathbb{N}\}$ contains infinitely many primes. Dirichlet proved the general fact that *if $a \in \mathbb{N}$, $b \in \mathbb{Z}$, and a and b are relatively prime, then $\{an + b : n \in \mathbb{N}\}$ contains infinitely many primes*, cf., *Remark 3.8.11a*.

In this book we shall refer to two other number theoretic issues where Dirichlet had seminal ideas. The first concerns the Dirichlet Box Principle, related to rational approximation of irrationals, and leading to the Kronecker Theorem which can be formulated in terms of trigonometric sums, e.g., *Exercises 3.40 and 3.41*. The second concerns a proof of Gauss' Law of Quadratic Reciprocity, e.g., *Remark 3.8.11*. The material of *Section 2.10* plays a role, as well as subtle issues concerning Gauss sums and the so-called Littlewood Flatness Problem, e.g., *Remark 3.8.11*.

3.2.4 Riemann (1826-1866).

Bernhard Riemann's life was tragic in its briefness and transcendental in its brilliance. The excerpts in [Kli72] about Riemann's ideas barely scratch the surface on the depth and breadth and lustre of his creativity, cf., [Edwa74] for implications of just one of his gems.

Riemann's *Habilitationsschrift* [Rie1873] was presented in 1854 but was only published in 1867 after his death. It is the first part of this work which has provided us with some of the material in *Sections 3.2.1– 3.2.3*. Next, Riemann developed the *Riemann integral*. His theory of integral was created to define Fourier coefficients and Fourier series expansions for a large class of functions. Finally, he developed the Riemann Localization Principle and several other important tools for

dealing with trigonometric series, e.g., [Zyg59], cf., [Ben71] for the relation between the *Riemann Localization Principle* and the notion of *support*.

The Riemann Localization Principle is a key technique in the study of *sets of uniqueness (U-sets)*. A set $E \subseteq [0,1)$ is a *U-set* if

$$\lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{-2\pi i n \gamma} = 0 \text{ off of } E \text{ implies } c_n = 0 \text{ for all } n \in \mathbb{Z}.$$

Using Riemann's theory, Cantor proved that *the empty set $\varnothing$ is a U-set*, cf., *Section 3.2.5.* Cantor's theorem was apparently known to Riemann, e.g., [Leb06, page 110]. At the other extreme, *if $|E| > 0$ then E is not a U-set*, e.g., *Exercise 3.37.*

As is well-known, *a bounded function $f : [a,b] \to \mathbb{C}$ is Riemann integrable if and only if f is continuous a.e.*, e.g., [Ben76, pages 94-96]. The concept of measure 0 and the notation "a.e." *(Definition A.4)* are now part of the Lebesgue theory (1902). Leading to Lebesgue, Vito Volterra (1881), at that time a student of Dini at the Scuola Normale Superiore in Pisa, constructed functions f whose derivative exists everywhere but for which f' is not Riemann integrable, e.g., [Ben76, pages 20-21], cf., the interesting examples in [Rie1873, Section 13]. Actually, H. J. Smith had solved the same problem in 1875; but Lebesgue was unaware of Smith's result in his thesis [Leb02], where he gives prominent mention of Volterra's example.

The point is that measure 0 was emerging in the late 19th century as an important idea. Norbert Wiener (1938) has made a case for formulating the notion of measure zero based on justifying Maxwell's and Gibbs' theory of statistical mechanics. He wrote that "the ideas of statistical randomness and phenomena of zero probability were current among the physicists and mathematicians in Paris around 1900, and it was in a medium, heavily ionized by these ideas that Borel and Lebesgue solved the mathematical problem of measure" [Wie81, Volume II, pages 794-806], cf., [Carl80].

Besides non-(Riemann) integrability, the issue of non-differentiability was prominent in this part of the 19th century. Weierstrass wrote, at least in a possibly edited version of a lecture he gave at the Royal Academy of Sciences on July 18, 1872, that "Riemann, as I learned

from some of his students, stated decisively (in 1861, or perhaps even earlier)" that

$$(3.2.4) \qquad\qquad f(x) = \sum_{n=1}^{\infty} \frac{\sin n^2 x}{n^2}$$

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decision

is an everywhere continuous nowhere differentiable function, e.g., Weierstrass' Mathematische Werke II, pages 71-74, cf., [BSt86] for a profound analysis of this area. Although (3.2.4) does have some points of differentiability, there are now many continuous nowhere differentiable functions including Weierstrass' lacunary Fourier series (1872),

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(\pi a^n x),$$

where $a > 1$ is an odd integer, $b \in (0,1)$ and $ab > 1 + 3\pi/2$, cf., [Ben76, pages 28-29] for other examples and [Dui91] for the relation to *selfsimilarity*.

Riemann convalesced and toured in Italy during the winter of 1862, arriving in Pisa in 1863. He became friendly with Betti and Beltrami. Betti, of "Betti number" fame, was Director of the Scuola Normale Superior, and there is an interesting Betti-Riemann correspondence at the Scuola. (The Scuola Normale was started by Napoleon, and is modeled after the École Normale Supériore in Paris.) Dini was a student at the Scuola at the time of Riemann's visit. He graduated in 1864, spent a year studying with Bertrand in Paris, and returned to the Scuola Normale where he spent the next 52 years. Besides Volterra, he counts Vitali as one of his students, e.g., [Ben76] for historical remarks and mathematical contributions of Vitali.

Riemann returned to Germany for the winter of 1864-1865, but then came back to Pisa. He died and was buried at Biganzolo in the northern part of Verbania (the Italian resort town on the western banks of Lago Maggiore just 15 miles south of the Swiss border). A mourner at the local cemetery will surely point you to the marker of "Il Tedesco".

3.2.5 Cantor (1845-1918).

Georg Cantor received his Ph.D. in 1867 at Berlin. His dissertation on quadratic Diophantine equations, related to some issue from Gauss' monumental *Disquisitiones Arithmeticae* [Gau66], was written under the direction of Kummer, cf., [Ben77] for remarks about Kummer, Fermat's Last Theorem, and ideals. Kronecker, who later became an intellectual adversary of Cantor's, was also a professor in Berlin at the time.

Cantor wrote several important papers on U-sets in the early 1870s, including his theorem quoted in *Remark 3.2.4*. In order to prove this result, that the empty set is a U-set, he first proved what is now known as the *Cantor-Lebesgue Lemma*: if $X \subseteq [0,1)$ *is a Lebesgue measurable set of positive measure, and if*

$$\forall \gamma \in X, \quad \lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{-2\pi i n \gamma} \in \mathbb{C},$$

then $\lim_{|n| \to \infty} c_n = 0$. Cantor actually proved the result for the case that X is a nondegenerate interval. Fatou first investigated the converse. In this regard, Lusin found a trigonometric series which was a.e. divergent and for which $\lim_{|n| \to \infty} c_n = 0$. Steinhaus clinched the converse by constructing such a series which was everywhere divergent and for which $\lim_{|n| \to \infty} c_n = 0$, e.g., [Bary64, Volume I, pages 176-177].

After proving that the empty set was a U-set, Cantor showed that finite sets and certain countably infinite sets are also U-sets. This work certainly influenced his later research on set theory and the infinite.

It was in 1874 that he gave his famous, correct, and controversial proof that of the fact that there are only countably many algebraic numbers. Recall that an *algebraic number* is a zero of a polynomial with integer coefficients.

In any case, Cantor tried to prove that all countable sets $E \subseteq [0,1)$ were U-sets; and this was finally achieved by F. Bernstein (1908) and W. H. Young (1909), cf., [But95]. Actually, Bernstein proved somewhat more, cf., *Sections 3.2.6* and *3.2.7*.

The remainder of Cantor's life, from the mid−1870s, was devoted to the study of the infinite, not only in mathematics as in [Can55], but often delving into various philosophical notions of infinity due to

the Greeks, the Scholastic philosophers, and his contemporaries, e.g.,
[Daub79]. Cantor certainly did not dote on philosophers. In a letter to
Bertrand Russell, who was then at Trinity College, Cambridge, Cantor
wrote (1911): "... and I am quite an adversary of Old Kant, who
in my eyes has done much harm and mischief to philosophy, even to
mankind; as you easily see by the perverted development of metaphysics
in Germany in all that followed him, as in Fichte, Schelling, Hegel,
Herbart, Schopenhauer, Hartman, Nietzche, etc. etc. on to this very
day. I never could understand why ... reasonable ... peoples ... could
follow yonder sophistical Philistine, who was so bad a mathematician."

3.2.6 Mensov (1892-1988).

Dmitrii Mensov proved a key result on U-sets in 1916 by finding a
non-U-set X with Lebesgue measure $|X| = 0$. He did this just after
graduating from Moscow University, where he wrote his thesis under
N. Lusin. Mensov's example stimulated a great deal of study about sets
of measure zero. Actually, on the basis of Mensov's example, Lusin and
Bary defined the notion of U-sets as such. Earlier, de la Vallée-Poussin
had proved that if a trigonometric series converged to $F \in L^1(\mathbb{T})$ off
a countable set E, then the series is the Fourier series of F. It was
generally felt that the same would be true for sets E with $|E| = 0$.
Mensov changed that perception. Mensov showed that there exists a
nontrivial trigonometric series which converges to $F = 0$ off a set of
measure zero. Since $F \in L^1(\mathbb{T})$ is the 0-function a.e., and since the
series has some nonzero coefficients, the series is not the Fourier series
of F. Needless to say, Mensov's example had a certain amount of shock
value, cf., [Men68], [Ben76, page 115] for other major results by Mensov.

Nina Bary (1923) asked for conditions on the coefficients $\{c_n\}$ of
trigonometric series to ensure that $c_n = 0$ for all n whenever

$$(3.2.5) \qquad \lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{-2\pi i n \gamma} = 0 \quad \text{a.e. on } \mathbb{T}.$$

In light of de la Vallée-Poussin's and Mensov's results of the previous
paragraph, it is interesting to observe that *if $\sum |c_n|^2 < \infty$ and (3.2.5)
is true then $c_n = 0$ for all n*, e.g., *Exercise 3.36*. There have been

deep results in this problem area, in the case $\Sigma|c_n|^2$ diverges, by Littlewood (1936), Wiener and Wintner (1938), A. C. Schaeffer (1939), Salem (1942), Ivašev-Musatov (1957), Brown and Hewitt (1980), and Körner [Kör87]. For example, Salem proved that *for each $\epsilon > 0$ there is a (bounded measure) $\mu \in M(\mathbb{T})\backslash\{0\}$ for which $|\operatorname{supp}\mu| = 0$ and*

$$(3.2.6) \qquad \exists C, N \quad \text{such that} \quad \forall |n| \geq N, \ |\overset{\vee}{\mu}[n]| \leq C\frac{1}{|n|^{\frac{1}{2}-\epsilon}},$$

e.g., [KS63, pages 106–112], cf., [Ben75, pages 96–97]. If the right side of (3.2.6) were $C(1/|n|^{\frac{1}{2}+\epsilon})$ then $\mu \in L^2(\mathbb{T})$, e.g., *Theorem 3.4.13*; this coupled with the condition $|\operatorname{supp}\mu| = 0$ implies μ is the 0-function.

With regard to (3.2.6) and the Lusin Conjecture of *Section 3.2.8*, deLeeuw, Kahane, and Katznelson proved the following result:

$$\forall \widehat{f} = F \in L^2(\mathbb{T}), \ \exists \widehat{g} = G \in C(\mathbb{T}) \text{ such that}$$
$$\forall n \in \mathbb{Z}, \quad |f[n]| \leq |g[n]| \text{ and} \|G\|_{L^\infty(\mathbb{T})} \leq 9\|F\|_{L^2(\mathbb{T})}$$

(Comptes Rendus Acad. Sci., Paris, 285(1977), 1001-1003).

3.2.7 Bary (1901-1961) and Rajchman (-1940).

What with Mensov's example, Alexander Rajchman (who died at Dachau in 1940) "seems to have been the first to realize that for sets of measure zero that occur in the theory of trigonometric series it is not so much the metric as the *arithmetic* properties that matter" (from Zygmund's biography of Salem in [Sal67]. Rajchman (1922) proved the existence of some uncountable, closed U-sets including the 1/3-Cantor set. He was motivated by some work of Hardy and Littlewood (Acta Math., 37(1914)), and Steinhaus (1920), on diophantine approximation to introduce "H-sets"; and proved that such sets are U-sets. In fact, the 1/3-Cantor set is an H-set. In a letter to Lusin, he also expressed his considered opinion, that any U-set is contained in a countable union of H-sets. Although this particular conjecture was proved false by Pyatetskii-Shapiro(1952), it was such questions that focused the direction of the subject.

Actually, Nina Bary had proved the existence of some uncountable, closed U-sets in 1921, and presented her results at Lusin's seminar at

the University of Moscow. They were unpublished at the time of Rajchman's paper. This does not undermine the importance of Rajchman's theorems, since Rajchman's approach illustrated the need for number theoretic (diophantine) properties in the construction of such sets.

Bary proved her first result on U-sets as an undergraduate, and made outstanding contributions to the subject throughout her life. One of her major results is that *the countable union of closed U-sets is a U-set*. The problem is open for the finite union of arbitrary U-sets. Another one of her theorems, which was proven in 1936–1937, asserts that *if α is rational and $E(\alpha)$ is the Cantor set with ratio of dissection α, then $E(\alpha)$ is a U-set if and only if $\beta = 1/\alpha$ is an integer*. This generalizes Rajchman's result about the 1/3-Cantor set C since $C = E(1/3)$. In general, $E(\alpha)$ is constructed by "throwing away" centered open intervals of length $\alpha(b - a)$, where $[a, b]$ is any remaining closed interval at a given step, and where the first step begins with $[a, b] = [0, 1]$, e.g., [KS63, Chapitre I].

A *Pisot-Vigayraghavan* (P-V) number is a real algebraic integer $\beta > 1$ with the property that all the other roots of its minimal polynomial have modulus less than 1. Bary's theorem on Cantor sets of uniqueness has the following spectacular sequel announced by Salem (1943): *if $\alpha \in (0, 1/2)$ then $E(\alpha)$ is a U-set if and only if $\beta = 1/\alpha$ is a P-V number*, e.g., [Bary64], [Ben76, pages 116–117], [Mey72], [Sal63] for the proof, a history of the proof, and recent developments.

3.2.8 The Lusin Conjecture.

In his dissertation of 1915 (actually he published a Comptes Rendus Acad. Sci., Paris note on the relevant material), Lusin conjectured that the Fourier series of every $F \in L^2(\mathbb{T})$ is convergent a.e. This is the *Lusin Conjecture*.

As background for the Lusin Conjecture, du Bois-Reymond (1872) constructed functions $F \in C(\mathbb{T})$ whose Fourier series diverge at some points, cf., [Rog59, pages 75–77], [Zyg59, Volume I, Chapter VIII] and *Exercise 3.45*. Further, just prior to Lusin, there were contributions in this general area by Fatou(1906), Jerosch and Weyl (1908), Weyl (1909), Fejér (1911), W. H. Young (1912), Hobson(1913), Plancherel (1913), and Hardy (1913).

In 1926, Kolmogorov constructed functions $F \in L^1(\mathbb{T}) \backslash L^2(\mathbb{T})$ whose Fourier series diverge *everywhere*! His proof used Kronecker's Theorem, which we shall discuss in *Section 3.2.10*. There were subsequent relevant "log estimates" by Kolmogorov and Seliverstov (1925), Plessner (1926), and Littlewood and Paley (1931). Finally, Lennart Carleson (1966) proved that *if* $F \in L^2(\mathbb{T})$ *then* $S(F) = F$ *a.e.* [Carl66], cf., [Fef73] for a conceptually different proof and [Moz71] for a superb exposition. R. A. Hunt (1968) used the method of Carleson's proof and the theory of interpolation of operators to extend Carleson's result to $L^p(\mathbb{T})$, $p > 1$, cf., [Ash76, pages 20–37] for an elegant presentation by Hunt, and [Ben76, pages 208–210] for a connection between techniques used by Carleson and the FTC. Also, for perspective vis a vis du Bois-Reymond's example and Carleson's Theorem, we have Kahane and Katznelson's Theorem that *if* $E \subseteq \mathbb{T}$ *is a set of measure zero then there is* $F \in C(\mathbb{T})$ *such that* $S(F)(\gamma)$ *diverges for all* $\gamma \in E$, e.g., [Kat76, Chapter 2].

We close this section with remarks by Carleson on the occasion of receiving the 1984 Steele Prize. They concern his proof and a remark about the FFT, cf., *Section 3.9*.

> "When I was a student at Harvard in 1950–1951, A. Zygmund and R. Salem were also in Cambridge and I learned very much from them. They also encouraged me to try to use Blaschke products as examples of a Fourier series which diverges a.e. I worked hard at that then, and all through the years I tried different ideas. Then finally, in 1964 or so, I realized the basic reason why there should exist an example. Very briefly we can describe the main feature of the trigonometric system $\cos nx$, $n \leq 2^m$, by writing down a matrix of ± 1 giving the sequence of $\text{sign}(\cos nx)$ which can occur. This matrix is essentially $2^m \times 2^m$, i.e., very few sequences of signs occur which, of course, is very favorable for examples of divergence. (This is also the basic idea behind the fast Fourier transform.) To my great astonishment, it now turned out that for a random $2^m \times 2^m$ matrix there is no example and then a proof of the convergence theorem

came naturally."

3.2.9 The Dirichlet Box Principle.

a. The *Dirichlet Box Principle* asserts that if Q boxes contain $Q+1$ objects, then at least one of the boxes contains more than one object. This fact may not seem to be at the usual Dirichlet level of brilliance, but it has been a staple in the method of proof of many results since he first made use of it.

An adaptation of the Dirichlet Box Principle is even used in Wiles' proof of Fermat's Last Theorem. In this case the objects are *Hecke rings* and an infinite sequence of sets of boxes is created. The assertion, in the part of the proof due to Taylor and Wiles, is that there are Hecke rings in every set of boxes.

b. Originally, Dirichlet used the box principle to give a new proof of the fact that *if $x \in (0,1)$ is irrational then*

$$\forall \epsilon > 0, \quad \exists p, q \in \mathbb{N} \quad \text{such that}$$

(3.2.7)

$$|x - \frac{p}{q}| < \epsilon \quad \text{and} \quad |x - \frac{p}{q}| < \frac{1}{q^2}.$$

The pairs p, q can be chosen to be relatively prime. (3.2.7) is an elementary result in Diophantine approximation, and Dirichlet's proof (in part c) has the advantage of being applicable to d-dimensional problems, e.g., [HW65, Theorem 201]. The first inequality of (3.2.7) follows from basic properties of $\mathbb{R}$; and the second inequality gives insight into the rapidity of rational approximation to irrationals.

c. To prove (3.2.7), let $Q > 1$, and consider the Q boxes $[\frac{n}{Q}, \frac{n+1}{Q}]$, $n = 0, \ldots, Q-1$, and the $Q+1$ numbers, 0, $x - [x]$, $2x - [2x], \ldots, Qx - [Qx]$. By the Dirichlet Box Principle there are integers $0 \leq q_1 < q_2 \leq Q$ and $p \in \mathbb{N}$ for which $|qx - p| < 1/Q$, where $q = q_2 - q_1 \in (0, Q] \cap \mathbb{N}$ and $p = [q_2 x] - [q_1 x]$. Thus the second inequality of (3.2.7) is valid. This part of the proof also works for rational x.

Since $\epsilon > 0$ is given in (3.2.7), we choose $Q = Q(\epsilon) = [\frac{1}{\epsilon}] + 1$ for the above argument. In particular, $\epsilon \geq 1/Q$ and so $|x - \frac{p}{q}| < \epsilon/q < \epsilon$.

3.2.10 Kronecker Sets and U-Sets.

A significant refinement of (3.2.7), which is deeper than d-dimensional versions of (3.2.7), is *Kronecker's Theorem.* Kronecker (1884) proved that *if* $\{1, \gamma_1, \ldots, \gamma_d\} \subseteq \mathbb{R}$ *is linearly independent over the rationals, if* $\{\lambda_1, \ldots, \lambda_d\} \subseteq \mathbb{R}$, *and if* $\epsilon, N > 0$, *then there are integers* $q > N$ *and* $p_1, \ldots, p_d$ *such that*

$$(3.2.8) \qquad \forall j = 1, \ldots, d, \quad |q\gamma_j - p_j - \lambda_j| < \epsilon.$$

Dirichlet's analysis was for the case $\lambda_j = 0$. With the same hypotheses, the conclusion (3.2.8) can be reworded to assert the existence of q for which

$$(3.2.9) \qquad \forall j = 1, \ldots, d, \quad |e^{2\pi i q\gamma_j} - e^{2\pi i \lambda_j}| < \epsilon.$$

There are several different proofs, e.g., [Ben75, Theorem 3.2.7], [HW65, Chapter 23], [KK64], [Kat76, pages 181–183].

Because of (3.2.9) we say that a closed set $E \subseteq \mathbb{T}$ is a *Kronecker set* if for each $\epsilon > 0$ and continuous function $F : E \to \mathbb{C}$, for which $|F| = 1$ on E, there is $q \in \mathbb{Z}$ such that

$$\sup_{\gamma \in E} |e^{2\pi i q\gamma} - F(\gamma)| < \epsilon.$$

In 1962, Paul Malliavin proved that if $E \subseteq \mathbb{T}$ is closed and if every closed subset of E is a set of spectral synthesis, then E is a U-set. In 1965, Nicholas Varopoulos proved that Kronecker sets satisfy the hypothesis of Malliavin's result; and, hence, *Kronecker sets are U-sets.* This is the "tip of the iceberg," cf., [Rud62], [KS63], [Kah70], [Ben71], [LP71], [Mey72].

3.3 Integration and differentiation of Fourier series

3.3.1 Example. INTEGRATION OF SERIES

If $\sum_{n=1}^{\infty} F_n$ is a uniformly convergent series of continuous functions F_n on $[0, 1]$, then

$$(3.3.1) \qquad \int_0^1 \left(\sum_{n=1}^{\infty} F_n(\gamma) \right) d\gamma = \sum_{n=1}^{\infty} \int_0^1 F_n(\gamma) \, d\gamma,$$

e.g., [Apo57]. On the other hand, it is well known that the hypotheses, $\{F_n : n = 1, \ldots\} \subseteq C[0,1]$ and $\sum_{n=1}^{\infty} F_n = F \in C[0,1]$, where the convergence is pointwise on $[0,1]$, are *not* sufficient to ensure (3.3.1), e.g., [Har49], [Ben76, Section 3.3]. For example, if we let $F_1(\gamma) = \gamma(1 - \gamma)$ and $F_n(\gamma) = n^2\gamma(1 - \gamma)^n - (n - 1)^2\gamma(1 - \gamma)^{n-1}$ for $n > 1$, then it is easy to see that each $F_n \in C[0,1]$ (and each $F_n \in C(\mathbb{T})$ since $F_n(0) = F_n(1) = 0$), $\sum_{n=1}^{\infty} F_n \equiv F = 0$ on $[0,1]$, $\int_0^1 \left(\sum_{n=1}^{\infty} F_n(\gamma) \right) d\gamma = 0$, and

$$\sum_{n=1}^{\infty} \int_0^1 F_n(\gamma)\, d\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \int_0^1 F_n(\gamma)\, d\gamma$$

$$= \lim_{N \to \infty} \int_0^1 N^2 \gamma (1 - \gamma)^N \, d\gamma = \lim_{N \to \infty} \frac{N^2}{(N + 1)(N + 2)} = 1,$$

cf., *Section 3.4* where we discuss approximate identities for $L^1(\mathbb{T})$.

The following theorem is a remarkable feature of Fourier series. It asserts that (3.3.1) is valid when the series $\sum_{n=1}^{\infty} F_n$ is replaced by the Fourier series of any function in $L^1(\mathbb{R})$. In particular, the Fourier series to be integrated can diverge everywhere, as in Kolmogorov's result mentioned in *Section 3.2.8*, cf., *Exercise 3.45*.

3.3.2 Theorem. INTEGRATION OF FOURIER SERIES

Let $F \in L^1(\mathbb{T}_{2\Omega})$. *The Fourier series* $S(F)$ *of* F *, with Fourier coefficients* $f = \{f[n]\}$, *can be integrated term by term, i.e.,*

$$(3.3.2) \qquad \forall \alpha, \beta \in \widehat{\mathbb{R}}, \quad \int_\alpha^\beta S(F)(\gamma)\, d\gamma = \int_\alpha^\beta F(\gamma)\, d\gamma,$$

where the left side of (3.3.2) denotes

$$\sum f[n] \int_\alpha^\beta e^{-\pi i n \gamma / \Omega}\, d\gamma.$$

Proof. Define

$$G(\gamma) = \frac{1}{2\Omega} \int_0^\gamma (F(\lambda) - f[0])\, d\lambda$$

for $\gamma \in [0, 2\Omega)$. Consequently, $G(0) = G(2\Omega) = 0$, and G can be extended 2Ω-periodically to $\hat{\mathbb{R}}$ with the property that $G \in AC_{\text{loc}}(\mathbb{R})$. Let $g = \{g[n]\}$ be the sequence of Fourier coefficients of G. As such, we can apply the Dirichlet Theorem, properly modified as in *Remark 3.1.7b*, to assert that $S(G) = G$ on $\mathbb{T}_{2\Omega}$, i.e.,

$$(3.3.3) \qquad \forall \gamma \in \mathbb{T}_{2\Omega}, \quad G(\gamma) = g[0] + \sum{}' g[n] e^{-\pi i n \gamma / \Omega},$$

where $\sum'$ denotes summation over $\mathbb{Z} \backslash \{0\}$. For $n \neq 0$, we compute

$$g[n] = \frac{1}{2\Omega} \int_0^{2\Omega} \left[\frac{1}{2\Omega} \int_0^\gamma \left(F(\lambda) - f[0] \right) d\lambda \right] e^{\pi i n \gamma / \Omega} \, d\lambda$$

$$= \frac{1}{(2\Omega)^2} \frac{\Omega}{\pi i n} e^{\pi i n \gamma / \Omega} \int_0^\gamma \left(F(\lambda) - f[0] \right) d\lambda \Big|_{\gamma=0}^{2\Omega}$$

$$- \frac{1}{(2\Omega)^2} \frac{\Omega}{\pi i n} \int_0^{2\Omega} \left(F(\gamma) - f[0] \right) e^{\pi i n \gamma / \Omega} \, d\gamma$$

$$= \frac{-f[n]}{2\pi i n}.$$

The last equation follows since $\frac{1}{2\Omega} \int_0^{2\Omega} F(\lambda) \, d\lambda = f[0]$ and $\int_0^{2\Omega} e^{\pi i n \gamma / \Omega} \, d\gamma = 0$. Integration by parts is allowable by the (local) absolute continuity. Combining the above computation for $g[n]$ with (3.3.3), we obtain

$$(3.3.4) \qquad \forall \gamma \in \mathbb{T}_{2\Omega}, \quad G(\gamma) = g[0] - \sum{}' \frac{f[n]}{2\pi i n} e^{-\pi i n \gamma / \Omega},$$

so that, since $G(0) = 0$,

$$(3.3.5) \qquad\qquad\qquad g[0] = \sum{}' \frac{f[n]}{2\pi i n}.$$

In particular, the series $\sum' f[n]/n$ converges, cf., *Remark 3.1.7a* about symmetric convergence.

By definition of G, (3.3.4) becomes

$$(3.3.6) \quad \frac{1}{2\Omega} \int_0^\gamma F(\lambda) \, d\lambda = \frac{\gamma}{2\Omega} f[0] + g[0] - \sum{}' \frac{f[n]}{2\pi i n} e^{-\pi i n \gamma / \Omega}.$$

By definition of $\int_0^\gamma S(F)(\lambda)\, d\lambda$ we have

$$\frac{1}{2\Omega} \int_0^\gamma S(F)(\lambda)\, d\lambda = \frac{1}{2\Omega} \sum f[n] \int_0^\gamma e^{-\pi i n\lambda/\Omega}\, d\lambda$$

$$= \frac{\gamma}{2\Omega} f[0] - \frac{1}{2\Omega} \sum{}' f[n] \frac{\Omega}{\pi i n} (e^{-\pi i n\gamma/\Omega} - 1)$$

$$= \frac{\gamma}{2\Omega} f[0] - \frac{1}{2\Omega} \sum{}' f[n] \frac{\Omega}{\pi i n} e^{-\pi i n\gamma/\Omega} + g[0].$$

Consequently, (3.3.2) is obtained for the interval $[0, \gamma] \subseteq [0, 2\Omega]$ by combining (3.3.6) with this last calculation.

The case for the interval $[\alpha, \beta] \subseteq [0, 2\Omega]$ is obtained by writing $\int_\alpha^\beta = \int_\alpha^0 + \int_0^\beta = \int_0^\beta - \int_0^\alpha$; and then making the natural adjustments for other values of α and β. $\square$

The following result is a consequence of (3.3.5).

3.3.3 Corollary. A PROPERTY OF FOURIER COEFFICIENTS $f \in A(\mathbb{Z})$

Let $F \in L^1(\mathbb{T}_{2\Omega})$. The series,

$$\sum{}' \frac{f[n]}{n}$$

converges, where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F.

3.3.4 Example. NECESSARY CONDITIONS FOR INTEGRABILITY OF TRIGONOMETRIC SERIES

a. The Riemann-Lebesgue Lemma asserts that if $F \in L^1(\mathbb{T}_{2\Omega})$ then $\lim_{|n| \to \infty} f[n] = 0$, where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F. On the other hand, suppose we are given a trigonometric series $\sum c_n e^{-2\pi i n\gamma}$ for which $\lim_{|n| \to \infty} c_n = 0$. Is there any way we can determine if this series is or is not the Fourier series of some function $F \in L^1(\mathbb{T}_{2\Omega})$, cf., *Section 3.2.6?*

We *can* assert that the trigonometric series,

(3.3.7) $$\sum_{n=2}^{\infty} \frac{\sin(\pi n\gamma/\Omega)}{\log n},$$

is not the Fourier series of an element $F \in L^1(\mathbb{T}_{2\Omega})$ even though the coefficients tend to 0 at $\pm\infty$, à la the necessary conditions given by the Riemann-Lebesgue Lemma. To verify this claim we argue as follows. First, we write (3.3.7) as

$$-\frac{1}{2i} \sum_{|n|\geq 2} \frac{\text{sgn } n}{\log |n|} e^{-\pi i n \gamma/\Omega},$$

where $\text{sgn } n = n/|n|$. Then we apply *Corollary 3.3.3* to observe that if (3.3.7) were a Fourier series (of an element $F \in L^1(\mathbb{T}_{2\Omega})$) then

$$\sum_{|n|\geq 2} \frac{1}{|n| \log |n|} < \infty,$$

which is false by the integral test, e.g., *Exercise 3.10*. Thus, we have constructed a sequence $f \in c_0(\mathbb{Z})\backslash A(\mathbb{Z})$, cf., *Example 1.4.4* where we constructed the analogue of (3.3.7) for the case $C_0(\mathbb{R})\backslash A(\mathbb{R})$. An example $F \in C(\mathbb{T}_{2\Omega})\backslash A(\mathbb{T}_{2\Omega})$ is constructed in *Exercise 3.43*, cf., *Exercise 3.45* and du Bois-Reymond's example mentioned in *Section 3.2.8*.

b. It does turn out, however, that the series (3.3.7) converges pointwise for each $\gamma \in \widehat{\mathbb{R}}$, e.g., *Exercise 3.29b*.

3.3.5 Theorem. DIFFERENTIATION OF FOURIER SERIES

Let $F \in AC(\mathbb{T}_{2\Omega})$. Then $F' \in L^1(\mathbb{T}_{2\Omega})$ (ordinary differentiation) and
$$S'(F) = S(F'),$$
where $S'(F)$ denotes the term by term differentiated series

$$-\sum{}' \frac{\pi i n}{\Omega} f[n] e^{-\pi i n \gamma/\Omega},$$

and where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F.
Proof. Clearly, $F' \in L^1(\mathbb{T}_{2\Omega})$, e.g., *Remark A.21*. By the absolute continuity and 2Ω-periodicity of F, and by the FTC, we compute

$$(F')^{\vee}[n] = -\frac{1}{2\Omega}\frac{\pi i n}{\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{\pi i n \gamma/\Omega} \, d\gamma = -\frac{\pi i n}{\Omega} \overset{\vee}{F}[n]$$

for $n \neq 0$. If $n = 0$, then

$$(F')^{\vee}[0] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F'(\gamma)\, d\gamma = 0$$

since the absolute continuity again allows us to use FTC. The result is obtained.   □

3.3.6 Example. $\zeta(2)$

   a.   Let $F(\gamma) = \frac{\pi - \gamma}{2}$ on $[0, 2\pi)$, and consider F as an element of $L^1(\mathbb{T}_{2\pi})$.

Figure 3.1

We shall compute $S(F)$. F is odd on $(-\pi, \pi)$, and we have

$$S(F)(\gamma) = \sum_{n=1}^{\infty} b_n \sin n\gamma,$$

where

$$\forall n \geq 1, \quad b_n = \frac{1}{\pi} \int_{0}^{2\pi} \frac{1}{2}(\pi - \gamma) \sin n\gamma\, d\gamma.$$

We calculate

$$b_n = -\frac{1}{2\pi} \left[-\frac{\gamma}{n} \cos n\gamma \,\Big|_{0}^{2\pi} + \frac{1}{n} \int_{0}^{2\pi} \cos n\gamma\, d\gamma \right] = \frac{1}{n}.$$

Thus,

$$S(F)(\gamma) = \sum_{n=1}^{\infty} \frac{\sin n\gamma}{n}.$$

By the Dirichlet Theorem, $S(F)(\gamma) = F(\gamma)$ on $\widehat{\mathbb{R}} \backslash \cup \{2\pi n\}$ and $S(F)(2\pi n) = 0$.

b. Since $F \in L^1(\mathbb{T}_{2\pi})$, we have

$$\forall \gamma \in [0, 2\pi], \quad \int_0^\gamma S(F)(\lambda)\, d\lambda = \int_0^\gamma F(\lambda)\, d\lambda,$$

by *Theorem 3.3.2.* This becomes

$$\forall \gamma \in [0, 2\pi], \quad \sum_{n=1}^\infty \frac{1}{n} \int_0^\gamma \sin n\lambda\, d\lambda = \frac{\gamma\pi}{2} - \frac{\gamma^2}{4}$$

and the left side is $-\sum_{n=1}^\infty \frac{1}{n^2}(\cos n\gamma - 1)$. Thus

$$\forall \gamma \in [0, 2\pi], \quad \frac{\gamma}{4}(2\pi - \gamma) = -\sum_{n=1}^\infty \frac{1}{n^2}(\cos n\gamma - 1).$$

c. Integrating both sides of the last expression in part *b* we obtain

$$\int_0^{2\pi} \frac{\gamma}{4}(2\pi - \gamma)\, d\gamma$$

$$= 2\pi \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{n=1}^\infty \frac{1}{n^2} \int_0^{2\pi} \cos n\gamma\, d\gamma = 2\pi \sum_{n=1}^\infty \frac{1}{n^2}.$$

Consequently,

$$(3.3.8) \qquad \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

d. The Riemann ζ-function $\zeta(s) = \sum_{n=1}^\infty 1/n^s$, discussed in *Example 2.4.6g*, was defined by Pietro Mengoli (1625-1686) in 1650. He showed that the harmonic series $\zeta(1)$ diverges. The problem of evaluating $\zeta(k)$, for integers $k \geq 2$, attracted the attention of British mathematicians, including James Gregory (1638-1675). Henry Oldenburg (c.1615-1677), first Secretary of the Royal Society in London, wrote to Gottfried Leibniz (1646–1716) asking him to evaluate $\zeta(2)$. This occurred during Leibniz's visit to London in 1673. In 1696, Leibniz admitted his inability to solve this problem. Earlier, in 1689, Jakob

Bernoulli (1654-1705), Daniel's uncle, had also tried and apparently given-up summing $\zeta(2)$, cf., [Pól54, pages 17–22], [Kli72, pages 448–449], [Ebe83] for tantalizing historical remarks and incisive analysis.

Around 1736, Euler was able to state (3.3.8) by an ingenious argument outlined in *Exercise 3.28*, cf., *Exercise 3.38*. His *Introductio in Analysin Infinitorum* (1748) contains (3.3.8) and many similar results. In light of our discussion in *Sections 3.2.1-3.2.3*, it should also be pointed out that the *Introductio* also "defines" a *function* of a "variable quantity" as "any analytic expression whatsoever made up from that variable quantity and from numbers or constant quantities."

The following result can be used to prove the Classical Sampling Theorem (*Theorem 3.10.10*), e.g., [Ben92b, pages 447–449]. The proof of *Theorem 3.3.7* is similar to that of *Theorem 3.3.2*, e.g., *Exercise 3.15*.

3.3.7 Theorem. INTEGRATION OF FOURIER SERIES – A REFINEMENT
   Let $F \in L^1(\mathbb{T}_{2\Omega})$ and $G \in BV(\mathbb{T}_{2\Omega})$. Then

$$(3.3.9) \quad \forall \alpha, \beta \in \widehat{\mathbb{R}}, \quad \int_\alpha^\beta S(F)(\gamma)G(\gamma)\,d\gamma = \int_\alpha^\beta F(\gamma)G(\gamma)\,d\gamma,$$

where the left side of (3.3.9) denotes

$$\sum f[n] \int_\alpha^\beta G(\gamma)e^{-\pi i n \gamma/\Omega}\,d\gamma,$$

and where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F.

3.3.8 Remark. INTEGRATION OF FOURIER SERIES – EVOCATIONS
   a. If $\beta - \alpha = 2\Omega$ in *Theorem 3.3.7*, then (3.3.9) becomes *Parseval's formula*,

$$\lim_{N \to \infty} \sum_{|n| \leq N} f[n]g[-n] = \frac{1}{2\Omega} \int_\alpha^{\alpha+2\Omega} F(\gamma)G(\gamma)\,d\gamma,$$

where $g = G^\vee$. With the adjustment as in *Proposition 1.10.4*, we obtain *Parseval's formula* in the form,

$$(3.3.10) \qquad \lim_{N \to \infty} \sum_{|n| \leq N} f[n]\overline{g[n]} = \int_{\mathbb{T}_{2\Omega}} F(\gamma)\overline{G(\gamma)}\,d\gamma,$$

where $F \in L^1(\mathbb{T}_{2\Omega})$, $G \in BV(\mathbb{T}_{2\Omega})$, and $\{f[n]\}$ and $\{g[n]\}$ are the Fourier coefficients of F and G, respectively, cf., *Theorem 3.4.12b* and *Exercise 3.47* for other statements of *Parseval's formula*.

b. With respect to *Corollary 3.3.3*, we recall the following Hardy and Littlewood Theorem: *if $F \in L^1(\mathbb{T}_{2\Omega})$ and $f[n] = 0$ for all $n < 0$, where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F, then*

$$\sum_{n=0}^{\infty} \frac{|f[n]|}{n+1} \leq \pi \|F\|_{L^1(\mathbb{T}_{2\pi})} < \infty.$$

This result is difficult to prove. One proof involves a fundamental factorization theorem for the so-called *Hardy space* $H^1(\mathbb{T}_{2\Omega})$, as well as the following *double series theorem* due to Hilbert: *if $f, g \in \ell^2(\mathbb{N} \cup \{0\})$ then*

$$\sum_{j,k=0}^{\infty} \frac{|f[j]g[k]|}{j+k+1} \leq \pi \|f\|_{\ell^2(\mathbb{N} \cup \{0\})} \|g\|_{\ell^2(\mathbb{N} \cup \{0\})},$$

where π is the best possible constant, as proved by Schur, e.g., [Hel83, pages 94–99], cf., *Exercise 3.34* for related material on Hilbert transforms of sequences.

c. It is well known that the desirable statement,

(3.3.11) $$\lim_{N \to \infty} \|S_N(F) - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0,$$

is *not* true for all $F \in L^1(\mathbb{T}_{2\Omega})$, e.g., *Example 3.4.9*.

On the other hand, a sequence $\{F_N\} \subseteq L^1(\mathbb{T}_{2\Omega})$ converges to $F \in L^1(\mathbb{T}_{2\Omega})$ *weakly*, i.e.,

$$\forall G \in L^\infty(\mathbb{T}_{2\Omega}), \quad \lim_{N \to \infty} \int_{\mathbb{T}_{2\Omega}} (F_N(\gamma) - F(\gamma))G(\gamma)\,d\gamma = 0,$$

if and only if

(3.3.12) $$\lim_{N \to \infty} \int_A (F_N(\gamma) - F(\gamma))\,d\gamma = 0$$

for every Lebesgue measurable set $A \subseteq \mathbb{T}_{2\Omega}$, e.g., [RN55, page 89]. If we have weak convergence, or, equivalently, (3.3.12), then (3.3.11) is true for $S_N(F) = F_N$ if $\{F_N\}$ converges to F in measure, e.g., [Ben76,

page 226]. (Of course, norm convergence always implies convergence in measure.) Further, Dieudonné and Grothendieck proved that (3.3.12) is true for all Lebesgue measurable sets $A \subseteq \mathbb{T}_{2\Omega}$ if and only if it is true for all open sets $A \subseteq \mathbb{T}_{2\Omega}$, e.g., [Ben76, page 225].

Because of this general relationship between weak and norm convergence, and because we would like to have (3.3.11) (or at least know how close we are to it), we note the following reformulation of *Theorem 3.3.2*: *Theorem 3.3.2* is (3.3.12) for all *intervals* A in the case $F_N = S_N(F)$. Further, *Theorem 3.3.7* gives (3.3.12) in this case for all finite unions A of intervals.

3.4   The $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ theories

We showed in *Remark 3.1.3* that $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$, and have already noted in *Section 3.2.8* that Fourier series of $L^2(\mathbb{T})$ functions converge a.e., whereas there are $L^1(\mathbb{T})$ functions whose Fourier series diverge at every point. Conceptually there are deeper differences between $L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ than the fact that larger spaces may allow more instances of unusual behavior; briefly, $L^1(\mathbb{T})$ has *algebraic* properties and $L^2(\mathbb{T})$ has *geometric* properties which characterize their Fourier analysis.

3.4.1 Definition. CONVOLUTION

a. Let $F, G \in L^1(\mathbb{T}_{2\Omega})$. The *convolution* of F and G, denoted by $F * G$, is

$$F * G(\gamma) = \int_{\mathbb{T}_{2\Omega}} F(\gamma - \lambda)G(\lambda)\, d\lambda = \int_{\mathbb{T}_{2\Omega}} F(\lambda)G(\gamma - \lambda)\, d\lambda.$$

(Recall that "$\int_{\mathbb{T}_{2\Omega}}$" designates "$\frac{1}{2\Omega}\int_{\alpha}^{\alpha+2\Omega}$" for any fixed $\alpha \in \widehat{\mathbb{R}}$.) As with $L^1(\mathbb{R})$ and *Exercise 1.31*, it is not difficult to prove that $F * G \in L^1(\mathbb{T}_{2\Omega})$ and

$$\forall F, G \in L^1(\mathbb{T}_{2\Omega}),$$

(3.4.1)

$$\|F * G\|_{L^1(\mathbb{T}_{2\Omega})} \le \|F\|_{L^1(\mathbb{T}_{2\Omega})} \|G\|_{L^1(\mathbb{T}_{2\Omega})}.$$

b. $L^1(\mathbb{T}_{2\Omega})$ is a *commutative algebra* taken with the operations of addition and convolution, i.e., $L^1(\mathbb{T}_{2\Omega})$ is a vector complex vector space

under addition, and convolution is distributive with respect to addition, as well as being associative, and commutative.

3.4.2 Proposition.

Let $F, G \in L^1(\mathbb{T}_{2\Omega})$, with corresponding sequences $f = \{f[n]\}, g = \{g[n]\} \in A(\mathbb{Z})$ of Fourier coefficients, i.e., $\hat{f} = F$ and $\hat{g} = G$. Then $fg = \{f[n]g[n]\} \in A(\mathbb{Z})$ is the sequence of Fourier coefficients of $F * G \in L^1(\mathbb{T}_{2\Omega})$, i.e.,

$$\forall n \in \mathbb{Z}, \quad f[n]g[n] = \int_{\mathbb{T}_{2\Omega}} F * G(\gamma) e^{\pi i n \gamma / \Omega} \, d\gamma.$$

The proof of *Proposition 3.4.2* is the same as that of *Proposition 1.5.2*.

3.4.3 Definition. APPROXIMATE IDENTITY

An *approximate identity* is a family $\{K_{(\lambda)} : \lambda > 0\} \subseteq L^1(\mathbb{T}_{2\Omega})$ of functions with the properties

a. $\forall \lambda > 0, \quad \int_{\mathbb{T}_{2\Omega}} K_{(\lambda)}(\gamma) d\gamma = 1,$

b. $\exists C \geq 1$ such that $\forall \lambda > 0, \quad \|K_{(\lambda)}\|_{L^1(\mathbb{T}_{2\Omega})} \leq C,$

c. $\forall \eta \in (0, \Omega], \quad \lim_{\lambda \to \infty} \frac{1}{2\Omega} \int_{\eta \leq |\gamma| \leq \Omega} |K_{(\lambda)}(\gamma)| d\gamma = 0.$

3.4.4 Theorem. APPROXIMATE IDENTITY THEOREM

a. Let $F \in C(\mathbb{T}_{2\Omega})$ and let $\{K_{(\lambda)}\} \subseteq L^1(\mathbb{T}_{2\Omega})$ be an approximate identity. Then

$$(3.4.2) \qquad \lim_{\lambda \to \infty} \|F - F * K_{(\lambda)}\|_{L^\infty(\mathbb{T}_{2\Omega})} = 0.$$

b. Let $F \in L^1(\mathbb{T}_{2\Omega})$ and let $\{K_{(\lambda)}\} \subseteq L^1(\mathbb{T}_{2\Omega})$ be an approximate identity. Then

$$(3.4.3) \qquad \lim_{\lambda \to \infty} \|F - F * K_{(\lambda)}\|_{L^1(\mathbb{T}_{2\Omega})} = 0,$$

cf., Theorem 1.6.9a.

The proof of *Theorem 3.4.4a* follows the proof of *Theorem 1.6.9a*, but for the case of $\|\dots\|_{L^\infty(\mathbf{T}_{2\Omega})}$ instead of $\|\cdots\|_{L^1(\mathbb{R})}$. In fact, by the uniform continuity of F, (1.6.6) can be replaced by the statement that for each $\epsilon > 0$ there is $\eta > 0$ for which

$$\forall |\lambda| < \eta, \quad \|F - \tau_\lambda F\|_{L^\infty(\mathbf{T}_{2\Omega})} < \epsilon/C.$$

This allows us to prove

$$\overline{\lim_{\lambda \to \infty}} \|F - F * K_{(\lambda)}\|_{L^\infty(\mathbf{T}_{2\Omega})} \leq \epsilon$$

analogous to the proof of *Theorem 1.6.9a*.

The proof of *Theorem 3.4.4b* follows from part *a*, e.g., *Exercise 3.16*.

3.4.5 Example. THE DIRICHLET AND FEJÉR KERNELS

a. The *Dirichlet function D_N* on $\mathbf{T}_{2\Omega}$ is defined by

$$(3.4.4) \qquad\qquad \forall \gamma \in \widehat{\mathbb{R}}, \quad D_N(\gamma) = \sum_{|n| \leq N} e^{-\pi i n \gamma / \Omega}.$$

It is not difficult to show that the trigonometric polynomial D_N can be written as

$$(3.4.5) \qquad\qquad D_N(\gamma) = \frac{\sin(N + \frac{1}{2})\frac{\pi\gamma}{\Omega}}{\sin\frac{\pi\gamma}{2\Omega}},$$

where $D_N(2k\Omega)$ is $2N + 1$ for $k \in \mathbb{Z}$, e.g., *Exercise 3.11a*. The family $\{D_N : N \in \mathbb{N} \cup \{0\}\}$ is the *Dirichlet kernel* on $\mathbf{T}_{2\Omega}$, cf., the Dirichlet kernels on $\mathbb{R}$ and $\mathbb{Z}$ in *Remark 1.6.4* and *Remark 3.1.2*, respectively. Using the notation of *Definition 3.4.3*, we see that $K_{(\lambda)} \equiv D_N$ is not an approximate identity since $D_N \notin L^1(\mathbf{T}_{2\Omega})$.

b. The *Fejér function W_N* on $\mathbf{T}_{2\Omega}$ is defined by

$$(3.4.6) \qquad \forall \gamma \in \widehat{\mathbb{R}}, \quad W_N(\gamma) = \sum_{|n| \leq N} \left(1 - \frac{|n|}{N+1}\right) e^{-\pi i n \gamma / \Omega}.$$

It is not difficult to show that the trigonometric polynomial W_N can be written as

$$W_N(\gamma) = \frac{D_0(\gamma) + \dots + D_N(\gamma)}{N+1}$$

$$(3.4.7)$$

$$\frac{1}{N+1}\left(\frac{\sin(N+1)\frac{\pi\gamma}{2\Omega}}{\sin\frac{\pi\gamma}{2\Omega}}\right)^2,$$

where $W_N(2k\Omega)$ is $N + 1$ for $k \in \mathbb{Z}$, e.g., *Exercise 3.11b*. The family $\{W_N : N \in \mathbb{N} \cup \{0\}\}$ is the *Fejér kernel* on $\mathbb{T}_{2\Omega}$, cf., the Fejér kernel on $\mathbb{R}$ and $\mathbb{Z}$ in *Remark 1.6.4* and *Remark 3.1.2*, respectively.

c. *The Fejér kernel is an approximate identity.* To see this, first note that $W_N \geq 0$ by (3.4.7), and

$$\int_{\mathbb{T}_{2\Omega}} W_N(\gamma)\, d\gamma = 1$$

by the definition (3.4.6) and the fact that $\int_{\mathbb{T}_{2\Omega}} e^{-\pi i n\gamma/\Omega}\, d\gamma = 0$ for $n \neq 0$. Thus, parts *a* and *b* of *Definition 3.4.3* are valid. Finally, we obtain part *c* of *Definition 3.4.3* by (3.4.7) and the estimate

$$\frac{1}{2\Omega} \int_{\eta \leq |\gamma| \leq \Omega} |W_N(\gamma)|\, d\gamma \leq \frac{2(\Omega - \eta)}{2\Omega(N+1)} \sup_{\eta \leq |\gamma| \leq \Omega} \frac{1}{\sin^2 \frac{\pi\gamma}{2\Omega}}$$

$$\leq \frac{\Omega - \eta}{\Omega(N+1)} \frac{1}{\sin^2 \frac{\pi\eta}{2\Omega}}.$$

In *Section 3.2.8*, we mentioned du Bois-Reymond's example (1872) of a function $F \in C(\mathbb{T})$ for which $\{S_N(F)(0)\}$ diverges, cf., *Exercise 3.45*. At the risk of being an alarmist over 100 years after the fact, it is not an exaggeration to say that this example dampened some of the optimism for a comprehensive theory of the representation of functions by trigonometric series. Fejér's result (1904), stated below in *Theorem 3.4.6a*, came none too soon, e.g., [Bir73, pages 150–156] for a translation of the relevant parts of Fejér's original paper.

3.4.6 Theorem. FEJÉR THEOREM

a. *Let $F \in C(\mathbb{T}_{2\Omega})$, and let $f = \{f[n]\} \in A'(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ be its sequence of Fourier coefficients. Then*

$$(3.4.8) \qquad F * W_N(\gamma) = \sum_{|n| \leq N} \left(1 - \frac{|n|}{N+1}\right) f[n] e^{-\pi i n\gamma/\Omega}$$

and

$$(3.4.9) \qquad \lim_{N \to \infty} \|F - F * W_N\|_{L^\infty(\mathbb{T}_{2\Omega})} = 0.$$

b. Let $F \in L^1(\mathbb{T}_{2\Omega})$. Then

$$\lim_{N \to \infty} \int_{\mathbb{T}_{2\Omega}} \left| F(\gamma) - \sum_{|n| \leq N} \left(1 - \frac{|n|}{N+1} \right) f[n] e^{-\pi i n \gamma / \Omega} \right| d\gamma = 0,$$

cf., Theorem 1.6.9b.

Proof. a. (3.4.8) follows from a direct computation of the left side, cf., *Exercise 3.12a*. (3.4.9) follows by combining *Example 3.4.5c* with *Theorem 3.4.4a*. Part *b* follows from part *a*, e.g., *Exercise 3.16*.   □

3.4.7 Corollary. UNIQUENESS

Let $F \in L^1(\mathbb{T}_{2\Omega})$ and assume $f[n] = 0$ for each $n \in \mathbb{Z}$, where $f = \{f[n]\}$ is the sequence of Fourier coefficients of F. Then F is the 0-function, cf., Theorem 1.6.9c.

3.4.8 Remark. WEIERSTRASS APPROXIMATION THEOREM

a. The *Weierstrass Approximation Theorem* (1885) asserts that if $F \in C[\alpha, \beta]$ then there is a sequence $\{P_N\}$ of polynomials for which

$$(3.4.10) \qquad\qquad \lim_{N \to \infty} \|F - P_N\|_{L^\infty[\alpha, \beta]} = 0.$$

(3.4.10) can be derived from (3.4.9) in the following way. By translation we can take $F \in C[-\Omega, \Omega]$ without loss of generality. Next choose c such that $G(-\Omega) = G(\Omega)$ where $G(\gamma) \equiv F(\gamma) - c\gamma$ for $\gamma \in [-\Omega, \Omega]$. In fact, let

$$c = \frac{F(\Omega) - F(-\Omega)}{2\Omega}.$$

Apply *Theorem 3.4.6* to G considered as an element of $C(\mathbb{T}_{2\Omega})$. Finally, uniformly approximate the trigonometric polynomials $G * W_N$ on $[-\Omega, \Omega]$ by polynomial approximants of their Taylor series expansions.

b. There are extensive developments of the Weierstrass Theorem, many of which have evolved from Stone's celebrated Stone-Weierstrass Theorem (1937). We refer to [BD81] and [Bur84], replete with ingenuity and scholarship, for recent contributions to the Stone-Weierstrass Theorem in a functional analytic uniform algebra setting.

In *Remark 3.3.8c* we asserted that the partial sums $S_N(F)$ do not necessarily converge to F in $L^1(\mathbb{T}_{2\Omega})$. We shall now prove this assertion.

3.4.9 Example. LEBESGUE CONSTANTS AND AN L^1-CONVERGENCE PROBLEM

a. Analogous to (3.4.8), a direct computation shows that

$$\forall F \in L^1(\mathbb{T}_{2\Omega}), \quad S_N(F)(\gamma) = F * D_N(\gamma),$$

cf., *Exercise 3.12.*

b. The *Lebesgue constants* are defined as $\|D_N\|_{L^1(\mathbb{T}_{2\Omega})}$ for each N. We shall prove that there is a *bounded sequence* $\{C(N) : N \geq 2\}$ such that

$$(3.4.11) \qquad \forall N \geq 2, \quad \|D_N\|_{L^1(\mathbb{T}_{2\Omega})} = \frac{4}{\pi^2} \log N + C(N),$$

cf., *Remark 3.3.8b.*

To see this we use (3.4.5) as follows. First,

$$\|D_N\|_{L^1(\mathbb{T}_{2\Omega})} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} |\sin(2N+1)x| \left| \frac{1}{\sin x} - \frac{1}{x} + \frac{1}{x} \right| dx,$$

so that

$$(3.4.12) \qquad \left| \|D_N\|_{L^1(\mathbb{T}_{2\Omega})} - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{|\sin(2N+1)x|}{x} dx \right|$$

$$\leq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left| \frac{1}{\sin x} - \frac{1}{x} \right| dx = C_1,$$

by the triangle inequality, where $C_1 < \infty$. In fact, it is easy to check that $\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = 0$.

Next, letting $t = (2N + 1)x$ and dividing the domain of integration according to the sign of the sine, we have

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{|\sin(2N + 1)x|}{x} \, dx = \frac{2}{\pi} \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin t|}{t} \, dt$$

$$+ \frac{2}{\pi} \int_{N\pi}^{N\pi + \frac{\pi}{2}} \frac{|\sin t|}{t} dt$$

(3.4.13)

$$= \frac{2}{\pi} \int_0^\pi \sin u \left(\sum_{k=1}^{N-1} \frac{1}{u + k\pi} \right) \, du$$

$$+ \frac{2}{\pi} \int_0^\pi \frac{\sin u}{u} \, du + \frac{2}{\pi} \int_{N\pi}^{N\pi + \frac{\pi}{2}} \frac{|\sin u|}{u} \, du.$$

Since both $\sin u \geq 0$ and

$$\sum_{k=2}^{N} \frac{1}{k\pi} \leq \sum_{k=1}^{N-1} \frac{1}{u + k\pi} \leq \sum_{k=1}^{N-1} \frac{1}{k\pi}$$

on $[0, \pi]$, we can use the integral test to compute

$$\frac{4}{\pi^2} \left(-1 + \log(N + 1) \right) \leq \frac{2}{\pi} \int_0^\pi \sin u \left(\sum_{k=1}^{N-1} \frac{1}{u + k\pi} \right) du$$

(3.4.14)

$$\leq \frac{4}{\pi^2} \left(1 + \log(N - 1) \right).$$

We obtain (3.4.11) by combining (3.4.12), (3.4.13), and (3.4.14).
c. Consider the linear mappings

$$L_N : L^1(\mathbb{T}_{2\Omega}) \quad \longrightarrow \quad L^1(\mathbb{T}_{2\Omega})$$

$$F \quad \longmapsto \quad S_N(F).$$

The norm of L_N, defined in *Definition B.6* is

$$\|L_N\| = \sup_{\|F\|_{L^1(\mathbb{T}_{2\Omega})} \leq 1} \|L_N(F)\|_{L^1(\mathbb{T}_{2\Omega})} = \sup_{\|F\|_{L^1(\mathbb{T}_{2\Omega})} \leq 1} \|F * D_N\|_{L^1(\mathbb{T}_{2\Omega})},$$

and so $\|L_N\| \le \|D_N\|_{L^1(\mathbb{T}_{2\Omega})}$ by (3.4.1) and part a. To prove the opposite inequality we first note that

$$\|L_N\| \ge \|L_N(W_n)\|_{L^1(\mathbb{T}_{2\Omega})} = \|D_N * W_n\|_{L^1(\mathbb{T}_{2\Omega})}.$$

Then, by *Theorem 3.4.4b* and *Example 3.4.5c*, we have

$$\lim_{n\to\infty} \|D_N * W_n\|_{L^1(\mathbb{T}_{2\Omega})} = \|D_N\|_{L^1(\mathbb{T}_{2\Omega})}.$$

Consequently,

$$(3.4.15) \qquad \forall N \ge 1, \quad \|L_N\| = \|D_N\|_{L^1(\mathbb{T}_{2\Omega})}.$$

Combining (3.4.15) and (3.4.11) with the Uniform Boundedness Principle *(Theorem B.8)*, we can assert that there are functions $F \in L^1(\mathbb{T}_{2\Omega})$ for which $\sup_N \|S_N(F)\|_{L^1(\mathbb{T}_{2\Omega})} = \infty$. In particular, for such functions we do *not* have $\lim_{N\to\infty} \|S_N(F) - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0$.

We begin our discussion of the $L^2(\mathbb{T})$ theory with the following definition *(Definition 3.4.10)* and background *(Remark 3.4.11)*.

3.4.10 Definition. ORTHONORMAL BASIS
   a. A sequence $\{E_n\} \subseteq L^2(\mathbb{T}_{2\Omega})$ is *orthonormal* in $L^2(\mathbb{T}_{2\Omega})$ if

$$\forall m, n \in \mathbb{Z}, \quad \int_{\mathbb{T}_{2\Omega}} E_m(\gamma)\overline{E_n(\gamma)}\, d\gamma = \delta(m, n),$$

where

$$\delta(m, n) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \ne n. \end{cases}$$

An orthonormal sequence $\{E_n\} \subseteq L^2(\mathbb{T}_{2\Omega})$ is an *orthonormal basis* (ONB) for $L^2(\mathbb{T}_{2\Omega})$ if

$$(3.4.16) \qquad \begin{aligned} &\forall F \in L^2(\mathbb{T}_{2\Omega}), \quad \exists\{c_n\} \subseteq \mathbb{C} \quad \text{such that} \\ &\qquad F = \sum c_n E_n \quad \text{in} \quad L^2(\mathbb{T}_{2\Omega}), \end{aligned}$$

cf., (3.4.18).

b. Using Hölder's Inequality, it is an elementary calculation to show that

$$(3.4.17) \qquad \lim_{n \to \infty} \int_{\mathbf{T}_{2\Omega}} F_n(\gamma)\overline{G_n(\gamma)} \, d\gamma = \int_{\mathbf{T}_{2\Omega}} F(\gamma)\overline{G(\gamma)} \, d\gamma$$

when $F, G, F_n, G_n \in L^2(\mathbf{T}_{2\Omega})$ and

$$\lim_{n \to \infty} \|F - F_n\|_{L^2(\mathbf{T}_{2\Omega})} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|G - G_n\|_{L^2(\mathbf{T}_{2\Omega})} = 0,$$

e.g., *Exercise 3.9*.

c. In the case of an ONB $\{E_n\}$, the coefficients c_n in (3.4.16) are of the form

$$(3.4.18) \qquad \forall n \in \mathbb{Z}, \quad c_n = \int_{\mathbf{T}_{2\Omega}} F(\gamma)\overline{E_n(\gamma)} \, d\gamma.$$

This follows from part *b*.

3.4.11 Remark. INTEGRAL EQUATIONS AND THE RIESZ-FISCHER THEOREM

a. In *Example 3.1.10f* we mentioned Abel in conjunction with elliptic functions. In fact, this work was in the realm of *integral equations*, and Abel (1823) solved the "tautochrone" equation

$$\int_0^y \frac{f(x)}{\sqrt{y - x}} \, dx = g(y),$$

for a given forcing function g, by computing

$$f(x) = \frac{1}{\pi} \int_0^x \frac{g'(y)}{\sqrt{x - y}} \, dy.$$

In the late 19th century it was realized that many problems in mathematical physics could be transformed into solving integral equations of the form

$$(3.4.19) \qquad \int_{\mathbf{T}} F(\gamma)K(\gamma, \lambda) \, d\gamma = G(\lambda)$$

e.g., [CH53], [Die81]. The Dirichlet problem in potential theory was solved in particular cases by Neumann. Vito Volterra (1896) used the

Neumann method to solve a certain type of integral equation, and this led to Ivar Fredholm's (still) eminently readable and fundamental paper on integral equations in Acta Mathematica (27(1903), 365–390).

b. Hilbert and (Erhard) Schmidt, in the periods 1904–1912 and 1905–1908, respectively, made great strides in solving (3.4.19), and, in the process, established some of the fundamental ideas of functional analysis. They also set the stage for the Riesz-Fischer Theorem in the following way.

Suppose $\{E_n\} \subseteq L^2(\mathbb{T})$ is *orthonormal*. Assume we can write K, F, and G of (3.4.19) as $K = \sum\limits_{m,n} k(m,n)\overline{E}_m E_n$, $F = \sum f(n)E_n$, $G = \sum g(n)E_n$. For a given kernel K and forcing function G, the goal is to find F. Formally,

$$\int\limits_{\mathbb{T}} F(\gamma)K(\gamma,\lambda)\,d\gamma = \sum_m \sum_{p,q} f(m)k(p,q)E_q(\lambda)\int\limits_{\mathbb{T}} E_m(\gamma)\overline{E_p(\gamma)}\,d\gamma$$

$$= \sum_q \left(\sum_m f(m)k(m,q)\right) E_q(\lambda),$$

and so (3.4.19) leads to the infinite system of linear equations

(3.4.20) $$\forall q \in \mathbb{Z}, \quad \sum_m f(m)k(m,q) = g(q).$$

Suppose $g \in \ell^2(\mathbb{Z})$, and $k : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ has the property that

$$\sum_{m,n} |k(m,n) - \delta(m,n)|^2 < \infty.$$

Then classical methods yield the construction of a unique solution $f \in \ell^2(\mathbb{Z})$ of (3.4.20), e.g., [GG81, pages 70–74].

c. Once the sequence $f \in \ell^2(\mathbb{Z})$ of part *b* is found, then the major problem in solving (3.4.19) in terms of $F = \sum f(n)E_n$ is accomplished by means of F. Riesz' Theorem: *if $\{E_n\} \subseteq L^2(\mathbb{T})$ is orthonormal and $f \in \ell^2(\mathbb{Z})$, then there is $F \in L^2(\mathbb{T})$ for which*

$$\forall n \in \mathbb{Z}, \quad f(n) = \int_{\mathbb{T}} F(\gamma)\overline{E_n(\gamma)}\,d\gamma.$$

This is Riesz' formulation of the *Riesz-Fischer Theorem* (1906-1907). Fischer's formulation is that $L^2(\mathbb{T})$ *is complete*, i.e., *if* $\{F_n\} \subseteq L^2(\mathbb{T})$ *is a Cauchy sequence in the L^2-norm then there is* $F \in L^2(\mathbb{T})$ *for which* $\lim \|F - F_n\|_{L^2(\mathbb{T})} = 0$. Zygmund refers to the Riesz-Fischer Theorem as "a great achievement of the Lebesgue theory".

Fischer's formulation is a special case of *Theorem A.18*, which itself is a staple in a basic real variables course, e.g., [Ben76, pages 232–233], [HS65, pages 192–194], [Rud66, pages 66–67]. We shall use *Theorem A.18* in the $L^2(\mathbb{T})$ theory which follows.

_____

3.4.12 Theorem. ONB, PARSEVAL FORMULA, AND CONVERGENCE
   a. ONB. $\{e^{-\pi i n \gamma/\Omega}\}$ *is an ONB for* $L^2(\mathbb{T}_{2\Omega})$.

   b. Parseval Formula. Let $F, G \in L^2(\mathbb{T}_{2\Omega})$, *and consider the pairings* $f \longleftrightarrow F$, $g \longleftrightarrow G$. *Then*

$$\int_{\mathbb{T}_{2\Omega}} F(\gamma)\overline{G(\gamma)}\, d\gamma = \sum f[n]\overline{g[n]},$$

and, in particular,

$$\|F\|_{L^2(\mathbb{T}_{2\Omega})} = \left(\int_{\mathbb{T}_{2\Omega}} |F(\gamma)|^2 d\gamma\right)^{1/2} = \left(\sum |f[n]|^2\right)^{1/2} = \|f\|_{\ell^2(\mathbb{Z})}.$$

   c. Convergence. For all $F \in L^2(\mathbb{T}_{2\Omega})$,

$$(3.4.21) \qquad\qquad \lim_{N\to\infty} \|F - S_N(F)\|_{L^2(\mathbb{T}_{2\Omega})} = 0.$$

Proof. i. $\{e^{-\pi i n \gamma/\Omega}\}$ is orthornormal in $L^2(\mathbb{T}_{2\Omega})$ by direct calculation, and $\overline{\mathrm{span}}\{e^{-\pi i n \gamma/\Omega}\} = L^2(\mathbb{T}_{2\Omega})$ by the Fejér Theorem (*Theorem 3.4.6a*) and the method of *Exercise 3.16*.

   ii. For any $F \in L^2(\mathbb{T}_{2\Omega})$ and any N

$$0 \le \|F - S_N(F)\|^2_{L^2(\mathbb{T}_{2\Omega})} = \|F\|^2_{L^2(\mathbb{T}_{2\Omega})} - \sum_{|n|\le N} |f[n]|^2,$$

and so

$$(3.4.22) \qquad \sum_{|n|\leq N} |f[n]|^2 \leq \|F\|^2_{L^2(\mathbb{T}_{2\Omega})}.$$

This is *Bessel's Inequality* (1828). Further, if $N > M$ then

$$\|S_N(F) - S_M(F)\|^2_{L^2(\mathbb{T}_{2\Omega})} = \sum_{M < |n| \leq N} |f[n]|^2,$$

and so, by (3.4.22), $\{S_N(F)\}$ is a Cauchy sequence in $L^2(\mathbb{T}_{2\Omega})$. Thus, $\sum f[n]e^{-\pi i n\gamma/\Omega}$ converges to some $K \in L^2(\mathbb{T}_{2\Omega})$ since $L^2(\mathbb{T}_{2\Omega})$ is complete.

iii. By *i*, if $G \in L^2(\mathbb{T}_{2\Omega})$ and

$$\forall n \in \mathbb{Z}, \quad \int_{\mathbb{T}_{2\Omega}} G(\gamma)e^{\pi i n\gamma/\Omega}d\gamma = 0,$$

then (3.4.17) allows us to conclude that $\|G\|_{L^2(\mathbb{T}_{2\Omega})} = 0$, and so G is the 0-function.

iv. Now, for any $F \in L^2(\mathbb{T}_{2\Omega})$ and corresponding K as in *ii*, we have (by (3.4.17) again)

$$\int_{\mathbb{T}_{2\Omega}} (F(\gamma) - K(\gamma))e^{\pi i n\gamma/\Omega}d\gamma$$
$$= f[n] - \lim_{N\to\infty} \sum_{|m|\leq N} f[m] \int_{\mathbb{T}_{2\Omega}} e^{-\pi i(m-n)\gamma/\Omega}d\gamma = 0.$$

Therefore, by *iii*, $F = K$ a.e.; and so (3.4.21) is obtained, giving part *c* as well as part *a*.

v. Part *b* follows from (3.4.17), (3.4.21), and the calculation

$$\int_{\mathbb{T}_{2\Omega}} F(\gamma)\overline{G(\gamma)}d\gamma = \lim_{N\to\infty} \int_{\mathbb{T}_{2\Omega}} S_N(F)(\gamma)\overline{S_N(G)(\gamma)}d\gamma$$

$$= \lim_{N\to\infty} \sum_{|m|,|n|\leq N} f[m]\overline{g[n]} \int_{\mathbb{T}_{2\Omega}} e^{-\pi i(m-n)\gamma/\Omega}d\gamma$$

$$= \sum f[n]\overline{g[n]}.$$

$\square$

In light of Bessel's Inequality (3.4.22), we know that if $F \in L^2(\mathbb{T})$ for the pairing $f \longleftrightarrow F$, then $f \in \ell^2(\mathbb{Z})$. Riesz's formulation of the Riesz-Fischer Theorem completes the picture as follows:

3.4.13 Theorem. $\ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{T})$

There is a unique linear bijection $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}_{2\Omega})$ with the properties

i. $\forall f \in \ell^2(\mathbb{Z}), \quad \|f\|_{\ell^2(\mathbb{Z})} = \|\mathcal{F}f\|_{L^2(\mathbb{T}_{2\Omega})}$,

ii. $\forall F \in L^2(\mathbb{T}_{2\Omega}), \quad f \equiv \mathcal{F}^{-1}F$ is the sequence of Fourier coefficients of F.

Proof. In light of *Corollary 3.4.7* and *Theorem 3.4.12*, it is sufficient to prove that for any sequence $\{c_n\} \in \ell^2(\mathbb{Z})$ there is $F \in L^2(\mathbb{T}_{2\Omega})$, uniquely determined a.e., such that $\{c_n\}$ is the sequence of Fourier coefficients of F.

If we define $S_N(\gamma) = \sum\limits_{|n| \leq N} c_n e^{-\pi i n \gamma / \Omega}$, then $\{S_N\} \subseteq L^2(\mathbb{T}_{2\Omega})$ is a Cauchy sequence since

$$\|S_N - S_M\|_{L^2(\mathbb{T}_{2\Omega})} = \sum_{M < |n| \leq N} |c_n|^2$$

when $N > M$. By the completeness of $L^2(\mathbb{T}_{2\Omega})$ there is a unique $F \in L^2(\mathbb{T}_{2\Omega})$ for which

$$\lim_{N \to \infty} \|F - S_N\|_{L^2(\mathbb{T}_{2\Omega})} = 0.$$

Further, for each n and for $N \geq |n|$, we have

$$|f[n] - c_n| = \left| \int_{\mathbb{T}_{2\Omega}} (F(\gamma) - S_N(\gamma)) e^{\pi i n \gamma / \Omega} d\gamma \right|$$

$$\leq \|F - S_N\|_{L^2(\mathbb{T}_{2\Omega})}$$

by Hölder's Inequality. Thus, $c_n = f[n]$. $\square$

3.5 $A(\mathbb{T})$ and the Wiener Inversion Theorem

Besides the formidable task of finding an effective intrinsic characterization of $A(\mathbb{T})$, e.g., *Examples 1.4.4* and *3.3.4*, the space $A(\mathbb{T})$

is worthy of careful attention because of its algebraic properties and the ramifications of those properties, cf., the introductory paragraph of *Section 3.4.* Fortunately, there is an accessible masterpiece on the subject by Kahane [Kah70]. Our modest goal in this section will be to state these algebraic properties, and to prove Wiener's theorem on the inversion of absolutely convergent Fourier series. We shall refer to this result as Wiener's Inversion Theorem.

3.5.1 Definition. CONVOLUTION

a. Let $f, g \in \ell^1(\mathbb{Z})$. The *convolution* of f and g, denoted by $f * g$, is

$$f * g[n] = \sum_{k=-\infty}^{\infty} f[n-k]g[k] = \sum_{k=-\infty}^{\infty} f[k]g[n-k].$$

More simply than the cases of $L^1(\mathbb{R})$ and $L^1(\mathbb{T}_{2\Omega})$, we see that $f * g \in \ell^1(\mathbb{Z})$ since

$$\sum_{n=-\infty}^{\infty} |f * g[n]| \leq \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f[n-k]g[k]| = \sum |f[n]| \sum |g[k]|.$$

Rewriting this expression, we have

$$(3.5.1) \qquad \forall f, g \in \ell^1(\mathbb{Z}), \quad \|f * g\|_{\ell^1(\mathbb{Z})} \leq \|f\|_{\ell^1(\mathbb{Z})}\|g\|_{\ell^1(\mathbb{Z})}.$$

b. $\ell^1(\mathbb{Z})$ is a *commutative algebra* taken with the operations of addition and convolution, i.e., $\ell^1(\mathbb{Z})$ is a complex vector space under addition, and convolution is distributive with respect to addition, as well as being associative, and commutative.

Further, $\ell^1(\mathbb{Z})$ has a *unit* u under convolution. u is defined by $u[n] = \delta(0, n)$, so that

$$\forall f \in \ell^1(\mathbb{Z}), \quad f * u = u * f = f$$

since

$$\forall n \in \mathbb{Z}, \quad \sum_{k=-\infty}^{\infty} f[n-k]\delta(0, k) = f[n].$$

A straightforward calculation yields the following result.

3.5.2 Proposition.

Let $\Omega > 0$ and consider $A(\mathbb{T}_{2\Omega})$ (Definition 3.1.8a).

a. Let $F, G \in A(\mathbb{T}_{2\Omega})$ and let $f = \{f[n]\}$, $g = \{g[n]\} \in \ell^1(\mathbb{Z})$ be the sequences of Fourier coefficients of F and G, i.e., $\hat{f} = F$ and $\hat{g} = G$. Then $f*g \in \ell^1\mathbb{Z}$ is the sequence of Fourier coefficients of $FG \in A(\mathbb{T}_{2\Omega})$, i.e.,

$$\forall n \in \mathbb{Z}, \quad (f*g)[n] = \int_{\mathbb{T}_{2\Omega}} F(\gamma)G(\gamma)e^{\pi i n\gamma/\Omega}\,d\gamma.$$

(Recall that "$\int_{\mathbb{T}_{2\Omega}}$" designates "$\frac{1}{2\Omega}\int_\alpha^{\alpha+2\Omega}$" for any fixed $\alpha \in \widehat{\mathbb{R}}$.)

b. $A(\mathbb{T}_{2\Omega})$ is a commutative algebra under the operations of addition and (ordinary pointwise) multiplication of functions. The function $U \equiv 1 \in A(\mathbb{T}_{2\Omega})$ is the multiplicative unit of $A(\mathbb{T}_{2\Omega})$, and

$$(3.5.1)' \qquad \forall F, G \in A(\mathbb{T}_{2\Omega}), \quad \|FG\|_{A(\mathbb{T}_{2\Omega})} \le \|F\|_{A(\mathbb{T}_{2\Omega})}\|G\|_{A(\mathbb{T}_{2\Omega})}.$$

3.5.3 Example. THE $A(\mathbb{T})$ NORM

a. Let $F \in A(\mathbb{T}_{2\Omega})$ and let $f = \{f[n]\} \in \ell^1(\mathbb{Z})$ be the sequence of Fourier coefficients of F. Then

$$(3.5.2) \qquad \|F\|_{A(\mathbb{T}_{2\Omega})} = |f[0]| + {\sum}' \frac{1}{|n|}|nf[n]|.$$

Using (3.5.2) and *Exercise 3.26* we can conclude that *if $F \in A(\mathbb{T}_{2\Omega})$ and $F' \in L^2(\mathbb{T}_{2\Omega})$ then*

$$(3.5.3) \qquad \|F\|_{A(\mathbb{T}_{2\Omega})} \le |f[0]| + \frac{\Omega}{\sqrt{3}}\|F'\|_{L^2(\mathbb{T}_{2\Omega})}.$$

b. Let F_ϵ be the 2Ω-periodic triangle function, $\max(1 - |\gamma|/\epsilon, 0)$, defined in *Remark 3.1.2b*. As we stated there, its sequence $\{w_{(\epsilon)}\}$ of Fourier coefficients is an element of $\ell^1(\mathbb{Z})$, and so $F_\epsilon \in A(\mathbb{T}_{2\Omega})$. Now define the 2Ω-periodic trapezoid function $V_\epsilon = 2F_{2\epsilon} - F_\epsilon$. Note that

$$(3.5.4) \qquad \|V_\epsilon\|_{A(\mathbb{T}_{2\Omega})} \le 3.$$

In fact, since $w_{(\epsilon)} \ge 0$ in $\mathbb{Z}$, we have

$$\|V_\epsilon\|_{A(\mathbb{T}_{2\Omega})} \le 2\|F_{2\epsilon}\|_{A(\mathbb{T}_{2\Omega})} + \|F_\epsilon\|_{A(\mathbb{T}_{2\Omega})}$$
$$= 2\sum w_{(2\epsilon)}[n] + \sum w_{(\epsilon)}[n] = 2F_{2\epsilon}(0) + F_{2\epsilon}(0) = 3.$$

[handwritten margin note: much too little vertical spacing]

We shall show that *if $F \in A(\mathbb{T}_{2\Omega})$ and $F(0) = 0$* (and so $F(2\Omega n) = 0$ for all $n \in \mathbb{Z}$) *then*

$$(3.5.5) \qquad \lim_{\epsilon \to 0} \|FV_\epsilon\|_{A(\mathbb{T}_{2\Omega})} = 0.$$

First, $FV_\epsilon \in A(\mathbb{T}_{2\Omega})$ by *Proposition 3.5.2*. If $F \in C^1(\mathbb{T}_{2\Omega})$ then $F \in A(\mathbb{T}_{2\Omega})$ by (3.5.3). Also, the pointwise a.e. derivative $(FV_\epsilon)'$ is not only supported by $[-2\epsilon, 2\epsilon]$ on $[-\Omega, \Omega)$, but is uniformly bounded independent of ϵ. In fact,

$$\|(FV_\epsilon)'\|_{L^\infty(\mathbb{T}_{2\Omega})} \leq \|F'\|_{L^\infty(\mathbb{T}_{2\Omega})} + \sup_{\gamma \in [-2\epsilon, -\epsilon] \cup [\epsilon, 2\epsilon]} \frac{1}{\epsilon} |F(\gamma)|,$$

and the second term on the right side is bounded independent of ϵ since $F(\gamma) = F(\gamma) - F(0)$ and the mean value theorem applies. Further, it is clear that

$$\lim_{\epsilon \to 0} \int_{-2\epsilon}^{2\epsilon} F(\gamma) V_\epsilon(\gamma) \, d\gamma = 0.$$

Thus, remembering the support of $(FV_\epsilon)'$, we can apply (3.5.3) again to obtain (3.5.5) for $F \in C^1(\mathbb{T}_{2\Omega})$.

For arbitrary $F \in A(\mathbb{T}_{2\Omega})$, we define the trigonometric polynomials

$$F_N(\gamma) = \sum_{1 \leq |n| \leq N} F^\vee[n] e^{-\pi i n \gamma / \Omega} + a_N[0],$$

where $a_n[0] \equiv -\sum_{1 \leq |n| \leq N} F^\vee[n]$. Thus, $F_N \in C^1(\mathbb{T}_{2\Omega})$ and $F_N(0) = 0$ so that the result of the previous paragraph can be used to obtain

$$\lim_{\epsilon \to 0} \|F_N V_\epsilon\|_{A(\mathbb{T}_{2\Omega})} = 0.$$

Thus, since

$$\|FV_\epsilon\|_{A(\mathbb{T}_{2\Omega})} \leq \|(F - F_N)V_\epsilon\|_{A(\mathbb{T}_{2\Omega})} + \|F_N V_\epsilon\|_{A(\mathbb{T}_{2\Omega})},$$

we can further apply *Proposition 3.5.2* and (3.5.4) to compute

$$\varlimsup_{\epsilon \to 0} \|FV_\epsilon\|_{A(\mathbb{T}_{2\Omega})} \leq 3\|F - F_N\|_{A(\mathbb{T}_{2\Omega})}.$$

The left side is independent of N and the right side is

$$3\left\|F^\vee[0] - \sum_{|n|>N} F^\vee[n]e^{-\pi i n\gamma/\Omega} + \sum_{1\le|n|\le N} F^\vee[n]\right\|_{A(\mathbb{T}_{2\Omega})}$$
$$\le 3\left|\sum_{|n|\le N} F^\vee[n]\right| + 3\sum_{|n|>N}|F^\vee[n]|.$$

As $N \to \infty$, the first term tends to 0 since $F(0) = 0$, and the second term tends to 0 since $F \in A(\mathbb{T}_{2\Omega})$.

The proof of (3.5.5) is complete.

There are far reaching generalizations of (3.5.5) related to spectral synthesis and the ideal structure of $L^1(\mathbb{T})$, e.g., [Ben75, Section 1.2].

3.5.4 Proposition.
$A(\mathbb{T}_{2\Omega}) = L^2(\mathbb{T}_{2\Omega}) * L^2(\mathbb{T}_{2\Omega}).$

Proof. The inclusion $L^2(\mathbb{T}_{2\Omega}) * L^2(\mathbb{T}_{2\Omega}) \subseteq A(\mathbb{T}_{2\Omega})$ is a consequence of Parseval's Formula and Hölder's Inequality. In fact,

$$\sum |f[n]g[n]| \le \left(\sum |f[n]|^2\right)^{1/2}\left(\sum|g[n]|^2\right)^{1/2} < \infty,$$

where $F, G \in L^2(\mathbb{T}_{2\Omega})$ and $\hat{f} = F$, $\hat{g} = G$; and so $fg \in \ell^1(\mathbb{Z})$.

For the inclusion, $A(\mathbb{T}_{2\Omega}) \subseteq L^2(\mathbb{T}_{2\Omega}) * L^2(\mathbb{T}_{2\Omega})$, let $F \in A(\mathbb{T}_{2\Omega})$, where $\hat{f} = F$. For each n, we can write $f[n] = (g[n])^2$ for some $g[n] \in \mathbb{C}$; and we define the sequence $g = \{g[n]\}$. $g \in \ell^2(\mathbb{Z})$ since $f \in \ell^1(\mathbb{Z})$, and, hence, $F = \hat{g} * \hat{g} \in L^2(\mathbb{T}_{2\Omega}) * L^2(\mathbb{T}_{2\Omega})$. $\qquad\square$

3.5.5 Remark. FACTORIZATION
a. The factorization, $A(\mathbb{T}) = L^2(\mathbb{T})*L^2(\mathbb{T})$, or, equivalently, $\ell^1(\mathbb{Z}) = \ell^2(\mathbb{Z})\ell^2(\mathbb{Z})$, of *Proposition 3.5.4* is elementary. A consequence of *Proposition 3.5.2* is the inclusion $A(\mathbb{T})A(\mathbb{T}) \subseteq A(\mathbb{T})$. In this context, we observe that $A(\mathbb{T}) = A(\mathbb{T})A(\mathbb{T})$, or, equivalently, $\ell^1(\mathbb{Z}) = \ell^1(\mathbb{Z}) * \ell^1(\mathbb{Z})$, is also valid since $U \equiv 1 \in A(\mathbb{T})$.

Further, $A(\mathbb{Z}) = A(\mathbb{Z})A(\mathbb{Z})$ and $A(\hat{\mathbb{R}}) = A(\hat{\mathbb{R}})A(\hat{\mathbb{R}})$. However, these two results are far from trivial and are due to Salem and Rudin, respectively. Paul Cohen (1959) proved that $A(\Gamma) = A(\Gamma)A(\Gamma)$ for any

locally compact abelian group Γ, cf., [Koo64], [Ptá72] for elegant proofs of the Cohen Factorization Theorem. Using Salem's Theorem, or an argument with convex functions, we also have $A(\mathbb{T}) = L^1_+(\mathbb{T}) * A(\mathbb{T})$, where $L^1_+(\mathbb{T}) = \{F \in L^1(\mathbb{T}) : F \geq 0\}$.

b. Although the proof that $A(\mathbb{T}) = A(\mathbb{T})A(\mathbb{T})$ is elementary, we have the following more intricate relationship: *if $F \in A(\mathbb{T})$ never vanishes then*

$$\forall H \in A(\mathbb{T}), \ \exists G \in A(\mathbb{T}) \ \text{ such that } \ H = FG.$$

In particular, *if $F \in A(\mathbb{T})$ never vanishes then $1/F \in A(\mathbb{T})$*. This last fact is *Wiener's Inversion Theorem*. There are Banach algebra proofs [Ben75, pages 22–23], a "spectral radius" proof [Ben75, pages 23–24], extensive generalizations which are documented and compared in [Ben75], and classical proofs going back to Wiener's original techniques [Wie81, Volume II, pages 519–623, esp., page 532], [Wie33, page 91]. We shall proceed in this last direction in *Sections 3.5.6–3.5.9*.

c. Before proving Wiener's Inversion Theorem, let us point out that a modified version of it is a natural component in the proof of Wiener's Tauberian Theorem, *Theorem 2.9.12* and *Remark 2.9.13*, cf., the historical remark in [Ben75, pages 142–143] and the proofs in [Ben75, Sections 1.1–1.4]. Also, as indicated earlier, these results are fundamental in spectral synthesis, e.g., [Ben75, Section 2.5]. We shall apply Wiener's Inversion Theorem in *Section 3.6*.

3.5.6 Definition. LOCAL MEMBERSHIP

Let $I \subseteq A(\mathbb{T}_{2\Omega})$ be an *ideal* in the algebra $A(\mathbb{T}_{2\Omega})$, i.e., I is a subalgebra of $A(\mathbb{T}_{2\Omega})$ and $FG \in I$ whenever $F \in A(\mathbb{T}_{2\Omega})$ and $G \in I$.

A function $F : \mathbb{T}_{2\Omega} \to \mathbb{C}$ *belongs to I locally at $\gamma \in \mathbb{T}_{2\Omega}$* if

$$\exists G_\gamma \in I \ \text{ and } \ \exists N_\gamma = (\alpha, \beta) \subseteq \mathbb{T}_{2\Omega} \ \text{ such that } \ \gamma \in (\alpha, \beta) \ \text{ and }$$
$$\forall \lambda \in N_\gamma, \ \ G_\gamma(\lambda) = F(\lambda).$$

In this case we write $F \in I_{\text{loc}}(\gamma)$.

3.5.7 Theorem. LOCAL MEMBERSHIP THEOREM

Let $I \subseteq A(\mathbb{T}_{2\Omega})$ be an ideal and let $F : \mathbb{T}_{2\Omega} \to \mathbb{C}$ be a function. If $F \in I_{\mathrm{loc}}(\gamma)$ for each $\gamma \in \mathbb{T}_{2\Omega}$ then $F \in I$.

Proof. For each $\gamma \in \mathbb{T}_{2\Omega}$ choose G_γ and N_γ (as in *Definition 3.5.6*) for which $G_\gamma = F$ on N_γ. Clearly we can take N_γ centered at γ; and for each γ we choose a closed interval $C_\gamma \subseteq N_\gamma$, also centered at γ and whose length $|C_\gamma|$ equals $\frac{1}{2}|N_\gamma|$. Since $\mathbb{T}_{2\Omega}$ is a compact set we can find $\gamma_1, \cdots, \gamma_n$ so that

$$(3.5.6) \qquad\qquad \mathbb{T}_{2\Omega} = \bigcup_{j=1}^{n} C_j,$$

e.g., *Definition B.1*.

Next, choose $V_j \in A(\mathbb{T}_{2\Omega})$, $j = 1, \cdots, n$, where $V_j = 1$ on C_{γ_j} and $V_j = 0$ off N_{γ_j}, e.g., *Example 3.5.3*, cf., *Exercise 1.50* and the general construction in [Ben75, Proposition 1.1.4]. Since I is an ideal, we have $V_j G_j \in I$ for each $j = 1, \cdots, n$. It is also clear that

$$(3.5.7) \; \forall \gamma \in \mathbb{T}_{2\Omega} \; \text{and} \; \forall j = 1, \cdots, n, \quad V_j(\gamma)G(\gamma) = V_j(\gamma)F(\gamma).$$

Defining

$$(3.5.8) \qquad F_0 = F(1 - (1 - V_1)(1 - V_2) \cdots (1 - V_n)),$$

we see that $F_0 \in I$ because of (3.5.7) and the fact that the "1s" cancel when we compute the right side of (3.5.8). Finally, $F_0 = F$ on $\mathbb{T}_{2\Omega}$. In fact, if $\gamma \in \mathbb{T}_{2\Omega}$, then there is k for which $1 - V_k(\gamma) = 0$ because of (3.5.6).   □

3.5.8 Example. $2|f[0]| > \|F\|_{A(\mathbb{T}_{2\Omega})}$ IMPLIES $1/F \in A(\mathbb{T}_{2\Omega})$

Let $F \in A(\mathbb{T}_{2\Omega}) \backslash \{0\}$ and let $f = \{f[n]\}$ be the sequence of Fourier coefficients of F. Assume

$$(3.5.9) \qquad\qquad 2|f[0]| > \|F\|_{A(\mathbb{T}_{2\Omega})}.$$

We shall show that $1/F \in A(\mathbb{T}_{2\Omega})$. To this end we combine the series expansion, $1/(1 + G) = 1 - G + G^2 - G^3 + \cdots$, and (3.5.9) to compute for $G(\gamma) \equiv \sum' \frac{f[n]}{f[0]} e^{-\pi i n \gamma/\Omega}$ that

$$\frac{1}{F(\gamma)} = \frac{1}{f[0]}(1 - G(\gamma) + G(\gamma)^2 - G(\gamma)^3 + \cdots).$$

Thus, by (3.5.9) and *Proposition 3.5.2b,*

$$\|\frac{1}{F}\|_{A(\mathbb{T}_{2\Omega})} \leq \frac{1}{|f[0]|} \sum_{n=0}^{\infty} \|G\|_{A(\mathbb{T}_{2\Omega})}^n$$

$$= \frac{1}{|f[0]|} \frac{1}{1 - \|G\|_{A(\mathbb{T}_{2\Omega})}} = \frac{1}{2|f[0]| - \|F\|_{A(\mathbb{T}_{2\Omega})}} < \infty.$$

3.5.9 Theorem. WIENER INVERSION THEOREM

Let $F \in A(\mathbb{T}_{2\Omega})$.

a. If $F(\gamma_0) \neq 0$ then there is $G \in A(\mathbb{T}_{2\Omega})$ such that $F = G$ on some open interval N centered at γ_0 and $1/G \in A(\mathbb{T}_{2\Omega})$.

b. If F never vanishes then $1/F \in A(\mathbb{T}_{2\Omega})$.

Proof. a. Without loss of generality, let $\gamma_0 = 0$, and define

$$(3.5.10) \quad \forall \gamma \in \mathbb{T}_{2\Omega}, \quad G_\epsilon(\gamma) = F(0) + V_\epsilon(\gamma)(F(\gamma)) - F(0)),$$

where V_ϵ was defined in *Example 3.5.3.* Choose $\eta \equiv |F(0)|/3 > 0$, and apply (3.5.5) to find $\epsilon > 0$ for which $\|V_\epsilon(F - F(0))\|_{A(\mathbb{T}_{2\Omega})} < \eta$. For this ϵ, set $G = G_\epsilon$.

Since

$$\left| \int_{\mathbb{T}_{2\Omega}} V_\epsilon(\gamma)(F(\gamma) - F(0)) \, d\gamma \right| \leq \|V_\epsilon(F - F(0))\|_{A(\mathbb{T}_{2\Omega})},$$

we have

$$\left| \int_{\mathbb{T}_{2\Omega}} G(\gamma) \, d\gamma \right| = \left| F(0) + \int_{\mathbb{T}_{2\Omega}} V_\epsilon(\gamma)(F(\gamma) - F(0)) \, d\gamma \right|$$

$$(3.5.11)$$
$$\geq |F(0)| - \|V_\epsilon(F - F(0))\|_{A(\mathbb{T}_{2\Omega})} > |F(0)| - \eta.$$

On the other hand, it is immediate from (3.5.10) that

$$(3.5.12) \qquad \|G\|_{A(\mathbb{T}_{2\Omega})} < |F(0)| + \eta.$$

Combining (3.5.11) and (3.5.12) with the definition of η, we obtain

$$(3.5.13) \qquad 2 \left| \int_{\mathbb{T}_{2\Omega}} G(\gamma) \, d\gamma \right| > \frac{4}{3}|F(0)| > \|G\|_{A(\mathbb{T}_{2\Omega})}.$$

From (3.5.10) we see that $G = F$ on $N_0 = (-\epsilon, \epsilon)$, and, because of (3.5.13) and *Example 3.5.8*, $1/G \in A(\mathbb{T}_{2\Omega})$. This completes the proof of part *a*.

b. For each $\gamma \in \mathbb{T}_{2\Omega}$, we use part *a* to choose $G_\gamma \in A(\mathbb{T}_{2\Omega})$ such that $G_\gamma = F$ on some open interval N_γ centered at γ and $1/G_\gamma \in A(\mathbb{T}_{2\Omega})$.

Thus, $1/F \in A(\mathbb{T}_{2\Omega})_{\text{loc}}(\gamma)$ for each $\gamma \in \mathbb{T}_{2\Omega}$, and so $1/F \in A(\mathbb{T}_{2\Omega})$ by *Theorem 3.5.7*.   □

3.5.10 Remark. $A(\mathbb{T})$ AND $A(\widehat{\mathbb{R}})$

a. Wiener's concept of *local membership* leads one to investigate the relationship between $A(\mathbb{T})$ and $A(\widehat{\mathbb{R}})$. For example, it is natural to ask the question: supposing $F \in A(\mathbb{T})$, $F(\pm\frac{1}{2}) = 0$, and G is defined on $\widehat{\mathbb{R}}$ as

$$(3.5.14) \qquad G(\gamma) = \begin{cases} F(\gamma), & \text{if } |\gamma| \leq \frac{1}{2}, \\[2ex] 0, & \text{otherwise,} \end{cases}$$

is $G \in A(\widehat{\mathbb{R}})$? Wiener proved the following theorem. *Let $F : \mathbb{T} \to \mathbb{C}$ be a function vanishing on $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$ and define $G : \widehat{\mathbb{R}} \to \mathbb{C}$ by (3.5.14); then $F \in A(\mathbb{T})$ if and only if $G \in A(\widehat{\mathbb{R}})$, and*

$$\exists C_1(\epsilon), C_2(\epsilon) > 0 \quad \text{such that} \quad C_1(\epsilon)\|G\|_{A(\widehat{\mathbb{R}})} \leq \|F\|_{A(\mathbb{T})} \leq C_2(\epsilon)\|G\|_{A(\widehat{\mathbb{R}})}.$$

The proof is not difficult, e.g., [Wie33, pages 80–82].

b. The following extension of Wiener's result from part *a* was proved by Wik [Wik65]. *Let $F \in L^\infty(\mathbb{T})$ vanish on $[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$, define $G : \widehat{\mathbb{R}} \to \mathbb{C}$ by (3.5.14), and let $w : \mathbb{R} \to \mathbb{R}$ be an even positive function, which is increasing on $(0, \infty)$ and which satisfies the condition*

$$(3.5.15) \qquad \exists C > 0 \quad \text{such that} \quad \forall t \in \mathbb{R}\setminus\{0\}, w(2t) \leq Cw(t);$$

then

$$\sum |F^\vee[n]|w(n) < \infty$$

if and only if

$$\int G^\vee(t)w(t)\,dt < \infty.$$

In this statement, we use "w" to denote a so-called *weight*, not the Fejér function.

c. Using the result of part *b*, Wik [Wik65] proved that *if $F \in A(\mathbb{T})$*, $-1/2 < \alpha < \beta < 1/2$, $F(\alpha) = F(\beta) = 0$, *and*

$$\sum{}' |F^{\vee}[n]| \log |n| < \infty,$$

then $F_{\alpha,\beta} \in A(\mathbb{T})$ where

$$\forall \gamma \in [-\tfrac{1}{2}, \tfrac{1}{2}), \quad F_{\alpha,\beta}(\gamma) = F(\gamma)\mathbf{1}_{[\alpha,\beta]}(\gamma)$$

and $F_{\alpha,\beta}$ is defined 1-periodically on $\widehat{\mathbb{R}}$.

d. Condition (3.5.15) is a *doubling condition* for weights. Such conditions play a conceptually important role in an interesting and unresolved set of problems categorized as *weighted norm inequality problems*, e.g., [G-CRdeF85]. An example of a weighted norm inequality is

$$(3.5.16) \qquad \left(\int |\widehat{f}(\gamma)|^2 \, d\mu(\gamma) \right)^{1/2} \le C \left(\int |f(t)|^2 \, w(t) \, dt \right)^{1/2},$$

where $w > 0$ and μ is a positive measure, cf., *Definition 2.6.5* and *Theorem 3.7.2*, which are applicable raisons d'être for dealing with such inequalities. The problem is to characterize the relationship between w and μ so that (3.5.16) is valid for a large class of functions, e.g., [BH92] for measure weights μ.

e. Wik's Theorem from part *c* can be thought of in terms of local membership *or* weighted norm inequalities. In the context of *local membership*, we can obtain $F_{\alpha,\beta} \in A(\mathbb{T})$ by *Theorem 3.5.7* if we can show $F_{\alpha,\beta} \in A(\mathbb{T})_{\text{loc}}(\gamma)$ for $\gamma = \alpha, \beta$, since local membership is obvious for other values of γ. In the context of *weighted norm inequalities*, define $w(n) = \log |n|$ and $Fv = F_{\alpha,\beta}$, i.e., $v = \mathbf{1}_{[\alpha,\beta]}$ on $[-\tfrac{1}{2}, \tfrac{1}{2})$; and consider the weighted norm inequality,

$$(3.5.17) \qquad \|Fv\|_{A(\mathbb{T})} \le C \sum{}' |F^{\vee}[n]| w(n),$$

for all $F \in A(\mathbb{T})$ for which $F(\alpha) = F(\beta) = 0$. Then Wik's Theorem can be restated by saying that if the right side of (3.5.17) is finite then $Fv = F_{\alpha,\beta} \in A(\mathbb{T})$.

f. *Geometrical* considerations play a significant role in a class of weighted norm inequalities referred to as *restriction theory* , e.g., [Ash76, pages 107–117 by E. M. Stein], [Ste93]. Also, extensions of the *classical uncertainty principle inequality* (2.8.5) are critical in quantifying the implications of inequalities such as (3.5.16), e.g., [BF94, Chapter 7], [BL94].

3.6 Maximum entropy and spectral estimation

We shall discuss the Maximum Entropy Theorem and prove a spectral estimation theorem. In so doing we shall prove the Fejér-Riesz Theorem and indicate the role of $A(\mathbb{T})$ in such matters.

3.6.1 Remark. THE SPECTRAL ESTIMATION AND EXTENSION PROBLEM

a. We gave a qualitative statement of the spectral estimation problem in *Definition 2.8.6b*. We shall now aim to quantify that statement for both the stochastic setting of *Section 2.8* and the deterministic setting of *Section 2.9*. As a first step, we say that the *spectral estimation problem* is to estimate the power spectrum in terms of given autocorrelation data on a finite interval.

In the context of Fourier series, we are given $N > 0$ and data $X_N \equiv \{r_n : n = 0, \pm 1, \dots, \pm N\} \subseteq \mathbb{C}$, and the *extension problem* associated with spectral estimation is to find nonnegative functions $S \in L^1(\mathbb{T})$ for which $S^{\vee}[n] = r_n$ for $n = 0, \pm 1, \dots, \pm N$. Because of Herglotz's Theorem on $\mathbb{Z}$ (*Theorem 2.7.10*), X_N must satisfy some positive definiteness condition, e.g., *Definition 3.6.2*. With this stipulation on X_N there are generally *many* nonnegative solutions $S \in L^1(\mathbb{T})$ as Krein (1940) first showed in the setting of $\mathbb{R}$. After Krein, contributions to the extension problem were made by Chover, Doss, Dym, Gohberg, R. R.Goldberg, Rudin, et al.; and [Rud63] also analyzed the difficulties in extending Krein's Theorem to higher dimensions. The fact that there can be many solutions to the extension problem leads to Fourier uniqueness problems in the spirit of *Example 1.10.6*, e.g., [Pric85, pages 149–170].

b. From the point of view of spectral estimation there are various ways of choosing a specific solution S from part a, depending on the type of application. One procedure, the *Maximum Entropy Method* (MEM), involves choosing the function $S = S_{MEM}$ which maximizes a certain logarithmic integral associated with entropy, e.g., *Theorem 3.6.3*.

Mathematically, we shall see that this choice restricts us to $A(\mathbb{T})$, instead of $L^1(\mathbb{T})$. Physically, since entropy is a measure of disorder in a system, S_{MEM} represents maximum uncertainty with regard to what we do not know about the system, whereas it depends on all the known autocorrelation data X_N. Thus, the choice of S_{MEM} is a mathematical guarantee that the least number of assumptions has been made regarding the information content of the unmeasured data at $|n| > N$, e.g., [Chi78], [IEEE82]for expert physical rationales and expositions.

John Parker Burg invented MEM in 1967 and van den Bos (1971) [Chi78, pages 92-93] showed that the Maximum Entropy Method of choosing a *spectral estimator* S is equivalent to least squares linear prediction, used in speech processing, and autoregression, used in statistics, e.g., *Section 3.7*. MEM is also related to the maximum likelihood method, e.g., [Chi78, page 3 and pages 132–133]. There is a deep study of MEM and moment problems by Landau [Lan87], *Definition 2.7.8c*, as well as an important new mathematical contribution by Gabardo [Gab93], cf., our extension of MEM to $\mathbb{R}$ in *A quantitative maximum entropy theorem for the real line*, Integral Equations and Operator Theory, 10(1987), 761–779.

3.6.2 Definition. POSITIVE DEFINITE MATRICES

a. An $(N + 1) \times (N + 1)$ matrix $R = (r_{jk})$, where $r_{jk} \in \mathbb{C}$ and $0 \leq j, k \leq N$, is *Hermitian* if $r_{jk} = \overline{r_{kj}}$. An $(N + 1) \times (N + 1)$ matrix $R = (r_{jk})$ is *positive semidefinite* if

$$\forall\, c = (c_0, c_1, \ldots, c_N) \in \mathbb{C}^{N+1}, \langle Rc, c \rangle = \sum_{j,k} r_{jk} c_k \overline{c}_j \geq 0.$$

Positive semidefinite matrices are Hermitian, e.g., *Exercise 2.52*.

b. An $(N + 1) \times (N + 1)$ positive semidefinite matrix R is *positive definite* written $R >> 0$, if $\langle Rc, c \rangle = 0$ implies $c = 0$. Clearly, if $R >> 0$ then R is nonsingular and R^{-1} exists. In fact, if $Rc = 0$ then $\langle Rc, c \rangle = 0$, and so $c = 0$ by hypothesis; thus, $R : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ is a linear injection and we have the result.

c. Let R be an $(N+1) \times (N+1)$ matrix with eigenvalues $\{\lambda_0, \ldots, \lambda_N\}$. By definition, the *trace* of R is $\sum_{j=0}^{N} r_{jj}$. It can be shown that the trace of R equals $\sum_{j=0}^{N} \lambda_j$ and that the determinant of R is $\prod_{j=0}^{N} \lambda_j$. Also, R is Hermitian if and only if

$$\forall c, d \in \mathbb{C}^{N+1} , \quad \langle Rc, d \rangle = \langle c, Rd \rangle;$$

and the eigenvalues of an Hermitian matrix are real.

Finally, if R is Hermitian, then $R >> 0$ if and only if each eigenvalue $\lambda_j > 0$, e.g., [Str88], cf., part *a*.

d. Let $\{r_j : j = 0, \pm 1, \pm 2, \ldots, \pm N\} \subseteq \mathbb{C}$ satisfy the condition $r_j = \overline{r}_{-j}$ for each j, and define the $(N+1) \times (N+1)$ matrix $R = (r_{jk})$, where $j, k \geq 0$ and $r_{jk} \equiv r_{j-k}$. R is a *Toeplitz matrix*, i.e., it takes constant values on "diagonals of negative slope". R is Hermitian since $r_j = \overline{r}_{-j}$. From the previous discussion, $\sum_{j=0}^{N} \lambda_j = (N + 1)r_0$; and if $R >> 0$ then the determinant of R is positive, $r_0 > 0$, and $R^{-1} \equiv (c_{jk})$ exists, cf., *Exercise 3.52*.

e. *Let $S \in L^1(\mathbb{T}) \backslash \{0\}$ be nonnegative, and let $s = \{s[n]\}$ be the sequence of Fourier coefficients of S. Then, for each N, the $(N + 1) \times (N + 1)$ matrix $R = (s[j - k])$, where $0 \leq j, k \leq N$, is positive definite.* In fact,

$$\sum_{0 \leq j, k \leq N} s[j - k] c_k \overline{c}_j = \int_{\mathbb{T}} S(\gamma) | \sum_{j=0}^{N} c_j e^{-2\pi i j \gamma} |^2 \, d\gamma > 0,$$

for all $(c_0, \ldots, c_N) \in \mathbb{C}^{N+1} \backslash \{0\}$, since $\sum_{j=0}^{N} c_j e^{-2\pi i j \gamma} = 0$ for at most finitely many points.

3.6.3 Theorem. THE MAXIMUM ENTROPY THEOREM
    Let $\{r_j : j = 0, \pm 1, \pm 2, \ldots, \pm N\} \subseteq \mathbb{C}$ satisfy the condition $r_j = \overline{r}_{-j}$ for each j, and assume the $(N + 1) \times (N + 1)$ matrix $R = (r_{jk}) >>$

0, *where* $j, k \geq 0$ *and* $r_{jk} \equiv r_{j-k}$. *There is a unique function* $S \in A(\mathbb{T})$, *with Fourier coefficients* $\{s[n]\} \in \ell^1(\mathbb{Z})$, *satisfying the following properties:*

    a. $\forall |n| \leq N$, $s[n] = r_n$;

    b. $S > 0$ *on* $\mathbb{T}$, *and, hence (by Theorem 3.5.9)* $S^{-1} \in A(\mathbb{T})$;

    c. $S = |S_+|^2$ *where* $S_+ \in A(\mathbb{T})$ *has the form*

$$S_+(\gamma) = \sum_{n=0}^{\infty} s_+[n] e^{-2\pi i n \gamma};$$

    d. $\forall |n| > N$, $(S^{-1})^{\vee}[n] = 0$ *and*

$$S(\gamma) = 1 / \sum_{n=-N}^{N} \left(\frac{1}{c_{00}} \sum_{m=0}^{N} \overline{c}_{0m} c_{0,m-n} \right) e^{-2\pi i n \gamma},$$

where $R^{-1} = (c_{mn})$;

    e. For all $F \in A(\mathbb{T})$, *for which* $F > 0$ *on* $\mathbb{T}$ *and* $F^{\vee}[n] = r_n$ *when* $|n| \leq N$, *we have*

$$\int_{\mathbb{T}} \log F(\gamma) \, d\gamma \leq \int_{\mathbb{T}} \log S(\gamma) \, d\gamma,$$

and equality is obtained if and only if $F = S$.

Our proof of *Theorem 3.6.3* in [Pric85, pages 95–97] depends on the fact that *if the matrix* R *of Theorem 3.6.3 is positive definite and if* $(a_0, a_1, \ldots, a_N)^T = R^{-1}(1, 0, 0, \ldots, 0)^T$, *then*

$$\forall \gamma \in \mathbb{T}, \quad \sum_{n=0}^{N} a_j e^{-2\pi i n \gamma} \neq 0,$$

e.g., [GS58], [GL94], cf., [DG79] for an important extension. We shall not prove *Theorem 3.6.3* since we shall not prove this fact. Instead we shall prove *Theorem 3.6.6* below, which is essentially *Theorem 3.6.3* and which depends on the Fejér-Riesz Theorem.

3.6.4 Theorem. FEJÉR-RIESZ THEOREM

Let

$$A(\gamma) = \sum_{n=-N}^{N} a_n e^{-\pi i n\gamma/\Omega} \geq 0 \quad on \quad \mathbb{T}_{2\Omega}$$

and define $A_*(z) = \sum_{n=-N}^{N} a_n z^n$ *so that* $A_*(e^{-\pi i\gamma/\Omega}) = A(\gamma)$. *There is a unique trigonometric polynomial* $B(\gamma) = \sum_{n=0}^{N} b_n e^{-\pi i n\gamma/\Omega}$ *with the properties that*

(3.6.1) $A = |B|^2 \quad on \quad \mathbb{T}_{2\Omega},$

and if $B_*(z) = 0$ *then* $|z| \leq 1$.

Proof. Since A is real, the Fourier coefficients $\{a_n\}$ have the property that $\overline{a}_n = a_{-n}$ for $n = -N, \dots, 0, 1, \dots N$. In particular, $a_0 \in \mathbb{R}$. Without loss of generality, assume $a_{-N} \neq 0$.

Now define the polynomial P_* by

$$P_*(z) = z^N A_*(z) = a_{-N} \quad + \quad a_{-N+1}z + \cdots + a_0 z^N$$
$$+ \quad a_1 z^{N+1} + \cdots + a_N z^{2N}, \; z \in \mathbb{C}.$$

Clearly,

$$\forall z \in \mathbb{C}\backslash\{0\}, \quad z^{2N}\overline{P_*(1/\overline{z})} = P_*(z).$$

Let $P_*(z_0) = 0$. If $z_0 \neq 0$ then $1/\overline{z}_0$ is also a zero of P_*.

Figure 3.2

If $0 < |z_0| < 1$ then, by differentiation, we see that z_0 and $1/\bar{z}_0$ have the same multiplicity. If $|z_0| = 1$ then z_0 has even multiplicity since $A \geq 0$. We shall say that a polynomial has 0 *as a zero of order m* if its first m coefficients, that is, those of $1, z, \ldots, z^{m-1}$, are 0; and it has ∞ *as a zero of order m* if its last m coefficients are 0. Thus, if $z_0 = 0$ is a zero of order m for P_*, then P_* has ∞ as a zero of order m. ~~The previous discussion allows us to write~~ *Thus,*

$$(3.6.2) \qquad P_*(z) = C \prod_{j=1}^{p}(z - z_j)\left(z - \frac{1}{\bar{z}_j}\right) \prod_{j=1}^{q}(z - u_j)^2,$$

where $0 < |z_j| < 1$, $|u_j| = 1$, and $2p + 2q = 2N$, i.e., $p + q = N$, cf., *Exercise 3.28i*.

Since $A \geq 0$, (3.6.2) allows us to write

$$A(\gamma) = |A_*(z)| = |z^N P_*(z)| = |P_*(z)|$$

$(3.6.3)$

$$= |C| \prod_{j=1}^{p} |z - z_j|\,\Big|z - \frac{1}{\bar{z}_j}\Big| \prod_{j=1}^{q} |z - u_j|^2$$

for $z = e^{-\pi i \gamma/\Omega}$, where $p + q = N$. Because $|z - \frac{1}{\bar{z}_j}| = |\frac{z - z_j}{z_j}|$ for such z on the unit circle, (3.6.3) becomes

$$A(\gamma) = \left| C^{1/2} \prod_{j=1}^{p} (e^{-\pi i \gamma/\Omega} - z_j) z_j^{-1/2} \prod_{j=1}^{q} (e^{-\pi i \gamma/\Omega} - u_j) \right|^2.$$

(3.6.1) is obtained by setting

$$(3.6.4) \quad B(\gamma) = C^{1/2} \prod_{j=1}^{p} (e^{-\pi i \gamma/\Omega} - z_j) z_j^{-1/2} \prod_{j=1}^{q} (e^{-\pi i \gamma/\Omega} - u_j).$$

The claim about the zeros of B_* is immediate from (3.6.4). $\qquad \square$

3.6.5 Remark. Fejér-Riesz Theorem: Potpourri and Titillation

a. The Fejér-Riesz Theorem was proved by Fejér and F. Riesz, and published by Fejér (Jour. für Reine und Angew. Math., 146(1915),

53–82, especially pages 55–59), cf., [RN55, pages 117–118], [PS76, page 259], [GS58, pages 20–22], [Ach56, pages 152–153].

   b. The Fejér-Riesz Theorem is a critical component in the classical proof of the *Spectral Theorem for Unitary Operators* in a Hilbert space, e.g., [RN55, pages 280–284], cf., the historical note on operator-theoretic applications in [Bur79].

   c. Herglotz's Theorem (1911) on $\mathbb{Z}$ (*Theorem 2.7.10*) can be proved as a consequence of the Fejér-Riesz Theorem, e.g., [RN55, pages 115–118], where the context is in terms of the moment problems mentioned in *Definition 2.7.8c*.

   d. Using the Fejér-Riesz Theorem, it is elementary to prove that *if $F \in C(\mathbb{T}_{2\Omega})$ is nonnegative, then there is a sequence $\{B_N\}$ of trigonometric polynomials on $\mathbb{T}_{2\Omega}$ for which*

$$\lim_{N \to \infty} \|F - |B_N|^2\|_{L^\infty(\mathbb{T}_{2\Omega})} = 0,$$

cf., *Exercise 3.48*.

   e. *Krein Theorem.* Using the Fejér-Riesz Theorem, Krein proved that *if $f \in PW_\Omega$ is nonnegative then there is $b \in PW_{\Omega/2}$ for which $f = |b|^2$ on $\mathbb{R}$ and for which the zeros of the entire function,*

$$b(z) = \int_{-\Omega/2}^{\Omega/2} \widehat{b}(\gamma) e^{2\pi i z \gamma} \, d\gamma,$$

are in the half-plane $\operatorname{Im} z \geq 0$, e.g., [Ach56, page 154]. (*PW_Ω is the Paley-Wiener space defined in *Remark 1.10.8*.*) In fact, this result is true for a larger space than PW_Ω, e.g., [Ach56, pages 137–152].

   f. If $A(\gamma) = \sum_{n=-N}^{N} a_n e^{-2\pi i n \gamma} \geq 0$ on $\mathbb{T}$ and $a_0 = 1$ then $A(0) \leq 2N + 1$. This is an immediate consequence of the fact that $\{a_n\} >> 0$:

$$0 \leq P(0) = \sum_{n=-N}^{N} a_n \leq \sum_{n=-N}^{N} |a_n| \leq (2N + 1)a_0.$$

In his paper referenced in part *a*, Fejér proved that $A(0) \leq N + 1$.

   g. The fact that the zeros z_0 of the polynomial B_*, of *Theorem 3.6.4*, satisfy $|z_0| < 1$ for $A > 0$ on $\mathbb{T}_{2\Omega}$ is useful in filter design. A rational

function H, all of whose zeros and poles z_0 satisfy $|z_0| < 1$, is a *minimum phase filter* , e.g., [OS75, pages 345–353], [Dau92, pages 194 ff.].

h. *Daubechies Theorem.* One of the early stunning successes of *wavelet theory* was Daubechies' Theorem (1987): *for any $r \geq 0$, there is a constructible function $\psi \in C_c^r(\mathbb{R})$ for which $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$ is an ONB for $L^2(\mathbb{R})$, where*

$$\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n).$$

Her proof requires the Fejér-Riesz Theorem, e.g., [Dau92, Chapter 6], especially pages 167–174 for the role of *Theorem 3.6.4.*

i. The functions A_* and B_* of *Theorem 3.6.4* are called z-transforms of $\{a_n\}$ and $\{b_n\}$, respectively, e.g., *Exercise 3.22.*

3.6.6 Theorem. A SPECTRAL ESTIMATION THEOREM

Let $\{r_j : j = 0, \pm 1, \pm 2, \cdots \pm N\} \subseteq \mathbb{C}$ satisfy the condition $r_j = \overline{r}_{-j}$ for each j, and assume the $(N + 1) \times (N + 1)$ matrix $R = (r_{jk}) >> 0$, where $j, k \geq 0$ and $r_{jk} \equiv r_{j-k}$. Let $\Omega > 0$. There is a positive function $S \in A(\mathbb{T}_{2\Omega})$, with Fourier coefficients $\{s[n]\} \in \ell^1(\mathbb{Z})$, satisfying the following properties:

(3.6.5) $$\forall |n| \leq N, \quad s[n] = r_n,$$

and

(3.6.6) $$\forall \gamma \in \mathbb{T}_{2\Omega}, \quad S(\gamma) = 1/A(\gamma) = 1/\sum_{n=-N}^{N} a_n e^{-\pi i n\gamma/\Omega},$$

where A designates the sum in the denominator on the right side of (3.6.6).

Proof. i. In order to prove this result we shall proceed in the following devious way.

Given the hypotheses on R we shall *momentarily* assume both (3.6.5) and (3.6.6) for some nonnegative function $S \in L^1(\mathbb{T}_{2\Omega})$. Using these hypotheses *and* conclusions, we shall show in parts *ii–v* how to obtain the coefficients $\{a_n : |n| \leq N\}$ of (3.6.6) from the given data $\{r_n : |n| \leq N\}$.

In order to give an honest proof of the theorem, we work backwards. In particular, we take the hypotheses on R (without the conclusions of the theorem!), and solve the system of equations in part v to obtain $\{a_n\}$. Then we define S in terms of these a_n by means of (3.6.6). The calculations in parts *ii–v* allow us to conclude that S is nonnegative, $S \in A(\mathbb{T}_{2\Omega})$, and $\overset{\vee}{S}[n] = r_n$ for all $|n| \leq N$.

ii. We shall prove that S is not only in $L^1(\mathbb{T}_{2\Omega})$, but $S \in A(\mathbb{T}_{2\Omega})$. In fact, A is a trigonometric polynomial, and so it is a continuous function on $\mathbb{T}_{2\Omega}$ with at most finitely many zeros. However, S is not integrable over any small interval centered at such a zero, e.g., *Exercise 3.27*, and thus S and A are really positive on $\mathbb{T}_{2\Omega}$. Thus, $S = 1/A \in A(\mathbb{T}_{2\Omega})$ by Wiener's Theorem (*Theorem 3.5.9*).

iii. Since $A > 0$ on $\mathbb{T}_{2\Omega}$ we can apply the Fejér-Riesz Theorem (*Theorem 3.6.4*). Thus, there is a trigonometric polynomial $B(\gamma) = \sum_{n=0}^{N} b_n e^{-\pi i n \gamma/\Omega}$ with the properties that $A = |B|^2$ on $\mathbb{T}_{2\Omega}$, $b_0 \neq 0$, and for which

$$B_*(z) = \sum_{n=0}^{N} b_n z^n = 0 \quad \text{implies} \quad |z| < 1,$$

where $B_*(e^{-\pi i \gamma/\Omega}) \equiv B(\gamma)$. In fact, the proof of *Theorem 3.6.4* allows us to write

$$B_*(z) = C^{1/2} \prod_{j=1}^{N} (z - z_j) z_j^{-1/2},$$

where $\{z_j\}$ is the set of zeros of B_* and each z_j satisfies $0 < |z_j| < 1$. Therefore,

$$\overline{B_*\left(\frac{1}{\overline{z}}\right)} = \overline{C}^{1/2} \prod_{j=1}^{N} \left(\frac{1}{z} - \overline{z}_j\right) \overline{z}_j^{-1/2} = \sum_{n=0}^{N} \overline{b}_n z^{-n},$$

and its zeros $z = 1/\overline{z}_j$ are outside the unit circle. Hence,

(3.6.7) $$\frac{1}{\overline{B_*\left(\frac{1}{\overline{z}}\right)}} = \sum_{n=0}^{\infty} c_n z^{-n}, \quad |z| \leq 1.$$

Further, if $z = e^{-\pi i \gamma/\Omega}$ then

(3.6.8) $$\overline{B_*\left(\frac{1}{\overline{z}}\right)} = \overline{B(\gamma)}.$$

iv. Combining the factorization $A = |B|^2$ with (3.6.6), we obtain

$$(3.6.9) \qquad S(\gamma)B(\gamma) = \frac{1}{\overline{B(\gamma)}} \in A(\mathbb{T}_{2\Omega}).$$

Because of (3.6.7) and (3.6.8) the Fourier coefficients $(SB)^\vee[n]$ vanish if $n > 0$.

Since $(SB)^\vee[n] = \overset{\vee}{S} * \overset{\vee}{B}[n]$, we have from (3.6.9) that

$$(3.6.10) \qquad \forall n > 0, \quad \sum_{k=0}^{N} s[n-k]b_k = 0.$$

v. Because of property (3.6.5), the N cases of (3.6.10) for $n = 1, \ldots, N$ give rise to the N equations

$$(3.6.11) \qquad \begin{aligned} &r_1 b_0 + r_0 b_1 + r_{-1} b_2 + \cdots + r_{1-N} b_N = 0, \\ &r_2 b_0 + r_1 b_1 + r_0 b_2 + \cdots + r_{2-N} b_N = 0, \\ &\cdots \\ &r_N b_0 + r_{N-1} b_1 + r_{n-2} b_2 + \cdots + r_0 b_N = 0. \end{aligned}$$

We rewrite (3.6.11) as the matrix equation,

$$Rb^T = -b_0(r_1, \ldots, r_N)^T,$$

where $b = (b_1, b_2, \ldots, b_N)$. R is invertible since $R >> 0$, e.g., *Definition 3.6.2c,d.* Thus, we compute b, cf., *Example 3.7.9*; and then we compute $\{a_n : -N \leq n \leq N\}$ in terms of b since $A = |B|^2$. $\qquad \square$

3.6.7 Example. SPECTRAL ESTIMATION AND MAXIMUM ENTROPY

a. Equation (3.6.6) gives spectral (frequency) information about a digital signal in the case that relatively little data X is known about the signal. In fact, a small value of $A(\gamma) > 0$ in (3.6.6) allows one to guess that this value of γ is a frequency component of the signal which generates given autocorrelation data X. Of course, we very rarely get something for nothing, and so this method of spectral estimation can only be used effectively when, as indicated in *Remark 3.6.1b*, certain physical parameters make sense.

This method of spectral estimation is a form of the MEM, and should be compared with Fourier transform methods, e.g., *Proposition 2.8.8* and *Example 2.9.7*, which really give accurate spectral information but which usually require large data sets X.

b. Let $\{r_n : |n| \le N\} \subseteq \mathbb{C}$ satisfy the hypotheses of either the MEM Theorem (*Theorem 3.6.3*) or *Theorem 3.6.6*. The relationship between these theorems was alluded to in part a, and is quantified by the following suggestive calculation. The calculation itself was made early on in the development of MEM, e.g., [Chi78, page 55], [IEEE82, page 944]. It is a rationale (not a proof) for supposing that S has the form (3.6.6) in the case that $\int_{\mathbb{T}} \log S(\gamma)\, d\gamma$ is the largest (or smallest) value of $\{\int_{\mathbb{T}} \log F(\gamma)\, d\gamma\}$, when F ranges over all positive functions F in $A(\mathbb{T})$ for which $\overset{\vee}{F}[n] = r_n$, $|n| \le N$.

Let $|n| > N$ be fixed, and consider the continuous (complex) variable $r \equiv r_n$. Assuming $\int_{\mathbb{T}} \log S(\gamma)\, d\gamma$ is an optimum we have

$$0 = \frac{\partial}{\partial r} \int_{\mathbb{T}} \log S(\gamma)\, d\gamma = \int_{\mathbb{T}} \frac{1}{S(\gamma)} \frac{\partial S(\gamma)}{\partial r}\, d\gamma,$$

where γ is fixed in the expression $\frac{\partial S(\gamma)}{\partial r}$, cf., the proof of *Theorem 3.7.7*. Writing $S(\gamma) = \sum r_j e^{-2\pi ij\gamma}$ we have $\frac{\partial S(\gamma)}{\partial r} = e^{-2\pi in\gamma}$. Thus, $(1/S)^{\vee}[n] = 0$ for all $|n| > N$, and so S has the form (3.6.6).

3.7 Prediction and spectral estimation

An *extension* of the factorization given by the Fejér-Riesz Theorem (*Theorem 3.6.4*) is the Szegö Factorization Theorem: *let $A \in L^1(\mathbb{T})$ be nonnegative; then $\log A \in L^1(\mathbb{T})$ if and only if $A = |B|^2$ for some $B \in H^2(\mathbb{T})$, where*

$$H^2(\mathbb{T}) = \{F \in L^2(\mathbb{T}) : \forall n < 0, \quad \overset{\vee}{F}[n] = 0\}.$$

Proofs can be found in [GS58, Section 1.14], [Hof62, pages 48–54]. We shall not prove Szegö's Factorization Theorem, but mention it because Szegö's original proof (Math. Ann. 84(1921), 232–244) depended on

the following result which he proved in 1920 (Math. Zeit. 6(1920), 167–202). (The proof of the Szegö Factorization Theorem in [GS58, Chapter 1.14] is joint work of F. Riesz and Szegö which actually appeared before Szegö's original proof, cf., [MW57, pages 115 ff.].)

3.7.1 Theorem. SzEGÖ ALTERNATIVE

Let $W \in L^1(\mathbb{T})$ be nonnegative, and define the "geometric mean" of W as

$$g(W) = \begin{cases} \exp \int_{\mathbb{T}} \log W(\gamma) \, d\gamma, & \text{if } \log W \in L^1(\mathbb{T}), \\ 0, & \text{if } \log W \notin L^1(\mathbb{T}). \end{cases}$$

Then

$$\inf \int_{\mathbb{T}} |1 - P(\gamma)|^2 W(\gamma) \, d\gamma = g(W),$$

where the infimum is taken over all trigonometric polynomials P on $\mathbb{T}$ of the form

$$P(\gamma) = \sum_{n=1}^{N} a_n e^{-2\pi i n \gamma},$$

and where $N \geq 1$ and $a_1, \ldots, a_N$ vary.

The Szegö Alternative has been generalized in several directions, e.g., [Ach56, pages 256 ff.], [DM76], [Hel64, pages 19–24], [Hof62, pages 48–50]. It can also be reformulated as follows.

3.7.2 Theorem. KOLMOGOROV THEOREM (1940)

Let $W \in L^1(\mathbb{T})$ be nonnegative, and define the space,

$$L_W^2(\mathbb{T}) = \left\{ F : \|F\|_{L_W^2(\mathbb{T})} = \left(\int_{\mathbb{T}} |F(\gamma)|^2 W(\gamma) \, d\gamma \right)^{1/2} < \infty \right\}.$$

Then

$$(3.7.1) \qquad \overline{\operatorname{span}}\{e^{-2\pi i n \gamma} : n \leq 0\} = L_W^2(\mathbb{T})$$

if and only if $\log W \notin L^1(\mathbb{T})$, where $\overline{\operatorname{span}} X$ is defined as the closure in $L_W^2(\mathbb{T})$, taken with the $\| \ldots \|_{L_W^2(\mathbb{T})}$-norm, of the linear span of elements from X.

In the spirit of *Section 2.8*, we state the following definition.

3.7.3 Definition. STATIONARY SEQUENCES AND POWER SPECTRA

a. Let $f \in \ell^2(\mathbb{Z})$. The ℓ^2-*autocorrelation* of f is the sequence $p : \mathbb{Z} \to \mathbb{C}$ defined as

$$\forall n \in \mathbb{Z}, \quad p[n] = \sum_n f[n+m]\overline{f[m]}, ,$$

cf., *Example 2.7.9* for L^2-autocorrelation. By the Parseval Formula, $\widehat{p} = |F|^2 \in L^1(\mathbb{T})$. $S \equiv |F|^2$ is the *power spectrum* of f.

b. A sequence $x = \{x[n] : n \in \mathbb{Z}\}$ in a complex Hilbert space H is *stationary* if the *inner product*

$$r[n] = \langle x[n+k], x[k] \rangle, \quad n \in \mathbb{Z},$$

is independent of k, e.g., *Definition B.4*. The sequence r is the *autocorrelation* of x, and it is elementary to check that r is a positive definite function on $\mathbb{Z}$. By Herglotz's Theorem, there is $S \in M_+(\mathbb{T})$ for which $\widehat{r} = S$. S is the *power spectrum* of x, e.g., *Definition 3.10.3*.

3.7.4 Remark. PREDICTION AND KOLMOGOROV'S THEOREM

a. *Theorem 3.7.2* is valid in more general contexts, including replacing the weight W by any $\mu \in M_+(\mathbb{T})$, e.g., [Kol41], cf., [Ach56, pages 261–263].

b. Let H be a Hilbert space, and let $x \in H$ be a stationary sequence with *power spectrum* S. Define $H(x)$ as $\overline{\text{span}}\{x[n]\} \subseteq H$. Kolmogorov noted that *there is a unique linear mapping,*

$$Z : L_S^2(\mathbb{T}) \longrightarrow H(x),$$

defined on the exponentials as $Z(e^{2\pi i n \gamma}) = x[n]$, *which is an isometric isomorphism*, i.e., Z is a bijection and $\|F\|_{L_S^2(\mathbb{T})} = \|Z(F)\|$ for all $F \in L_S^2(\mathbb{T})$, where $\|\ldots\|$ is the norm on H.

c. Using *Theorem 3.7.2* and the result of part *b*, Kolmogorov solved the problem of "predicting the future from the whole past" [Kol41], cf., [DM76], [Ben92a]. We shall not go into the details of defining *deterministic sequences* which are required for a clear statement of the

prediction problem which Kolmogorov solved. However, one has the intuition of *prediction* from (3.7.1) in the sense that the *past information* $\{e^{2\pi i n \gamma} : n \leq 0\}$ is sufficient to *approximate* or *predict* any $F \in L^2_W(\mathbb{T})$, including exponentials $F(\gamma) = e^{2\pi i n \gamma}$ for *times* $n > 0$.

We shall now attempt to quantify *Remark 3.7.4c* for the practical matter of addressing prediction problems that arise in analyzing bio-electric traces, speech data, economic and weather trends, and a host of time series from a variety of subjects. We begin with the following example.

3.7.5 Example. PREDICTION ESTIMATES

a. Let $\Omega > 0$ and let $\widehat{f} = F$, where $F \in L^2(\mathbb{T}_{2\Omega})$ has Fourier coefficients $f = \{f[n]\} \in \ell^2(\mathbb{Z})$. Suppose that for a given value of n, $f[n]$ is not explicitly known, whereas $H_n \equiv \{f[n - k] : k \geq 1\}$ *is* known. When is it possible to predict, i.e., approximate or evaluate, $f[n]$ in terms of H_n? One way of addressing this question is to write

$$(3.7.2) \qquad \epsilon_N[n] = f[n] - \sum_{k=1}^{N} a_k f[n - k],$$

so that

$$\epsilon_N[n] = \int_{\mathbb{T}_{2\Omega}} F(\gamma) e^{\pi i n \gamma / \Omega} \, d\gamma - \sum_{k=1}^{N} a_k \int_{\mathbb{T}_{2\Omega}} F(\gamma) e^{\pi i (n-k)\gamma / \Omega} \, d\gamma,$$

from which we have the estimate

$$|\epsilon_N[n]| \leq \int_{\mathbb{T}_{2\Omega}} |F(\gamma)| \left| 1 - \sum_{k=1}^{N} a_k e^{-\pi i k \gamma / \Omega} \right| d\gamma$$

$$(3.7.3)$$

$$\leq \|F\|_{L^2(\mathbb{T}_{2\Omega})} \left(\int_{\mathbb{T}_{2\Omega}} \left| 1 - \sum_{k=1}^{N} a_k e^{-\pi i k \gamma / \Omega} \right|^2 d\gamma \right)^{1/2}.$$

In the transition from (3.7.2) to the right side of (3.7.3) we have squandered our information H_n. Further, $1 \notin \overline{\text{span}}\{e^{-\pi i k \gamma / \Omega} : k \geq 1\}$ in $L^2(\mathbb{T}_{2\Omega})$ by *Theorem 3.4.12*. This fact is corroborated by the Szegö

Alternative since the weight $W \equiv 1$ on the right side of (3.7.3) has the property that $\log W \in L^1(\mathbb{T}_{2\Omega})$.

b. We now adjust the calculation of part a by implementing the Parseval formula to compute

$$\|\epsilon_N\|_{\ell^2(\mathbb{Z})} = \left(\int_{\mathbb{T}_{2\Omega}} |F(\gamma)|^2 \left| 1 - \sum_{k=1}^{N} a_k e^{-\pi i k \gamma / \Omega} \right|^2 d\gamma \right)^{1/2}.$$

Of course, n is no longer fixed as it is in (3.7.2). On the other hand, if $\log W \notin L^1(\mathbb{T}_{2\Omega})$, where $W \equiv |F|^2$, then the Szegö Alternative (*Theorem 3.7.1*) or Kolmogorov's Theorem (*Theorem 3.7.2*) can be used to glean prediction theoretic information in the following way, cf., *Exercise 3.54*. For each fixed n, $|\epsilon_N[n]| \leq \|\epsilon_N\|_{\ell^2(\mathbb{Z})}$, and so $\inf |\epsilon_N[n]| = \inf \|\epsilon_N\|_{\ell^2(\mathbb{Z})} = 0$ in the case $\log |F| \notin L^1(\mathbb{T}_{2\Omega})$, where the infima are taken over all $N \geq 1$ and $a_1, \ldots, a_N \in \mathbb{C}$.

c. Our next adjustment of the calculation in part a deals with *analogue* signals $f \in PW_\Omega$, instead of *discrete* signals in $\ell^2(\mathbb{Z})$. We write

$$\epsilon_N(t) = f(t) - \sum_{k=1}^{N} a_k f(t - k\tfrac{\alpha}{\Omega})$$

for a fixed $\alpha > 0$ and for a fixed $t \in \mathbb{R}$. Then we compute

$$\epsilon_N(t) = \int \widehat{f}(\gamma) e^{2\pi i t \gamma} \, d\gamma - \sum_{k=1}^{N} a_k \int \widehat{f}(\gamma) e^{2\pi i (t - k\alpha/\Omega)\gamma} \, d\gamma,$$

cf., [Ben92b, Proposition 9]. Thus,

$$(3.7.4) \quad |\epsilon_N(t)| \leq \left(\frac{\Omega}{\alpha} \right)^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R})} \left(\int_{-\alpha}^{\alpha} \left| 1 - \sum_{k=1}^{N} a_k e^{-2\pi i k \gamma} \right|^2 d\gamma \right)^{\frac{1}{2}}.$$

As in (3.7.3), we have squandered information about t on the right side of (3.7.4). However, it can be shown that *if* $\alpha < 1/2$ *then*

$$(3.7.5) \qquad \overline{\mathrm{span}}\{ e^{-2\pi i k \gamma} : k \geq 1 \} = L^2[-\alpha, \alpha].$$

Thus,

$$\inf |\epsilon_N(t)| = 0,$$

where the infimum is taken over all $N \geq 1$ and $a_1, \ldots, a_N \in \mathbb{C}$.

The density result (3.7.5) is due to Carleman, e.g., [You80, pages 114-116], cf., *Remark 3.7.11*.

d. Suppose we are given a fixed discrete "time" n, resp., a fixed continuous "time" t. In part b, resp., part c, we have shown in theory how to predict the value $f[n]$, resp., $f(t)$, in terms of its known values at previous times. This prediction requires f to satisfy certain conditions; and the prediction itself is made in terms of a given error bound ϵ. In fact, in both cases it can be proved that coefficients a_j, $j = 1, \ldots, N$, exist with the property that $|\epsilon_N[n]| < \epsilon$, resp., $|\epsilon_N(t)| < \epsilon$.

We wish to design an applicable tool based on the idea of *Example 3.7.5*. We begin with the definition of *linear prediction*.

3.7.6 Definition. LINEAR PREDICTION MODELS

a. Let $M, N \geq 1$. A *pole-zero linear prediction model* of a sequence $f \in \ell^2(\mathbb{Z})$ is an equation of the form

$$(3.7.6) \qquad f[n] = \sum_{k=1}^{N} a_k f[n-k] + \sum_{j=0}^{M} b_j u[n-j],$$

for each n, where $u \in \ell^2(\mathbb{Z})$, $\{a_k\}, \{b_k\} \subseteq \mathbb{R}$, and $b_0 \neq 0$.

$f[n]$ can be thought of as the output of a *linear translation invariant system* with some unknown input u, including its past and present values, as well as input consisting of past values of f, viz., $f[n-1], \ldots, f[n-N]$. In this point of view, the goal is to estimate the *system parameters* $\{a_k\}, \{b_k\} \subseteq \mathbb{R}$, so that (3.7.6) is a meaningful predictor of f.

b. If $b_1 = b_2 = \cdots = b_M = 0$ then (3.7.6) is an *all-pole model*, or, equivalently, an *autoregressive* (AR) *model*. If $a_1 = a_2 = \cdots = a_N = 0$ then (3.7.6) is an *all-zero model*, or, equivalently, a *moving average* (MA) *model*. The full pole-zero model (3.7.6) is also called an *autoregressive moving average* (ARMA) *model*.

c. In the following, as in *Example 3.7.5*, we shall deal with the all-pole model. This type of linear prediction goes back to Yule's work (1927) on sun spot analysis, cf., the beginning of *Section 2.9* for another

remark on sun spots. The mathematics underlying the effectiveness of linear prediction is Gauss' technique of linear least squares estimation from 1795, cf., *Proposition 1.10.9.* Major contributions were made by Kolmogorov and Wiener, independently, in the early 1940s, e.g., [Wie49] including Appendix C by Norman Levinson. Linear prediction is a staple in time series analysis, e.g., [BSS88], [JN84], [Mak75], [Pri81]. We mention, for example, the introduction of linear prediction into speech analysis and data compression by Fant, Jury, Atal and Schroeder, and Itakura and Saito in the 1960s, e.g., [MG82].

3.7.7 Theorem. LEAST SQUARES METHOD FOR ALL-POLE MODEL

Let $N \geq 1$, assume $f \in \ell^2(\mathbb{Z}) \backslash \{0\}$ is real-valued with ℓ^2-autocorrelation r, and consider the sequence of equations

$$(3.7.7) \qquad \forall n \in \mathbb{Z}, \quad f[n] = \sum_{k=1}^{N} a_k f[n-k] + u[n].$$

There are unique coefficients $a_1^*, \ldots, a_N^* \in \mathbb{R}$ and a sequence $\epsilon_N \in \ell^2(\mathbb{Z})$ defined by

$$\epsilon_N[n] = f[n] - \sum_{k=1}^{N} a_k^* f[n-k]$$

for each n, such that

$$\forall a = (a_1, \ldots, a_N) \in \mathbb{R}^N,$$

(3.7.8)

$$0 \leq r[0] - \sum_{k=1}^{N} a_k^* r[k] = \|\epsilon_N\|_{\ell^2(\mathbb{Z})}^2 \leq \|u\|_{\ell^2(\mathbb{Z})}^2,$$

where u is defined by (3.7.7). (Thus, u depends on $a \in \mathbb{R}^N$ as opposed to the interpretation of Definition 3.7.6).

Proof. a. For a given $n \in \mathbb{Z}$ and $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, we define the approximation $f_a[n]$ of $f[n]$ as

$$f_a[n] = \sum_{k=1}^{N} a_k f[n-k].$$

Clearly, $(f - f_a) \in \ell^2(\mathbb{Z})$, and we set

$$E_N(a) = \sum_n \left(f[n] - \sum_{k=1}^N a_k f[n-k] \right)^2 < \infty.$$

b. For each k, we have the formal calculation

$$(3.7.9) \qquad \frac{\partial E_N(a)}{\partial a_k} = -2 \sum_n \left(f[n] - \sum_{j=1}^N a_j f[n-j] \right) f[n-k].$$

By Hölder's Inequality, the right side converges uniformly on any bounded interval of the a_k-axis. Thus, the right side of (3.7.9) is in fact $\frac{\partial E_N(a)}{\partial a_k}$.

c. A *necessary* condition in order that E_N have a local minimum (or maximum) at $a^* \in \mathbb{R}^N$ is that

$$(3.7.10) \qquad \forall k = 1, \ldots, N, \quad \frac{\partial E_N(a^*)}{\partial a_k} = 0.$$

A *sufficient* condition in order that E_N (which satisfies (3.7.10)) have a local minimum at $a^* \in \mathbb{R}^N$ is that $D_{N-k}(a^*) > 0$ for each $k = 0, 1, \ldots, N$, where $D_0(a) = 1$ and $D_{N-k}(a)$ is the determinant obtained from the $N \times N$ matrix,

$$\left(\frac{\partial^2 E_N(a)}{\partial a_m \partial a_n} \right), \quad m, n = 1, \ldots, N,$$

by deleting the last k rows and columns, and taking the determinant of the resulting $(N-k) \times (N-k)$ matrix, e.g., [Apo57].

d. In our case, by (3.7.9), we have

$$\forall a \in \mathbb{R}^N, \quad \frac{\partial^2 E_N(a)}{\partial a_m \partial a_n}$$

$$(3.7.11)$$

$$= 2 \sum_j f[j-m] f[j-n] = 2r[m-n] = 2r[n-m].$$

Since $f \in \ell^2(\mathbb{Z})$, we compute $\widehat{r} = |F|^2 \in L^1(\mathbb{T})$ by the Parseval Formula; and the $N \times N$ matrix $R = (r[m-n])$, $m, n = 1, \ldots, N$, is positive

definite, as we showed in *Definition 3.6.2e*. An elementary characterization of positive definite matrices is that $D_0(a), D_1(a), \ldots, D_N(a)$ be positive [Per52, Theorem 9–26].

Thus, our goal is to find values $a^* \in \mathbb{R}^N$ for which (3.7.10) is valid; and, by the previous paragraph and the sufficient conditions for minima in part c, these values will in fact be minimizers of E_N.

e. In order to obtain a candidate a^* for a minimizer, we rewrite (3.7.9) and (3.7.10) as the system of N linear equations in N unknowns $a_1, \ldots, a_N$:

$$\forall 1 \le k \le N,$$

(3.7.12)

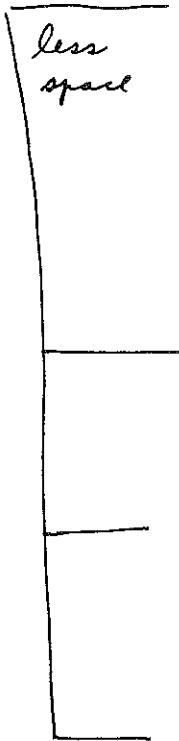$$\sum_{j=1}^{N} a_j \sum_n f[n-j]f[n-k] = \sum_n f[n]f[n-k],$$

or, equivalently,

(3.7.13) $\qquad \forall 1 \le k \le N, \quad \sum_{j=1}^{N} a_j r[k-j] = r[k].$

Since $D_N(a) > 0$ by (3.7.11) and the discussion in part d, the system (3.7.13) has a unique solution $a^* = (a_1^*, \ldots, a_N^*) \in \mathbb{R}^N$.

Thus, the inequality in (3.7.8) is obtained.

f. Expanding the square in the definition of E_N, and substituting the minimizer a^* into (3.7.12), we obtain

$$0 \leq E_N(a^*) = \sum_n f[n]^2 - 2 \sum_{k=1}^N a_k^* \sum_n f[n]f[n-k]$$

$$+ \sum_n \left(\sum_{k=1}^N a_k^* f[n-k] \right)^2$$

$$= \sum_n f[n]^2 - 2 \sum_{k=1}^N a_k^* \sum_n f[n]f[n-k] + \sum_{k=1}^N a_k^* \sum_n f[n]f[n-k]$$

$$= \sum_n f[n]^2 - \sum_{k=1}^N a_k^* \sum_n f[n]f[n-k]$$

$$= r[0] - \sum_{k=1}^N a_k^* r[k].$$

This completes (3.7.8). □

3.7.8 Example. SPECTRAL ESTIMATION AND THE ALL-POLE MODEL

a. A typical and important issue in many problems and fields is to find the *spectral peaks* or fundamental frequencies in a given signal $f \in \ell^2(\mathbb{Z})$. In theory, the graph of the Fourier series $F = \hat{f}$ will provide this spectral information. In reality, there are potential problems. For example, trigonometric polynomial approximations of $\hat{f}$, or approximations such as *Proposition 2.8.8*, may be inadequate because they are either too good or too bad! In the former case, a very good approximation of $\hat{f}$ may have so much spectral information from "noise" embedded in f that desirable information about pure tones (in f) is obscured when observing $\hat{f}$. In the latter case, the approximation may not be developed enough to specify relevant frequencies in $\hat{f}$.

b. In cases, such as those hypothesized in part a, where the Fourier transform can not be directly used to observe some fundamental frequencies in a signal, there are other methods which sometimes provide spectral information. The all-pole model is one of these methods. The prediction estimates in *Example 3.7.5* show that the *prediction error* ϵ_N in *Theorem 3.7.7* tends to 0 as $N \to \infty$ in many cases. The all-pole

model in (3.7.7) of *Theorem 3.7.7* shows that such a model and its corresponding prediction error can be used to specify spectral peaks by the following process and rationale.

By taking the Fourier transforms of the sequences f and u, (3.7.7) becomes

$$(3.7.14) \qquad F(\gamma) \left(1 - \sum_{k=1}^{N} a_k e^{-2\pi i k \gamma} \right) = U(\gamma),$$

where F and U are Fourier series with Fourier coefficients $\{f[n]\}$ and $\{u[n]\}$, respectively. With the minimization effected by *Theorem 3.7.7*, equation (3.7.14) becomes

$$F(\gamma) = \frac{\widehat{\epsilon}_N(\gamma)}{1 - \sum_{k=1}^{N} a_k^* e^{-2\pi i k \gamma}}.$$

Assume $0 < A \leq |\widehat{\epsilon}_N(\gamma)| \leq B$, which is reasonable for many applications, e.g., [Chi78]. Suppose F is continuous and $|F(\gamma_0)|$ is large in comparison to $|F(\gamma)|$ for values of γ near γ_0, i.e., suppose F has a spectral peak at γ_0. Define the polynomial

$$P_*(z) = 1 - \sum_{k=1}^{N} a_k^* z^k, \quad z \in \mathbb{C}.$$

We write $P(\gamma) = P_*(e^{-2\pi i \gamma})$. Then $|P(\gamma_0)|$ is small. This simple observation is the basis of the *all-pole model method of spectral estimation*.

In order to describe this method, we consider the following procedure for a given $N \geq 1$ and a given sequence f. First, choose a threshold $\epsilon > 0$ and consider the annular region $A_\epsilon = \{z : 1 - \epsilon \leq |z| \leq 1 + \epsilon\}$. The choice of ϵ can be adjusted according to the amount of spectral information desired. Next, compute $\{a_k^* : k = 1, \ldots, N\}$, cf., *Example 3.7.9*. Compute the zeros z_0 of P_*, e.g., in MATLAB use the *roots* command. If $z_0 \in A_\epsilon$, compute γ_0 by taking the projection $e^{-2\pi i \gamma_0}$ of z_0 to the unit circle.

$F(\gamma_0)$ is a candidate for a spectral peak.

c. As a caveat for our presentation in part *b*, we note that we have been precise about certain matters, e.g., *Theorem 3.7.7*, but quite cavalier about others. For example, the structure of u is important for the type of application at hand; and the proper behavior of $\hat{u}$ is important for the success of the all-pole model method of spectral estimation in that application, e.g., [JN84, Section 2.4], [Pri81].

We should also mention that we are not using the all-pole method to predict values of f so much as to determine its spectral behavior. In fact, in order to compute a^* in *Theorem 3.7.7* we assume knowledge of each $f[n]$.

3.7.9 Example. LEVINSON RECURSION ALGORITHM

a. Norman Levinson (1947) was the first to use the *structure* of a Toeplitz matrix (*Definition 3.6.2d*) to solve the system of linear equations (3.7.13) recursively, e.g., [Wie49, Appendix B]. This system not only plays a role in linear prediction (*Theorem 3.7.7*), but was also essential in the proof of *Theorem 3.6.6*, e.g., (3.6.11), which is associated with MEM. (Recall the equivalence of these methods noted in *Remark 3.6.1b*.) In statistics, equations (3.6.11) and (3.7.13) are called the Yule-Walker Equations. In making use of the Toeplitz structure, Levinson's algorithm has led to numerically realistic computations of the prediction coefficients $a_1^*, \ldots, a_N^*$. For example, the classical Gauss elimination method requires $N^3 + KN^2$ multiplications or divisions, whereas even Levinson's original method only required $N^2 + KN$ such operations.

b. If $\{c_k : k \geq 0\} \subseteq \mathbb{C}$ and $\exp(\sum_{k=0}^{\infty} c_k z^k) = \sum_{k=0}^{\infty} b_k z^k$, then one example of the Lebedev-Milin Inequalities is the inequality

$$|b_n|^2 \leq \exp\left(\sum_{k=1}^{n} k|c_k|^2 - \sum_{k=1}^{n} \frac{1}{k}\right), \quad b_0 = 1.$$

These inequalities have applications in univalent function theory, as well as in number theory and spectral synthesis, e.g., our construction of *idelic pseudo-measures* in *Zeta functions for idelic pseudo-measures*, Ann. Scuola Norm. Sup., 6(1979), 367–377.

It turns out that

$$(3.7.15) \qquad b_{k+1} = \sum_{j=0}^{k} \left(1 - \frac{j}{k+1}\right) c_{k+1-j} b_j$$

[Pou84]; and that this recursion formula can be implemented numerically to deal with linear prediction problems. For example, in dealing with the Fejér-Riesz Theorem (*Theorem 3.6.4*), $A = |B|^2$, we let $\{c_k\}$ be the sequence of Fourier coefficients of $\log A$, where $A \geq 0$ is the given nonnegative polynomial. Then, by way of a standard argument in complex analysis, we obtain $b_0 = e^{c_0/2}$ and (3.7.15) for $k = 0, \ldots, N-1$, where $b_0, \ldots, b_N$ are the Fourier coefficients of B, e.g., [Pou84].

3.7.10 Remark. RAMIFICATIONS OF SZEGÖ FACTORIZATION ON $\mathbb{R}$

a. We first stated and discussed the Paley-Wiener Logarithmic Integral Theorem in *Example 1.6.5c*. It asserts that *if $\varphi \in L^2(\widehat{\mathbb{R}}) \setminus \{0\}$ is nonnegative, then there is $f \in L^2(\mathbb{R})$, for which* supp $f \subseteq [0, \infty)$ *and* $|\widehat{f}| = \varphi$ *a.e., if and only if*

$$\int \frac{|\log \varphi(\gamma)|}{1 + \gamma^2} \, d\gamma < \infty.$$

Although it is elementary to prove the Szegö Factorization Theorem on $\mathbb{R}$ from this theorem of Paley and Wiener, e.g., [Pric85, pages 156-157], they were, in fact, motivated in their research by a result of Carleman on quasi-analytic functions [PW33], [PW34, Theorem XII], cf., [Koo88], [Rud66, Chapter 19] for the theory of quasi-analytic functions.

b. By an approximate identity argument, the Paley-Wiener Logarithmic Integral Theorem can be used to prove the following result. *If $\mu \in M(\mathbb{R})$, then* supp $\mu \subseteq [T, \infty)$ *if and only if*

$$\int \frac{|\log |\widehat{\mu}(\gamma)||}{1 + \gamma^2} \, d\gamma < \infty.$$

c. The result for compactly supported measures analogous to that of part *b* is due to Beurling and Malliavin: *let W be a continuous*

function for which $|W| > 1$ *and* $\log |W|$ *is uniformly continuous; then the condition,*

$$\int \frac{\log |W(\gamma)|}{1 + \gamma^2} \, d\gamma < \infty,$$

is necessary and sufficient that for all $\epsilon > 0$ *there exists* $\mu \in M(\mathbb{R})$ *such that* $\operatorname{supp} \mu \subseteq [-\epsilon, \epsilon]$ *and* $\hat{\mu} K \in L^\infty(\hat{\mathbb{R}})$ [BM62], [Mal79].

3.7.11 Remark. CLOSURE THEOREMS FOR SETS OF EXPONENTIALS

In *Example 3.7.5* we saw a role for the closure theorems of Szegö, Kolmogorov, and Carleman. There are other landmark contributions by Paley and Wiener [PW34], Levinson [Lev40], and Beurling and Malliavin [BM67], as well as deep results by others, cf., the superb expositions of [Red77], [You80], [Koo88]. We shall close this section with some perspective on such theorems.

a. Equation (3.7.5) was one of the first substantial results of an area which culminated in the work of Beurling and Malliavin [BM67]. Beurling and Malliavin solved the following *closure problem* for a given discrete subset $D \subseteq \mathbb{R}$: find the upper bound $\Omega \geq 0$ of the set of $\alpha \geq 0$ for which

$$(3.7.16) \qquad \overline{\operatorname{span}}\{e^{-2\pi i t \gamma} : t \in D\} = L^2[-\alpha, \alpha].$$

Their solution includes writing Ω in terms of a density condition on D.

b. Density results such as (3.7.16) are a *weak* form of *sampling theorems*– weak but *not* necessarily elementary. In fact, if (3.7.16) is valid and $f \in PW_\alpha$, then there is a sequence of trigonometric polynomials P_n, where

$$P_n(\gamma) = \sum_{t \in D_n} a_{t,n} e^{-2\pi i t \gamma} \quad \text{and} \quad D_n \subseteq D,$$

for which

$$\lim_{n \to \infty} \|\hat{f} - P_n\|_{L^2[-\alpha,\alpha]} = 0.$$

Distributionally, each $P_n = \hat{p}_n$ where $p_n = \sum_{t \in D_n} a_{t,n} \delta_t$. Thus,

$$\|\hat{f} - P_n\|_{L^2[-\alpha,\alpha]} = \|\hat{f} - P_n \mathbf{1}_{[-\alpha,\alpha]}\|_{L^2(\hat{\mathbb{R}})}$$

$$= \|f - p_n * d_{2\pi\alpha}\|_{L^2(\mathbb{R})},$$

and so

$$\lim_{n\to\infty} \sum_{t\in D_n} a_{t,n}\tau_t d_{2\pi\alpha} = f \quad \text{in} \quad L^2(\mathbb{R}),$$

cf., the sampling theorems in [Ben92b], [BF94], [BSS88], [Hig85].

c. Let $\mu \in M_+(\mathbb{R})$ and define

$$L^1_\mu(\mathbb{R}) = \{f : \|f\|_{L^1_\mu(\mathbb{R})} = \int |f(t)|\,d\mu(t) < \infty\},$$

cf., *Remarks 2.7.4*a and 3.5.10d.

By *Theorem 2.7.6*, $\mu = w + \mu_s$ where $w \in L^1(\mathbb{R})$ is nonnegative and μ_s is the sum of the discrete and continuous singular parts of μ. Krein proved the following L^1-version of Kolmogorov's Theorem (*Theorem 3.7.2*) on $\mathbb{R}$:

$$\overline{\text{span}}\{e^{-2\pi it\gamma} : \gamma \le 0\} = L^1_\mu(\mathbb{R})$$

if and only if

$$\int \frac{|\log w(t)|}{1+t^2}\,dt = \infty.$$

d. Suppose $w \ge 1$ on $\mathbb{R}$. In particular, $w \notin L^1(\mathbb{R})$, whereas $w \in L^1(\mathbb{R})$ in part c. Assume w is even, $1 = w(0) \le w(t)$, and $w(u+t) \le w(u)w(t)$ for $u,t \in \mathbb{R}$. $L^1_w(\mathbb{R}) \subseteq L^1(\mathbb{R})$ is an algebra under convolution, and Beurling (1938) posed the *spectral analysis question*: is every proper closed ideal $I \subseteq L^1_w(\mathbb{R})$ contained in a regular (i.e., $L^1_w(\mathbb{R})/I$ has a unit under convolution) maximal ideal? For an equivalent analytic means of posing this question, consider the following property of $I \equiv L^1_w(\mathbb{R})$:

(3.7.17) $\forall \gamma \in \widehat{\mathbb{R}},\ \exists f \in I$ such that $\widehat{f}(\gamma) \ne 0.$

Then the spectral analysis question is equivalent to finding conditions on w so that, whenever a closed ideal $I \subseteq L^1_w(\mathbb{R})$ satisfies (3.7.17), we can conclude that $I = L^1_w(\mathbb{R})$.

We have posed the spectral analysis question in this section since there is an equivalent dual formulation in terms of sets of exponentials in $L^\infty_{1/w}(\mathbb{R})$, e.g., [Ben75].

e. Beurling (1938) proved that if

$$(3.7.18) \qquad \int \frac{\log w(t)}{1 + t^2} \, dt < \infty$$

then the spectral analysis question has an affirmative answer [Beu89].

Let $w = 1$ on $\mathbb{R}$. Then (3.7.18) is satisfied and Beurling's Theorem reduces to the Wiener Tauberian Theorem : $f \in L^1(\mathbb{R})$ has a nonvanishing Fourier transform if and only if the closed principal ideal $I \subseteq L^1(\mathbb{R})$ generated by f is all of $L^1(\mathbb{R})$, cf., *Remark 2.9.13* and the formulation of Wiener's Tauberian Theorem in *Theorem 2.9.12*.

3.8   Discrete Fourier Transform

In *Section 1.1* we defined the Fourier transform $\widehat{f}$ of $f : \mathbb{R} \to \mathbb{C}$; in this case $\widehat{f}$ is defined on $\widehat{\mathbb{R}}$. In *Section 3.1* we defined the Fourier transform $\widehat{f}$ of $f : \mathbb{Z} \to \mathbb{C}$; in this case, and for a given $\Omega > 0$, $\widehat{f}$ is defined on $\mathbb{T}_{2\Omega} \equiv \widehat{\mathbb{R}}/(2\Omega\mathbb{Z})$, i.e., the Fourier series $\widehat{f}$ is a 2Ω-periodic function on $\widehat{\mathbb{R}}$ with Fourier coefficients $f = \{f[n]\}$. The next step is the following definition.

3.8.1 Definition. DISCRETE FOURIER TRANSFORM

a. Let N be a positive integer, and let $\mathbb{Z}_N$ be *the set of integers* $0, 1, \ldots, N-1$ *under addition modulo N*. This means that if $m, n \in \mathbb{Z}_N$ and the ordinary sum $m + n \le N - 1$, then the addition modulo N of m and n has the value $m + n$. However, if $m, n \in \mathbb{Z}_N$ and the ordinary sum $m + n > N - 1$, then the addition modulo N of m and n has the value $m + n - N$. For example, the addition table for $\mathbb{Z}_6$ is given in *Figure 3.3*.

When dealing with $\mathbb{Z}_N$ we shall *denote* the addition modulo N of $m, n \in \mathbb{Z}_N$ by $m + n \in \mathbb{Z}_N$.

In order to *define* functions f on $\mathbb{Z}_N$, we assign values $f[n]$ for $n = 0, 1, \ldots, N - 1$, and then we extend f as an N-periodic function on $\mathbb{Z}$. Thus, $f[m + nN] = f[m]$ for any $m \in \{0, 1, \ldots, N - 1\}$ and for all $n \in \mathbb{Z}$. In this case, we write $f : \mathbb{Z}_N \to \mathbb{C}$, cf., part *e*.

b. The *Fourier transform* of $f : \mathbb{Z}_N \to \mathbb{C}$ is the function $F : \mathbb{Z}_N \to \mathbb{C}$ defined as

$$(3.8.1) \quad \forall n \in \{0, 1, \ldots, N-1\}, \quad F[n] = \sum_{m=0}^{N-1} f[m]e^{-2\pi imn/N}.$$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Figure 3.3

Because of the setting $\mathbb{Z}_N$, F is called the *Discrete Fourier Transform* (DFT) of f. In this context we shall write

$$f \longleftrightarrow F, \quad \widehat{f} = F, \quad f = \overset{\vee}{F},$$

just as we did in *Definitions 1.1.2* and *3.1.1* for the cases of Fourier transforms (on $\mathbb{R}$) and Fourier series.

c. Letting $f : \mathbb{Z}_N \to \mathbb{C}$, and introducing the (standard) notation,

$$W_N = e^{-2\pi i/N},$$

the DFT F of f is defined as

$$(3.8.2) \qquad \forall n \in \mathbb{Z}_N, \quad F[n] = \sum_{m \in \mathbb{Z}_N} f[m] W_N^{mn}.$$

Clearly, (3.8.1) and (3.8.2) are equivalent.

We could have defined (3.8.1) for each $n \in \mathbb{Z}$ by our definition in part *a*; in fact,

$$\forall n \in \mathbb{Z}, \quad F[n] = \sum_{m=0}^{N-1} f[m]e^{-2\pi imn/N}$$

$$= \sum_{m=0}^{N-1} f[m]e^{-2\pi im(n+N)/N} = F[n + N].$$

In any case, $\mathbb{Z}_N$ is the natural domain for F.

d. $\mathbb{Z}_N$ is a commutative group under the operation of addition modulo N.

A *character* γ of $\mathbb{Z}_N$ is a homomorphism from $\mathbb{Z}_N$ into the multiplicative group $\{z \in \mathbb{C} : |z| = 1\}$, i.e.,

$$\forall m \in \mathbb{Z}_N, \quad |\gamma(m)| = 1$$

and

$$\forall m, n \in \mathbb{Z}_N, \quad \gamma(m + n) = \gamma(m)\gamma(n).$$

The set of characters of $\mathbb{Z}_N$ is denoted by $\widehat{\mathbb{Z}}_N$, and $\widehat{\mathbb{Z}}_N$ becomes a commutative group by defining the addition $\gamma_1 + \gamma_2$ of characters γ_1, γ_2 by means of the formula

$$\forall m \in \mathbb{Z}_N, \quad (\gamma_1 + \gamma_2)(m) = \gamma_1(m)\gamma_2(m).$$

In this setting it can be *proved* that $\widehat{\mathbb{Z}}_N = \mathbb{Z}_N$, e.g., [Rud62], cf., *Remark 3.1.3e*.

Algebraic considerations are fundamental in many aspects of harmonic analysis; but, in this book, I am coming closest to cheating the reader by their omission in my treatment of the DFT and FFT.

e. Let $L(\mathbb{Z}_N)$ denote the vector space of complex sequences on $\mathbb{Z}_N$, and define the DFT mapping

$$\mathcal{F}_N : L(\mathbb{Z}_N) \longrightarrow L(\widehat{\mathbb{Z}}_N),$$

$$f \longmapsto \widehat{f}.$$

Since $\mathbb{Z}_N$ is a finite set, $L(\mathbb{Z}_N)$ can be considered any one of the L^p-spaces on $\mathbb{Z}_N$. In fact, $L(\mathbb{Z}_N)$ is the N-dimensional space of all functions on $\mathbb{Z}_N$.

When one thinks of $L(\mathbb{Z}_N)$ as $L^2(\mathbb{Z}_N)$, we define the *inner product*

$$\forall f, g \in L(\mathbb{Z}_N), \quad \langle f, g \rangle = \frac{1}{N} \sum_{m=0}^{N-1} f[m]\overline{g[m]}.$$

f. In order to make the analogy between the DFT F of f on $\mathbb{Z}_N$ (N even) and Fourier series on $\mathbb{T}$ with Fourier coefficients on $\mathbb{Z}$, we could

consider the finite sets,

$$\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\} \text{ and } \{-\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}-1\},$$

corresponding to approximations of $\mathbb{T}$ and $\mathbb{Z}$, respectively.

The inversion theorem for the DFT is elementary.

3.8.2 Theorem. INVERSION FORMULA FOR THE DFT
   Let $N > 1$ and let $f : \mathbb{Z}_N \longrightarrow \mathbb{C}$ have DFT F. Then

$$(3.8.3) \quad \forall m = 0, 1, \ldots, N-1, \quad f[m] = \frac{1}{N} \sum_{n=0}^{N-1} F[n] e^{2\pi i m n / N}.$$

Proof. Note that $W_N^N = 1$, and if $N > 1$ then $W_N \neq 1$. Thus, since

$$(3.8.4) \qquad 1 + r + r^2 + \cdots + r^{N-1} = \frac{1 - r^N}{1 - r}, \quad r \neq 0,$$

we see that

$$(3.8.5) \qquad 1 + W_N + W_N^2 + \cdots + W_N^{N-1} = 0.$$

For fixed m, the right side of (3.8.3) is

$$\frac{1}{N} \sum_{n=0}^{N-1} F[n] e^{2\pi i m n / N} = \frac{1}{N} \sum_{n=0}^{N-1} F[n] W_N^{-mn}$$

$$(3.8.6)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{j=0}^{N-1} f[j] W_N^{jn} \right) W_N^{-mn} = \frac{1}{N} \sum_{j=0}^{N-1} f[j] \left(\sum_{n=0}^{N-1} W_N^{(j-m)n} \right).$$

If $j = m$ then $\sum_{n=0}^{N-1} W_N^{(j-m)n} = N$. If $j \neq m$ and $r \equiv W_N^{j-m}$, then $r \neq 1$ and (3.8.4) gives

$$(3.8.7) \qquad \sum_{n=0}^{N-1} W_N^{(j-m)n} = \frac{1}{1 - r}(1 - e^{2\pi i (m-j)n}) = 0.$$

Substituting this information into the right side of (3.8.6) gives $f[m]$.

$\square$

The simplicity of (3.8.5) or (3.8.7) in *Theorem 3.8.2* evaporates if one considers the *Gauss sum*,

$$(3.8.8) \qquad \mathcal{G}_N^{\pm} = \sum_{n=0}^{N-1} e^{\pm 2\pi i n^2/N},$$

e.g., *Example 3.8.6, Theorem 3.8.7, Theorem 3.8.9*, and *Theorem 3.8.10*.

3.8.3 Corollary.

$\mathcal{F}_N : L(\mathbb{Z}_N) \longrightarrow L(\mathbb{Z}_N)$ *is a linear bijection.*

3.8.4 Theorem. ONB AND PARSEVAL FORMULA

a. ONB. $\{\frac{1}{\sqrt{N}} W_N^n : n = 0, \ldots, N-1\}$ *is an ONB for* $L(\mathbb{Z}_N)$ *taken with the inner product defined in* Definition 3.8.1e.

b. Parseval Formula. Let $f, g \in L(\mathbb{Z}_N)$ *and consider the pairings* $f \longleftrightarrow F, g \longleftrightarrow G$. *Then*

$$(3.8.9) \qquad \sum_{m=0}^{N-1} f[m]\overline{g[m]} = \frac{1}{N} \sum_{n=0}^{N-1} F[n]\overline{G[n]},$$

and, in particular,

$$\left(\sum_{m=0}^{N-1} |f[m]|^2 \right)^{1/2} = \left(\frac{1}{N} \sum_{n=0}^{N-1} |F[n]|^2 \right)^{1/2}.$$

Proof. a. For a fixed $k, n \in \mathbb{Z}_N$, we consider W_N^k and W_N^n as functions on $\mathbb{Z}_N$ defined by $W_N^k[m] \equiv W_N^{mk}$. Then

$$(3.8.10)$$

$$\langle \frac{1}{\sqrt{N}} W_N^k, \frac{1}{\sqrt{N}} W_N^n \rangle = \frac{1}{N} \sum_{m=0}^{N-1} W_N^{mk} W_N^{-mn} = \frac{1}{N} \sum_{m=0}^{N-1} (e^{-2\pi i(k-n)/N})^m.$$

The right side of (3.8.10) is 1 if $k = n$. If $k \neq n$ then $r \equiv e^{-2\pi i(k-n)/N} \neq 1$ since $(k-n)/N \notin \mathbb{Z}$. Thus the right side of (3.8.10) is

$$(1 - r^N)/(1 - r) = 0$$

since $r^N = 1$. Therefore, $\{\frac{1}{\sqrt{N}} W_N^n : n \in \mathbb{Z}_N\}$ is orthonormal.

Linear independence follows from orthonormality. In fact, if $k \in \mathbb{Z}_N$ is fixed and $\sum_{n=0}^{N-1} a_n W_N^n = 0$, then

$$\forall m \in \mathbb{Z}, \quad \frac{1}{N} \sum_{n=0}^{N-1} a_n W_N^{mn} W_N^{-mk} = 0,$$

and so

$$\sum_{n=0}^{N-1} a_n \left(\frac{1}{N} \sum_{m=0}^{N-1} W_N^{mn} W_N^{-mk} \right) = 0.$$

By the orthonormality the left side is a_k.

The result follows since $L(\mathbb{Z}_N)$ is N-dimensional.

b. It is sufficient to prove (3.8.9); and (3.8.9) follows from *Theorem 3.8.2*, part a, and the fact that

$$\sum_{m=0}^{N-1} f[m]\overline{g[m]} = \frac{1}{N} \sum_{n=0}^{N-1} F[n] \sum_{k=0}^{N-1} \overline{G[k]} \left(\frac{1}{N} \sum_{m=0}^{N-1} W_N^{mn} W_N^{-mk} \right). \quad \square$$

3.8.5 Example. THE DFT MATRIX

a. The DFT $N \times N$-matrix $\mathcal{D}_N$ is defined as $(\frac{1}{\sqrt{N}} W_N^{mn})$, $m, n = 0, \ldots, N-1$, i.e.,

$$\mathcal{D}_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-2\pi i/N} & e^{-2\pi i2/N} & \cdots & e^{-2\pi i(N-1)/N} \\ 1 & e^{-2\pi i2/N} & e^{-2\pi i4/N} & \cdots & e^{-2\pi i2(N-1)/N} \\ \vdots & & & & \\ 1 & e^{-2\pi i(N-1)/N} & e^{-2\pi i2(N-1)/N} & \cdots & e^{-2\pi i(N-1)(N-1)/N} \end{pmatrix}.$$

Thus,

$$\mathcal{D}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

If $f \in L(\mathbb{Z}_N)$ is considered as a $1 \times N$ column vector then the DFT F of f is the $1 \times N$ column vector

$$F = \sqrt{N} \, \mathcal{D}_N f.$$

By *Theorem 3.8.2* we have

$$\mathcal{D}_N^{-1} = \frac{1}{\sqrt{N}} \overline{\mathcal{D}}_N,$$

where $\overline{\mathcal{D}}_N$ denotes complex conjugation of the entries of $\mathcal{D}_N$.

b. Note that the *trace (Definition 3.6.2)* of $\mathcal{D}_N$ is

$$\text{trace}(\mathcal{D}_N) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi i n^2/N} = \frac{1}{\sqrt{N}} \mathcal{G}_N^-,$$

which we shall evaluate in *Theorem 3.8.9* and *Theorem 3.8.10*.

c. Let $U_N^n : \mathbb{Z}_N \longrightarrow \mathbb{C}$, for fixed $n \in \mathbb{Z}_N$, be defined by $U_N^n[m] = \delta_{m,n}$ for $m \in \mathbb{Z}_N$. Clearly, $\{\sqrt{N}\, U_N^n\}$ is an ONB for $L(\mathbb{Z}_N)$, with the inner product defined in *Definition 3.8.1e.* $\{: m = 0, 1, \ldots, N-1\}$

It is easy to check that

$$(3.8.11) \qquad\qquad \mathcal{D}_N U_N^n = \frac{1}{\sqrt{N}} W_N^n,$$

where U_N^n and W_N^n are considered as $1 \times N$ column vectors and the left side is matrix multiplication. Similarly, a direct calculation shows that

$$(3.8.12) \qquad\qquad \mathcal{D}_N W_N^{-n} = \sqrt{N}\, U_N^n.$$

For example,

$$\mathcal{D}_5 W_5^{-2}$$

$$(3.8.13)$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & + & e^{2\pi i 2/5} & + & e^{2\pi i 4/5} & + & e^{2\pi i 6/5} & + & e^{2\pi i 8/5} \\ 1 & + & e^{2\pi i/5} & + & e^{2\pi i 2/5} & + & e^{2\pi i 3/5} & + & e^{2\pi i 4/5} \\ 1 & + & 1 & + & 1 & + & 1 & + & 1 \\ 1 & + & e^{-2\pi i/5} & + & e^{-2\pi i 2/5} & + & e^{-2\pi i 3/5} & + & e^{-2\pi i 4/5} \\ 1 & + & e^{-2\pi i 2/5} & + & e^{-2\pi i 4/5} & + & e^{-2\pi i 6/5} & + & e^{2\pi i 8/5} \end{pmatrix}.$$

The 1×5 column matrix on the right side of (3.8.13) is $\sqrt{5}\, U_5^2$. In fact, $\sum_{m=0}^{N-1} e^{-2\pi i m n/N} = 0$ for a fixed $n \in \mathbb{Z}_N$ by (3.8.4).

Combining (3.8.11) and (3.8.12) we see that

$$\forall n \in \mathbb{Z}_N, \quad \mathcal{D}_N^4 U_N^n = U_N^n.$$

Since $\{\sqrt{N}\, U_N^n\}$ is an ONB for $L(\mathbb{Z}_N)$ we conclude that

(3.8.14) $$\mathcal{D}_N^4 = I,$$

where I is the identity matrix, cf., *Example 1.10.12* where we did the analogous calculation for the Fourier transform on $\mathbb{R}$.

d. (3.8.14) leads naturally to investigating the *eigenvalue problem* for the DFT. In fact, because of (3.8.14), *the eigenvalues of $\mathcal{D}_N$ are* $\pm 1, \pm i$. The more difficult aspect of the eigenvalue problem is the *multiplicity problem*, viz., finding the eigenvectors of $\mathcal{D}_N$ and the dimension of the space of eigenvectors for each of the eigenvalues.

In any case the complete eigenvalue problem was essentially solved by Gauss, fundamental related calculations were exposited in E. Landau's classic book, Vorlesungen über Zahlentheorie (Volume 1, 1927, pages 164–165), and the explicit solution was recorded in [Goo62, page 261]. Recent comprehensive contributions are due to McClellan and Parks [MP72] and Auslander and Tolimieri, e.g., [AT79]. Using a technique due to Schur, e.g., [BSh66, pages 349–353], it is shown in [AT79, page 856] that *the solution of the multiplicity problem is equivalent to evaluating* trace $(\mathcal{D}_N)$ *for all N*.

In the following material, from *Example 3.8.6* to *Theorem 3.8.10*, we shall deal with the evaluation of the Gauss sum $\mathcal{G}_N^{\pm}$. ($\mathcal{G}_N^{\pm}$ was defined in (3.8.8)).

3.8.6 Example. $\mathcal{G}_N^{\pm} = 0$ IF 4 DIVIDES $N - 2$

a. If $N - 2 = 4k$, $k \geq 0$, then $N = 2(2k+1) \equiv 2M$, where $M \geq 1$ is odd. Then

$$(3.8.15) \quad \text{trace}(\mathcal{D}_N) = \sum_{m=0}^{M-1} e^{-2\pi i m^2/(2M)} + \sum_{m=0}^{M-1} e^{-2\pi i (m+M)^2/(2M)}.$$

ital ?

Expanding $(m+M)^2$ and using the fact that $e^{-2\pi i M/2} = -1$ for M odd, we see that

$$\sum_{m=0}^{M-1} e^{-2\pi i(m+M)^2/(2M)} = -\sum_{m=0}^{M-1} e^{-2\pi i m^2/(2M)} e^{-2\pi i m};$$

and, hence, the right side of (3.8.15) is 0.

b. Note that $\mathcal{G}_3^+ = i\sqrt{3}$ and $\mathcal{G}_3^- = -i\sqrt{3}$. In fact,

$$\sum_{m=0}^{2} e^{2\pi i m^2/3} = 1 + e^{2\pi i/3} + e^{2\pi i 4/3} = 1 + 2e^{2\pi i/3}$$

$$= 1 + 2\cos\frac{2\pi}{3} + 2i\sin\frac{2\pi}{3} = i\sqrt{3}$$

and

$$\sum_{m=0}^{2} e^{-2\pi i m^2/3} = 1 + 2\cos\frac{2\pi}{3} - 2i\sin\frac{2\pi}{3} = -i\sqrt{3}.$$

3.8.7 Theorem. $\left|\sum_{m=0}^{N-1} e^{2\pi i m^2/N}\right| = \sqrt{N}$, N ODD

Let $N \geq 1$ be an odd integer. Then

$$(3.8.16) \qquad \left|\sum_{m=0}^{N-1} e^{\pm 2\pi i m^2/N}\right| = \sqrt{N}.$$

Proof. Let $g[m] = e^{-2\pi i m^2/N}$, $m = 0, 1, \ldots, N-1$, and let $G = \hat{g}$. We shall prove (3.8.16) with the minus sign in the exponent. The plus sign case is a consequence of the same argument with the signs in the DFT properly adjusted. For $n = 0, \ldots, N-1$, we compute

$$G[2n] = \sum_{m=0}^{N-1} e^{-2\pi i m^2/N} e^{-2\pi i m(2n)/N}$$

$$(3.8.17) \qquad = e^{2\pi i n^2/N} \sum_{m=0}^{N-1} e^{-2\pi i(m+n)^2/N}$$

$$= e^{2\pi i n^2/N} \sum_{m=0}^{N-1} e^{-2\pi i m^2/N} = e^{2\pi i n^2/N} G[0].$$

The second step follows since $(m + n)^2 - n^2 = m^2 + 2mn$; and the third step follows since $W_N^{(m+N)^2} = W_N^{m^2}$ and by noting that the Gauss sum for either "$m + n$" or "m" has a domain of N consecutive integers. By (3.8.17) we have

(3.8.18)
$$\sum_{n=0}^{N-1} |G[2n]|^2 = N|G[0]|^2.$$

Since N is odd,

(3.8.19)
$$\sum_{n=0}^{N-1} |G[n]|^2 = \sum_{n=0}^{N-1} |G[2n]|^2.$$

For example, let $N = 5$ so that $G[2n]$, $n = 0, 1, 2$, consists of $G[0]$, $G[2]$, $G[4]$; and $G[2 \cdot 3] = G[1]$ and $G[2 \cdot 4] = G[3]$ since 5 divides $6 - 1$ and 5 divides $8 - 3$, respectively. The same phenomenon occurs for any odd $N \geq 3$ by properties of $\mathbb{Z}_N$.

Combining (3.8.18) and (3.8.19) with the Parseval Formula we obtain
$$N|G[0]|^2 = N \sum_{m=0}^{N-1} |g[m]|^2 = N^2.$$

(3.8.16) is obtained. $\square$

It is more difficult to evaluate $\mathcal{G}_N^{\pm}$ than $|\mathcal{G}_N^{\pm}|$. We shall first compute $\mathcal{G}_N^{+}$ for N odd in *Theorem 3.8.9*. Gauss gave the first proof of *Theorem 3.8.9* in 1805 after working "with all efforts" for four years, e.g., [BE81, pages 109–110]. There have been many other derivations of *Theorem 3.8.9*, e.g., [BE81]. We shall give Daniel Shanks' proof [Sha58].

Shanks originally devised ingenious finite term identities to prove deep infinite term identities of Euler and Gauss. The original idea for his proofs goes back to his PhD thesis at the University of Maryland in 1954. The identity by which he obtained Gauss' infinite term identity is *Lemma 3.8.8*, and in [Sha58] he used it to obtain Gauss' Theorem (*Theorem 3.8.9*).

3.8.8 Lemma. SHANKS FINITE IDENTITY

    If $x > 0$, $P_0(x) \equiv 1$, and

$$\forall N \in \mathbb{N}, \quad P_N(x) = \prod_{n=1}^{N} \left(\frac{1 - x^{2n}}{1 - x^{2n-1}} \right),$$

then

(3.8.20)
$$\sum_{n=1}^{2N} x^{n(n-1)/2} = \sum_{n=0}^{N-1} \frac{P_N(x)}{P_n(x)} x^{n(2N+1)}.$$

The proof of *Lemma 3.8.8* begins with the identity

$$(1 - x^{2N})x^{n(2N+1)} = (1 - x^{2N-1})x^{n(2N-1)}$$
$$+(1 - x^{2n+1})x^{(n+1)(2N-1)} - (1 - x^{2n})x^{n(2N-1)},$$

which we multiply by $P_{N-1}(x)/P_n(x)(1 - x^{2N-1})$.

3.8.9 Theorem. GAUSS COMPUTATION OF $\mathcal{G}_N^+$, N ODD

    Let $N \geq 1$ be an integer. Then

$$\frac{N-1}{4} \in \mathbb{N} \cup \{0\} \quad \text{implies} \quad \mathcal{G}_N^+ = \sqrt{N}$$

and

$$\frac{N-3}{4} \in \mathbb{N} \cup \{0\} \quad \text{implies} \quad \mathcal{G}_N^+ = i\sqrt{N}.$$

(These two cases include all odd integers N.)

Proof. We shall consider the case $(N - 1)/4 \in \mathbb{N} \cup \{0\}$. The other case is similar.

    Let $x = v^2$ and $v = e^{i\theta}$. Clearly,

(3.8.21)
$$P_M(x) = v^M \prod_{n=1}^{M} \left(\frac{v^{2n} - v^{-2n}}{v^{2n-1} - v^{1-2n}} \right).$$

Further, if $Q_0 \equiv 1$ and

$$Q_M = \prod_{n=1}^{M} \left(\frac{\sin 2n\theta}{\sin(2n - 1)\theta} \right),$$

then

$$(3.8.22) \qquad \sum_{n=1}^{2M} v^{n(n-1)} = \sum_{n=0}^{M-1} \frac{Q_M}{Q_n} v^{M+n(4M+1)}$$

and

$$(3.8.23) \qquad \sum_{n=1}^{2M+1} v^{n(n-1)} = \sum_{n=0}^{M} \frac{Q_M}{Q_n} v^{M+n(4M+1)}$$

by *Lemma 3.8.8*. For example, in the case of (3.8.22), the left side is precisely the left side of (3.8.20); and so

$$(3.8.24) \qquad \sum_{n=1}^{2M} v^{n(n-1)} = \sum_{n=0}^{M-1} \frac{P_M(x)}{P_n(x)} x^{n(2M+1)},$$

from which (3.8.22) is obtained by substituting (3.8.21) into the right side of (3.8.24).

Letting $N = 2K + 1$ and $\theta = 2\pi/N$, we have

$$v^{2K} = \exp i \left(2K(\frac{2\pi}{N}) \right) = \exp i \left((2K+1)(\frac{2\pi}{N}) - (\frac{2\pi}{N}) \right) = v^{-1}.$$

Thus,

$$\mathcal{G}_N^+ = \sum_{n=0}^{N-1} e^{2\pi i n^2/N} = \sum_{n=0}^{N-1} e^{i\theta n^2} = \sum_{n=0}^{N-1} v^{n^2}$$

$$(3.8.25) \qquad = \sum_{n=-K}^{K} v^{(K+n)^2} = v^{K^2} \sum_{n=-K}^{K} v^{2nK+n^2}$$

$$= v^{K^2} \sum_{n=-K}^{K} v^{n(n-1)} = v^{K^2} \left[\sum_{n=1}^{K} v^{n(n-1)} + \sum_{n=1}^{K+1} v^{n(n-1)} \right].$$

Finally, let $N = 4M + 1$ and $K = 2M$. Combining (3.8.25) with (3.8.22) and (3.8.23), we obtain

$$\mathcal{G}_N^+ = v^{K^2} \left[\sum_{n=1}^{K} v^{n(n-1)} + \sum_{n=1}^{K+1} v^{n(n-1)} \right]$$

$$= v^{4M^2} \left[\sum_{n=0}^{M-1} \frac{Q_M}{Q_n} v^{M+n(4M+1)} + \sum_{n=0}^{M} \frac{Q_M}{Q_n} v^{M+n(4M+1)} \right]$$

$$= \sum_{n=0}^{M-1} \frac{Q_M}{Q_n} v^{(M+n)(4M+1)} + \sum_{n=0}^{M} \frac{Q_M}{Q_n} v^{(M+n)(4M+1)}.$$

Note that $v^N = 1$ since $v^{2K} = v^{-1}$. Hence,

$$(3.8.26) \qquad \mathcal{G}_N^+ = 1 + 2 \sum_{n=0}^{M-1} \frac{Q_M}{Q_n}.$$

Also, $Q_0 = 1$ and $Q_N > 0$ for $n = 1, \ldots, M$ since $\theta = 2\pi/N$. In fact, for such $n, 0 < 2n\theta \le 4M\pi/N < \pi$. Thus, $\mathcal{G}_N^+ > 0$ if $\frac{N-1}{4} \in \mathbb{N} \cup \{0\}$. The result follows by (3.8.16). ((3.8.26) gives a cryptic way to write $\sqrt{N}$ as a sum of products of quotients of sines!) $\qquad \square$

In 1835, Dirichlet used Fourier series to evaluate $\mathcal{G}_N^{\pm}$ for all N. In this paper, Dirichlet also gave Gauss' proof of the *Law of Quadratic Reciprocity (Remark 3.8.11a)* once he (Gauss) had *Theorem 3.8.9*, e.g., Dirichlet's *Werke*, Chelsea Publishing Company, New York, pages 257–270. The following is Dirichlet's computation of $\mathcal{G}_N^{\pm}$. It not only shows Dirichlet's brilliance, but also the power of elementary harmonic analysis.

3.8.10 Theorem. DIRICHLET COMPUTATION OF $\mathcal{G}_N^{\pm}$

$$\mathcal{G}_N^{\pm} =$$

(3.8.27)

$$\begin{cases} (1+i)\sqrt{N}, resp., \ (1-i)\sqrt{N}, & if \ \ N/4 \in \mathbb{N}, \\ \sqrt{N}, & if \ \ (N-1)/4 \in \mathbb{N} \cup \{0\}, \\ 0, & if \ \ (N-2)/4 \in \mathbb{N} \cup \{0\}, \\ i\sqrt{N}, resp., \ -i\sqrt{N}, & if \ \ (N-3)/4 \in \mathbb{N} \cup \{0\}, \end{cases}$$

where the two values for two cases on the right side of (3.8.27) indicate $\mathcal{G}_N^+$ and $\mathcal{G}_N^-$, respectively.

Proof. We shall evaluate $\mathcal{G}_N^-$. The calculation of $\mathcal{G}_N^+$ is similar.

    i. Let $G(\gamma) = e^{-2\pi i \gamma^2/N}$ on $\widehat{\mathbb{R}}$. Then

$$\sum_{k=0}^{N-1} \left(\tau_{-k} G(0) + \tau_{-k} G(1) \right) = (1 + e^{-2\pi i/N})$$

$$+ \left(e^{-2\pi i/N} + e^{-2\pi i 2^2/N} \right) + \left(e^{-2\pi i 2^2/N} + e^{-2\pi i 3^2/N} \right)$$

$$+ \ldots + \left(e^{-2\pi i (N-1)^2/N} + e^{-2\pi i N^2/N} \right) = 2\mathcal{G}_N^-.$$

    Let

$$F(\gamma) = \tau_{-0} G(\gamma) + \tau_{-1} G(\gamma) + \ldots + \tau_{-(N-1)} G(\gamma)$$

for $\gamma \in \widehat{\mathbb{R}}$. Then

$$F(\gamma + 1) = F(\gamma) + e^{-2\pi i \gamma^2/N} \left(e^{-4\pi i \gamma} - 1 \right)$$

and so $F(1) = F(0)$. At this point, we shall only consider F defined on $[0, 1)$, and extend F as a 1-periodic function on $\widehat{\mathbb{R}}$.

    By the smoothness of G, and the fact that $F \in C(\mathbb{T})$, we can invoke Dirichlet's Theorem (*Theorem 3.1.6*) and *Remark 3.1.7b* to write

$$\mathcal{G}_N^- = \frac{F(1) + F(0)}{2} = F(0) = \sum f[m],$$

where the sequence $f = \{f[m]\}$ of Fourier coefficients of F is evaluated by

$$f[m] = \int_0^1 \sum_{k=0}^{N-1} \tau_{-k} G(\gamma) e^{2\pi i m \gamma} \, d\gamma$$

$$= \sum_{k=0}^{N-1} \int_k^{k+1} G(\lambda) e^{2\pi i m \lambda} \, d\lambda = \int_0^N e^{-2\pi i \lambda^2/N} e^{2\pi i m \lambda} \, d\lambda$$

for each $m \in \mathbb{Z}$. Completing the square, we obtain

$$f[m] = e^{\pi i m^2 N/2} \int_0^N e^{-2\pi i (\gamma - mN/2)^2/N} \, d\gamma,$$

and, hence,

$$(3.8.28) \qquad \mathcal{G}_N^- = \sum e^{\pi i m^2 N/2} \int_0^N e^{-2\pi i (\gamma - mN/2)^2/N} \, d\gamma.$$

If m is even then $e^{\pi i m^2 N/2} = 1$ and if m is odd then $e^{\pi i m^2 N/2} = e^{\pi i N/2} = i^N$. Separating the sum on the right side of (3.8.28) into even ($m = 2k$) and odd ($m = 2k + 1$) parts, and making the corresponding changes of variable, (3.8.28) becomes

$$\mathcal{G}_N^- = I_e + I_o,$$

where

$$I_e = \sum \int_{-kN}^{(1-k)N} e^{-2\pi i \lambda^2/N} \, d\lambda = \int e^{-2\pi i \lambda^2/N} \, d\lambda$$

and

$$I_o = i^N \sum \int_{(-k-\frac{1}{2})N}^{(1-k-\frac{1}{2})N} e^{-2\pi i \lambda^2/N} \, d\lambda = i^N \int e^{-2\pi i \lambda^2/N} \, d\lambda.$$

Using *Theorem 2.10.1*, we see that

$$\int e^{-2\pi i \lambda^2/N} \, d\lambda = \frac{\sqrt{N}}{2}(1 - i).$$

Combining this information we have the formula,

$$(3.8.29) \qquad \mathcal{G}_N^- = \frac{\sqrt{N}}{2}(1 - i)(1 + i^N).$$

   ii. (3.8.27) follows by letting $N = 4k$, $4k+1$, $4k+2$, $4k+3$, respectively, cf., *Example 3.8.6a* for the case $N = 4k + 2$ and *Example 3.8.6b* for the case $N = 4k + 3$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.8.11 Remark. GAUSS SUMS: POTPOURRI AND TITILLATION

   a. *Law of Quadratic Reciprocity.* The *Law of Quadratic Reciprocity* asserts that if p and q are distinct odd primes then

size ?

$$(3.8.30) \qquad \left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = e^{2\pi i (p-1)(q-1)/8},$$

where $\left(\frac{p}{q}\right)$ is the *Legendre Symbol*. It is definedas follows. If r and m are relatively prime then r is the *quadratic residue* of m if

$$\exists n \in \mathbb{N} \text{ such that } \frac{n^2 - r}{m} \in \mathbb{N} \cup \{0\},$$

and it is a *quadratic nonresidue* if there is no such n. Then $\left(\frac{q}{p}\right) = 1$ if p is a quadratic residue of q, and $\left(\frac{p}{q}\right) = -1$ if it is a quadratic nonresidue of q.

Equation (3.8.30) is important in Diophantine polynomial equations, e.g., [BE81], [HW65], as well as more advanced (and just as difficult!) number theoretic topics relates to the Riemann ζ-function, more general Dirichlet L-functions, etc., e.g., [BSh66], [Cha68], [Cha70].

Equation (3.8.30) was first stated by Euler (1783). Legendre had an incomplete proof since he used a property, only later proved by Dirichlet, about primes in arithmetic progressions, e.g., *Section 3.2.3*. Independently, Gauss discovered (3.8.30), which he called the "Theorema aureum", in March 1795 before his 18th birthday on April 30, 1777; and gave the rigorous proof by April 1796. He went on to give seven other proofs. His third proof is in [HW65, Chapter VI]. His fourth proof, published in 1809, used *Theorem 3.8.9*. This means of obtaining the Law of Quadratic Reciprocity by means of Gauss sums has been refined by magisterial lineage: Schaar (1848), Kronecker (1880), Hecke (1919), C. L. Siegel (1966), e.g., [Cha68, pages 34–42].

b. *Littlewood Flatness Problem.* Let $\mathcal{U}_N$ denote the class of *unimodular trigonometric polynomials* $U(\gamma) = \sum_{n=0}^{N} u_n e^{-2\pi i n \gamma}$, i.e., $|u_n| = 1$ for $n = 0, 1, \ldots, N$. In light of a question asked in [Lit66] and expanded upon in [Lit68, Problem 19], we state the following *Littlewood Flatness Problem*: determine whether or not there are unimodular polynomials $U_N \in \mathcal{U}_N$ and positive constants $\epsilon_N > 0$ tending to zero, as $N \to \infty$, such that

$$\forall \gamma \in \mathbb{T},$$

(3.8.31)
$$(1 - \epsilon_N)\sqrt{N+1} \leq |U_N(\gamma)| \leq (1 + \epsilon_N)\sqrt{N+1}$$

for all large N. It turns out that Gauss sums and their variants play a natural role in dealing with (3.8.31).

The inequalities (3.8.31) assert that there are trigonometric polynomials $\frac{1}{\sqrt{N+1}} U_N$, whose modulus is almost identically 1 on $\mathbb{T}$ and whose coefficients have moduli which are all identically 1. Also, using the *Parseval Formula*, they imply that

$$(3.8.32) \quad \forall N, \exists U_N \in \mathcal{U}_N \quad \text{such that} \quad \lim_{N \to \infty} \frac{\|U_N\|_{L^\infty(\mathbb{T})}}{\|U_N\|_{L^2(\mathbb{T})}} = 1.$$

There have been herculean attempts to prove (3.8.31), sometimes in concert with subtle failures. Finally, in 1980, Kahane [Kah80] proved (3.8.31) by showing that such polynomials U_N *exist*, cf., [QS96]. His proof has a fundamental probabilistic component, and it remains to *construct* the U_N. There *are* constructions in the more general case where it is only assumed that the moduli of the coefficients are bounded by 1, e.g., [BNe74], [Benk92].

c. *Antenna Theory.* The ratio $\|F\|_{L^\infty(\mathbb{T})}/\|F\|_{L^2(\mathbb{T})}$ is called the *crest factor* of F. This relationship between L^∞ ("maximum and minimum values") and L^2 ("energy") norms plays a role in a number of applications. Further, under certain constraints, trigonometric polynomials provide a natural model for arrays of energy transmitters and receptors in fields such as acoustics, electromagnetism, and seismology, e.g., [Sche60].

In particular, the space $\mathcal{U}_N$ combined with (3.8.32) can be used in *antenna array signal processing*, e.g., [Stei76], [Hay85]. The space $\mathcal{U}_N$ gives way to other models for other aspects of antenna theory. For example, transmitters and receptors can be placed at points $x_1, \ldots, x_N \in \mathbb{R}^3$; and space factors for the corresponding array outputs can be modeled by trigonometric polynomials $P(\gamma) = \sum_{n=1}^{N} \exp 2\pi i x_n \cdot \gamma$. Since each of the coefficients is 1, it becomes relevant to check how close P is to the δ measure, and to analyze the impact of the sidelobes of P outside of a neighborhood of the origin, e.g., [BHe93], cf., equation (UP) in *Remark 1.1.4* and *Example 2.4.8b*, as well as the case of absolute values for the examples considered in part *b* and *Theorem 2.10.3a*.

d. *Lagrange's Theorem and Surgery.* The quadratic forms $\sum r_{jk} c_k \overline{c_j}$ we have dealt with have had complex entries; and we have established central results in *harmonic analysis* in the case these forms are nonnegative. It turns out that quadratic forms with integer or rational number

entries have been central to the development of *number theory* and *algebra*. The illustrious Dedekind, Frobenuis, Minkowski, E. Noether, E. Artin, C. L. Siegel, Eichler, and Hasse are all major contributors.

Lagrange (1770) proved that *every positive integer is the sum of four squares of integers*. More generally, if $\{r_{jk} : j, k = 1, \ldots, N\} \subseteq \mathbb{N}$, it is natural to ask which integers $n \in \mathbb{N}$ are of the form $n = \sum r_{jk} c_k c_j$ for some sequence $c = \{c_j : 1, \ldots, N\} \subseteq \mathbb{N}$, and how many such "solutions" c there are, e.g., [HW65, Chapter 20]. The problem in this generality is unsolved. Many of the special cases that are known, as well as Minkowski's classification of quadratic forms over $\mathbb{Q}$, involve Gauss sums, e.g., [Scha85, Chapter 5]. These Gauss sums are of the form $\sum e^{2\pi i q(x)}$ where q is a quadratic form associated with some algebraic structure. For example, if b is a bilinear function $b : V \times V \rightarrow K$, where V is a vector space over a field K, then $q(x) = b(x, x)$ defines a quadratic form.

Surgery invariants can also be determined by evaluating Gauss sums of this type. The purpose of surgery in *differential topology* is to characterize simply connected manifolds in higher dimensions, and a basic Gauss sum was computed in this setting by J W. Morgan and D. P. Sullivan (Annals of Mathematics, 1974). The first use of Gauss sums in surgery problems was made by Edgar Brown.

3.9   Fast Fourier Transform

Let $N > 1$ and let $f : \mathbb{Z}_N \rightarrow \mathbb{C}$ have DFT F. As is apparent from *Example 3.8.5a*, $\{F[n] : n \in \mathbb{Z}_N\}$ can be computed with N^2 operations, where an *operation* is defined to mean a complex multiplication followed by a complex addition. The *Fast Fourier Transform* (FFT) is an algorithm to compute a DFT by $N \log_2 N$ operations in the case $N = 2^r$, e.g., *Example 3.9.2*. The fundamental paper on the FFT is due to Cooley and Tukey [CT65], and it involves an idea which they refer to as a *two step algorithm*.

It turns out that two step Fourier analysis algorithms have been used in various applications since early in the 19th century. The first published results are due to Francesco Carlini (1828) in his research

on hourly barometric variations. This historical fact, as well as the fact that Gauss had a general form of the FFT as early as 1805, is found in the fascinating article by M. T. Heideman, D. H. Johnson, and C. S. Burrus [HJB84] on the history of the FFT, cf., [DV90], [IEEE69], [OS75, Chapter 6], [RR72], [Walk91] for other developments. Gauss' results were published posthumously in 1866.

3.9.1 Theorem. Two Step FFT Algorithm

Let $N_1, N_2 > 1$ and let $f : \mathbb{Z}_N \to \mathbb{C}$ have DFT F, where $N = N_1 N_2$.

a.   Each $n = 0, 1, \ldots, N - 1$ has the unique representation $n = n_2 N_1 + n_1$, for some $n_1 = 0, 1, \ldots, N_1 - 1$ and some $n_2 = 0, 1, \ldots, N_2 - 1$; and each $m = 0, 1, \ldots, N - 1$ has the unique representation $m = m_1 N_2 + m_2$, for some $m_1 = 0, 1, \ldots, N_1 - 1$ and some $m_2 = 0, 1, \ldots, N_2 - 1$.

b.   In the format of part a, F can be written as

(3.9.1)

$$F[n_2 N_1 + n_1] = \sum_{m_2=0}^{N_2-1} \left(\sum_{m_1=0}^{N_1-1} f[m_1 N_2 + m_2] W_{N_1}^{m_1 n_1} W_N^{m_2 n_1} \right) W_{N_2}^{m_2 n_2}$$

for each $n = n_2 N_1 + n_1$.

c.   The total number of operations required to compute $\{F[n] : n = n_2 N_1 + n_1, \ n_1 \in \mathbb{Z}_{N_1}, \ and \ n_2 \in \mathbb{Z}_{N_2}\}$ is

$$N(N_1 + N_2).$$

Proof. Part *a* is immediate when, for the case of n, we think of the N_2 equispaced elements $0, N_1, 2N_1, \ldots, (N_2-1)N_1$ of $\mathbb{Z}_N$. In fact, for each fixed $n_2 N_1$ we obtain all the elements of $\mathbb{Z}_N$ which are greater than or equal to $n_2 N_1$ and less than $(n_2+1)N_1$ by adding $n_1 = 0, 1, \ldots, N_1 - 1$ to $n_2 N_1$. (If $n_2 = N_2 - 1$ then $(n_2 + 1)N_1 = 0$.)

b. For a given n, choose n_1, n_2 for which $n = n_2 N_1 + n_1$. Then

(3.9.2) $$F[n_2 N_1 + n_1] = \sum_{m_1, m_2} f[m_1 N_2 + m_2] W_N^{m_1 N_2 n} W_N^{m_2 n}.$$

We calculate

$$e^{-2\pi i m_1 N_2 n / N} = e^{-2\pi i m_1 n / N_1} = e^{-2\pi i m_1 (n_2 N_1 + n_1)/N_1}$$

and

$$e^{-2\pi i m_2(n_2 N_1 + n_1)/N} = e^{-2\pi i m_2 n_2 / N_2} e^{-2\pi i m_2 n_1 / N},$$

and so the right side of (3.9.2) is

$$\sum_{m_1, m_2} f[m_1 N_2 + m_2] W_{N_1}^{m_1 n_1} W_N^{m_2 n_1} W_{N_2}^{m_2 n_2}.$$

Writing out the domains of m_1 and m_2, this is precisely (3.9.1), which we shall also write as

$$F[n] = \sum_{m_2=0}^{N_2-1} F[m_2, m_1] W_{N_2}^{m_2 n_2},$$

where

$$F[m_2, m_1] = \sum_{m_1=0}^{N_1-1} f[m_1 N_2 + m_2] W_{N_1}^{m_1 n_1} W_N^{m_2 n_1}.$$

c. For fixed n_1, n_2, and m_2, we can compute

$$(3.9.3) \qquad \sum_{m_1=0}^{N_1-1} f[m_1 N_2 + m_2] W_{N_1}^{m_1 n_1}$$

by $N_1 - 1$ operations, where the first of these operations yields

$$f[0 \cdot N_2 + m_2] + f[1 \cdot N_2 + m_2] W_{N_1}^{1 \cdot n_1},$$

and the $(N_1 - 1)$st operation yields the complete sum (3.9.3). Multiplying this sum by $W_N^{m_2 n_1}$ we obtain $F[m_2, n_1]$ in $(N_1 - 1) + 1 = N_1$ operations.

Now, using the data $\{F[m_2, n_1]\}$ which required N_1 operations to compute, we can compute each

$$F[n] = \sum_{m_2=0}^{N_2-1} F[m_2, n_1] W_{N_2}^{m_2 n_2}$$

by an *additional* N_2 operations. Thus, for each pair (n_1, n_2), $F[n] = F[n_2 N_1 + n_1]$ can be computed by means of (3.9.1) in $N_1 + N_2$ operations. Since there are $N = N_1 N_2$ pairs (n_1, n_2), all the data $\{F[n] : n = 0, \ldots, N - 1\}$ can be computed by means of (3.9.1) in $N(N_1 + N_2)$ operations. $\square$

3.9.2 Example. ORDER $N \log_2 N$ ALGORITHM

a. Let $N = 2N_1$. Then each $n = 0, 1, \ldots, N-1$ can be written as $n = n_2 N_1 + n_1$ for $n_1 = 0, 1, \ldots, N_1 - 1$ and $n_2 = 0, 1$. Hence, if $F : \mathbb{Z}_N \to \mathbb{C}$, defined as

$$F[n] = \sum_{m=0}^{N-1} f[m] W_N^{mn},$$

is the DFT of $f : \mathbb{Z}_N \to \mathbb{C}$, then

$$F[n] = F[n_2 N_1 + n_1] = \sum_{m_1, m_2} f[2m_1 + m_2] W_N^{(2m_1 + m_2)n},$$

where $m_1 = 0, 1, \ldots, N_1 - 1$ and $m_2 = 0, 1$. We compute

$$F[n_2 N_1 + n_1] = \sum_{m_2=0}^{1} \sum_{m_1=0}^{N_1-1} f[2m_1 + m_2] W_{2N_1}^{(2m_1 + m_2)n}$$

$$= \sum_{m_2=0}^{1} \sum_{m_1=0}^{N_1-1} f[2m_1 + m_2] W_{N_1}^{m_1 n_1} W_N^{m_2 n_1} e^{-\pi i m_2 n_2},$$

since

$$e^{-2\pi i 2m_1 n/(2N_1)} = e^{-2\pi i m_1(n_2 N_1 + n_1)/N_1} = W_{N_1}^{m_1 n_1}$$

and

$$e^{-2\pi i m_2 n/(2N_1)} = e^{-2\pi i m_2(n_2 N_1 + n_1)/(2N_1)}$$

$$= e^{-\pi i m_2 n_2} e^{-2\pi i m_2 n_1/(2N_1)} = e^{-\pi i m_2 n_2} W_N^{m_2 n_1}.$$

Therefore,

$$F[n_2 N_1 + n_1] = \sum_{m_1=0}^{N_1-1} f[2m_1] W_{N_1}^{m_1 n_1} + e^{-\pi i n_2} W_N^{n_1} \sum_{m_1=1}^{N_1-1} f[2m_1 + 1] W_{N_1}^{m_1 n_1}.$$

Consequently, because

$$e^{-\pi i n_2} W_N^{n_1} = e^{-2\pi i n_2 N_1/(2N_1)} W_N^{n_1} = W_N^n,$$

we obtain the *FFT algorithm*,

(3.9.4)

$$F[n_2 N_1 + n_1] = \sum_{m_1=0}^{N_1-1} f[2m_1] W_{N_1}^{m_1 n_1} + W_N^n \sum_{m_1=1}^{N_1-1} f[2m_1 + 1] W_{N_1}^{m_1 n_1}$$

for each $n = n_2 N_1 + n_1$.

b. Let $\#(K)$ be the number of multiplications required to compute the DFT of $f : \mathbb{Z}_K \to \mathbb{C}$. Clearly, $\#(K) \leq (K-1)^2$.

In dealing with the right side of (3.9.4), we see that $2\#(N_1)$ multiplications are required to compute

$$s_e(n_1) = \sum_{m_1=0}^{N_1-1} f[2m_1] W_{N_1}^{m_1 n_1} \quad \text{and} \quad s_0(n_1) = \sum_{m_1=0}^{N_1-1} f[2m_1+1] W_{N_1}^{m_1 n_1}$$

for all $n_1 \in \mathbb{Z}_{N_1}$.

Now note that $W_N^n = \pm W_N^{n_1}$ depending on whether $n_2 = 0, 1$ in the representation $n = n_2 N_1 + n_1$. Thus, N_1 multiplications are required to compute $\{W_N^{n_1} s_0(n_1) : n_1 \in \mathbb{Z}_{N_1}\}$ for given data $\{W_N^{n_1}, \ s_0(n_1) : n \in \mathbb{Z}_{N_1}\}$. Consequently, equation (3.9.4) allows us to write

(3.9.5) $\#(2N_1) = 2\#(N_1) + N_1.$

c. Because of (3.9.5) and the fact that $\#(2) = 1$ (since the DFT of $\{f[0], f[1]\}$ is $\{f[0] + f[1], f[0] + f[1]W_2^1\}$), we have

$$\#(2) = 1$$
$$\#(4) = 2 + 2 = 4 = \tfrac{1}{2} 4 \log_2 4$$
$$\#(8) = 8 + 4 = 12 = \tfrac{1}{2} 8 \log_2 8$$

$$\cdots \ .$$

In fact, *if $N = 2^r$ then the number of multiplications required to compute the DFT on $\mathbb{Z}_N$ is*

(3.9.6) $\dfrac{N}{2} \log_2 N.$

To prove (3.9.6) we proceed by induction. We just checked the result for $N = 2, 4, 8$. Now assume (3.9.6) is true for $N = N_1 = 2^p$, i.e., make the induction hypothesis that $\#(N_1) = \frac{N_1}{2} \log_2 N_1$. Letting $N = 2N_1$, (3.9.5) and the induction hypothesis imply

$$\#(N) \ = 2\#(N_1) + N_1 = N_1 \log_2 N_1 + N_1$$
$$= \tfrac{N}{2}(\log_2 N_1 + 1) = \tfrac{N}{2} \log_2 N,$$

and the proof is complete.

d. The *FFT algorithm*, which achieves (3.9.6), is *defined* by means of (3.9.4).

Equation (3.9.4), or the more general (3.9.1), can be viewed in terms of replacing the DFT $N \times N$-matrix $\mathcal{D}_N$, having no 0-entries, in terms of a collection of matrices with many 0-entries, cf., *Example 3.9.3* and Carleson's remark at the end of *Section 3.2.8*.

In this spirit, C. Rader (1968) introduced an important idea, based on the fact that $\mathbb{Z}_p$ is a field, which ultimately led to Winograd's celebrated algorithms (1978) for computing the DFT on $\mathbb{Z}_p$, e.g., [RR72], [DV90].

3.9.3 Example. SPARSE FFT MATRICES

We shall quantify the remark in *Example 3.9.2d* about matrices with 0 entries.

The right side of (3.9.4) splits the domain of $f : \mathbb{Z}_N \to \mathbb{C}$, $N = 2N_1$, into its even and odd parts. This can be accomplished by a matrix operation. For example, if $N = 8$ and

$$\mathcal{C}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

then

(3.9.7) $\mathcal{C}_8 f^T = (f[0], f[2], f[4], f[6], f[1], f[3], f[5], f[7])^T.$

The rule for constructing $\mathcal{C}_N$ is obvious.

The next operation in (3.9.4) tells us that $\sqrt{N}\mathcal{D}_N$ is related to two copies of $\sqrt{N_1}\mathcal{D}_{N_1}$ embedded into an $N \times N$ array. These two copies address the even and odd parts of the domain of f, resp. Thus, we

introduce the $N \times N$ matrix

$$
\mathcal{B}_N = \left(\begin{array}{c|c}
\sqrt{N_1}\mathcal{D}_{N_1} & \\
\hline
 & \sqrt{N}\mathcal{D}_{N_1}
\end{array} \right)
$$

where the first and third quadrants are each $N_1 \times N_1$ 0-matrices. For example, if $N = 8$ then

$$
\mathcal{B}_8 \mathcal{C}_8 f^T = \left(s_e(0), s_e(1), s_e(2), s_e(3), s_o(0), s_o(1), s_o(2), s_o(3) \right)^T,
$$

where the "even" and "odd" sums s_e and s_0 are the DFTs defined in *Example 3.9.2b* for the case $N_1 = 4$.

Finally, we have to introduce a matrix which incorporates the factor W_N^n of $s_o(n_1)$ (when $n = n_2 N_1 + n_1$) and adds it to $s_e(n_1)$. Recall from *Example 3.9.2b* that $W_N^n = W_N^{n_1}$ if $n_2 = 0$ and $W_N^n = -W_N^{n_1}$ if $n_2 = 1$. Therefore, if $N = 8$ we set

$$
\mathcal{A}_8 = \left(\begin{array}{cccc|cccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & W_8^1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & W_8^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & W_8^3 \\
\hline
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -W_8^1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -W_8^2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -W_8^3
\end{array} \right)
$$

and obtain

$$
\mathcal{A}_8 \mathcal{B}_8 \mathcal{C}_8 f^T = F^T.
$$

The rule for constructing $\mathcal{A}_N$ is obvious, and so

$$
\mathcal{A}_N \mathcal{B}_N \mathcal{C}_N = \sqrt{N} \mathcal{D}_N
$$

(as easy as ...).

3.9.4 Remark. BUTTERFLIES AND BIT REVERSAL

a. Let $N = 2^r$ and let $f : \mathbb{Z}_N \to \mathbb{C}$ have DFT F. Besides the matrix formulation of the FFT algorithm (3.9.4) in *Example 3.9.3*, we can also formulate it in terms of diagrams whose basic components are called *butterflies.*

To define a butterfly, we begin by *rewriting* (3.9.4) as

$$(3.9.8) \quad F[k] = F_0[k] + W_N^k F_1[k], \quad k = 0, 1, \ldots, (1/2)N - 1$$

and

$$(3.9.9)$$
$$F[k + (1/2)N] = F_0[k] - W_N^k F_1[k], \quad k = 0, 1, \ldots, (1/2)N - 1,$$

where

$$(3.9.10)$$
$$F_0[k] = \sum_{m=0}^{\frac{N}{2}-1} f[2m] W_{N/2}^{mk} \quad \text{and} \quad F_1[k] = \sum_{m=0}^{\frac{N}{2}-1} f[2m+1] W_{N/2}^{mk},$$

and where we have again used the fact that $W_N^{n_2(N/2)+k} = \pm W_N^k$ depending of whether $n_2 = 0, 1$.

The DFT $F[k], k = 0, 1, \ldots, N - 1$, written in terms of the calculations (3.9.8)–(3.9.10), can be visualized as

$$(3.9.11) \qquad \begin{array}{ccc} F_0[k] & \longrightarrow & F_0[k] + W_N^k F_1[k] \\ & \times & \\ F_1[k] & \longrightarrow & F_0[k] - W_N^k F_1[k], \end{array}$$

where $k = 0, 1, \ldots, (1/2)N - 1$; and this diagram is called a *butterfly.*

b. The butterfly (3.9.11) can be viewed as a construction of the DFT $F : \mathbb{Z}_N \to \mathbb{C}$ in terms of the two DFTs, F_0 and F_1, on $\mathbb{Z}_{N/2}$.

In the same way, F_0 and F_1 can each be constructed in terms of pairs of DFTs on $\mathbb{Z}_{N/4}$. For example,

$$F_0[k] = F_{00}[k] + W_{N/2}^k F_{01}[k]$$

and

$$F_0[k + (1/4)N] = F_{00}[k] - W_{N/2}^k F_{01}[k],$$

for $k = 0, 1, \ldots, (1/4)N - 1$, where F_{00} is the DFT of

$$\{f[0], f[4], f[8], \ldots, f[N - 4]\}$$

and F_{01} is the DFT of $\{f[2], f[6], f[10], \ldots, f[N - 2]\}$.

Since $N = 2^r$ this procedure can be reduced to the consideration of 2^{r-1} DFTs on $\mathbb{Z}_2$. Each of these is the DFT of a pair $\{f[j], f[k]\}$, and the pairs are mutually disjoint. Clearly, the stepwise evolution of F from these 2^{r-1} DFTs on $\mathbb{Z}_2$ can be pictured and understood in terms of butterflies, e.g., [OS75, Chapter 6], [Walk91]. Computationally, it is convenient to compute the 2-point DFTs first, then the 4-point DFTs, etc.

c. Suppose $f : \mathbb{Z}_N \to \mathbb{C}$ is given and the computation of its DFT F is desired in the natural ordering $(F[0], F[1], \ldots, F[N - 1])$, i.e., for a given f, a computational device will compute an ordered N-tuple $(F_0, \ldots, F_{N-1})$, and we want to be sure that $F_k = F[k]$ for each k.

From (3.9.10) it is clear that if we begin with the DFTs of the pairs $\{f[0], f[1]\}$, $\{f[2], f[3]\}$, etc., we shall *not* obtain F in its naturing ordering. It turns out that f *can* be ordered in such a way so that the DFTs of consecutive pairs $\{f[j], f[k]\}$ in this ordering yield the desired natural ordering of F. The procedure is called *bit reversal*. For example, if $N = 8$ then the DFTs of $\{f[0], f[4]\}$, $\{f[2], f[6]\}$, $\{f[1], f[5]\}$, $\{f[3], f[7]\}$ will yield the ordered N-tuple $(F[0], F[1], \ldots, F[N - 1])$ when the halving procedure of part b is implemented.

Bit reversal is defined as follows for $N = 2^r$. At level $r = 1$ the bit reversal ordering of the set $\{0, 1\}$ is the ordered 2^1-tuple $(0, 1)$. At level $r = 2$ the bit reversal ordering of the set $\{0, 1, 2, 3\}$ is the ordered 2^2-tuple $(0, 2, 1, 3)$. Inductively, at level m suppose the set $\{0, 1, \ldots, 2^m - 1\}$ has as its bit reversal ordering the ordered $M = 2^m$-tuple,

$$(b_0, \ldots, b_{M-1}).$$

Then, by definition, at level $m + 1$, the *bit reversal* ordering of the set $\{0, 1, \ldots, 2M - 1\}$ is

$$(2b_0, 2b_1, \ldots, 2b_{M-1}, 2b_0 + 1, 2b_1 + 1, \ldots, 2b_{M-1} + 1).$$

For example the bit reversal orderings at levels 3 and 4 are

$$(0, 4, 2, 6, 1, 5, 3, 7)$$

and

$$(0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15),$$

respectively. The term "bit reversal" is appropriate since the coefficients of the binary expansions (part *d*) of integers are reversed at the critical step in the above process. This is the reason for the subscripts 0 and 1 in part *b*.

   d. The *binary expansion* of $n \in \{0, 1, \ldots, 2^r\}$ is

$$n = \sum_{j=1}^{r} \epsilon_j 2^{j-1},$$

where $\epsilon_j \in \{0, 1\}$. For example, if $\epsilon_0 = \epsilon_1 = \ldots = \epsilon_r = 0$ then $n = 0$, and if $\epsilon_0 = \epsilon_1 = \ldots = \epsilon_{r-1} = 1$ then $n = \sum_{j=1}^{r} 2^{j-1} = (2^r - 1)/(2 - 1) = 2^r - 1$. Thus, each such n is well-defined by an r-array $(\epsilon_1, \ldots, \epsilon_r)$ of 0s and 1s.

   e. Suppose $\{X_n^r\}$ is a "tree of spaces" where r designates the level, and where, for each fixed $r \geq 0$ there are $N = 2^r$ elements X_n^r, indexed by n. Using the binary expansion of part d we write

$$X_n^r = X_{(\epsilon_1, \ldots, \epsilon_r)}^r;$$

and, using the bit reversal ordering, the tree $\{X_n^r\}$ has the form

$$X_0^0$$

$$X_0^1 \qquad\qquad X_1^1$$

$$X_{(0,0)}^2 \quad X_{(0,2)}^2 \qquad\qquad X_{(1,0)}^2 \quad X_{(1,1)}^2$$

etc. At level $r - 1$ the space $X_{(\epsilon_1, \ldots, \epsilon_{m-1}, 0)}^{r-1}$ is the (single) parent of

$$X_{(\epsilon_1, \ldots, \epsilon_{m-1}, 0)}^r \quad \text{and} \quad X_{(\epsilon_1, \ldots, \epsilon_{m-1}, 1)}^r.$$

Tree models, including some where bit reversal ordering is essential, abound in mathematics and engineering. There are applications in image compression, channel crosstalk reduction, Walsh functions and the waveletpackets of Coifman, Meyer, and Wickerhauser, subband coding and the theory of nonlinear waveletpackets, frequency localization, multirate systems and bit allocation, C. Fefferman's proof of Carleson's Theorem proving the Lusin Conjecture, etc., e.g., [BA83], [BF94, Chapter 10], [BSa94], [Dau92], [Fef73], [Mey90].

f. With regard to the algorithmic butterflies of part *a*, recall the locally lacunary butterfly from *Example 1.4.4* in *Chapter 1*. Even *Chapter 2* has a lepidopteral connection: Laurent Schwartz is not only a world class mathematician but has seven butterflies named after him!

The extension of Carleson's Theorem (*Section 3.2.8*) to 2-dimensions is true for partial sums taken over squares $\{(m, n) : -N \leq m, n \leq N\}$. C. Fefferman's proof uses butterfly-shaped subsets of $\mathbb{Z} \times \mathbb{Z}$ in a fundamental way [Fef71], cf., the proofs by Sjölin and Tevzadze at about the same time.

3.9.5 Example. COMPUTATION OF W_N, $N = 2^r$

Let $C_r = \cos(2\pi/2^r)$ and $S_r = \sin(2\pi/2^r)$. If $r = 1$ then $C_1 = -1$ and $S_1 = 0$. Using the half-angle formulas,

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}} \quad \text{and} \quad \sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$

we can compute C_r and S_r for large values of r by the recurrence equations,

$$(3.9.12) \qquad C_{r+1} = \sqrt{\frac{1 + C_r}{2}} \quad \text{and} \quad S_{r+1} = \sqrt{\frac{1 - C_r}{2}}.$$

The problem with (3.9.12) from a computational point of view is the possibility that $\lim_{r\to\infty}(1 - C_r) = 0$ might lead to computational instability. On the other hand, since $\sin^2 x + \cos^2 x = 1$, the double angle formulas can be written as

$$\cos x = \sqrt{\frac{1 + \cos 2x}{2}} \quad \text{and} \quad \sin x = \frac{\sin 2x}{2 \cos x} = \frac{\sin 2x}{2\sqrt{\frac{1 + \cos 2x}{2}}};$$

d. $F(\gamma) = \gamma, \quad \gamma \in [-\Omega, \Omega)$.

e. $F(\gamma) = \gamma, \quad \gamma \in [0, 2\Omega)$.

f. $F(\gamma) = \gamma^2, \quad \gamma \in [-\Omega, \Omega)$.

g. $F(\gamma) = \gamma^2, \quad \gamma \in [-2\Omega, 0)$.

h. $F(\gamma) = \begin{cases} c \neq 0, & \gamma \in (0, \Omega), \\ 0, & \gamma \in [\Omega, 2\Omega]. \end{cases}$

i. $F(\gamma) = \gamma \cos \left(\dfrac{\pi \gamma}{\Omega} \right), \quad \gamma \in [-\Omega, \Omega)$.

j. $F(\gamma) = \gamma \sin \left(\dfrac{\pi \gamma}{\Omega} \right), \quad \gamma \in [-\Omega, \Omega)$.

k. $F(\gamma) = \left| \sin \left(\dfrac{\pi \gamma}{\Omega} \right) \right|, \quad \gamma \in [0, 2\Omega)$.

l. $F(\gamma) = \begin{cases} \cos \left(\frac{\pi \gamma}{\Omega} \right), & \gamma \in (0, \Omega), \\ 0, & \gamma \in [\Omega, 2\Omega]. \end{cases}$

m. $F(\gamma) = \begin{cases} \sin \left(\frac{\pi \gamma}{\Omega} \right), & \gamma \in (0, \Omega), \\ 0, & \gamma \in [\Omega, 2\Omega]. \end{cases}$

(handwritten margin note: spacing of "0")

3.2. Using MATLAB, graph the *Nth* partial sums, for $N = 1, 2, 4, 8$, of the Fourier series of the functions defined in parts *a, b, c, f, j, k* of *Exercise 3.1*.

3.3. Designate F in *Exercise 3.1a* by F_a, and similarly for parts $b, c, \cdots, m$.

a. Show that, formally,

$$S(F_b)' = S(F_a) \quad \text{and} \quad S(F_f)' = 2S(F_d),$$

where $S(F)'$ is the term by term differentiation of $S(F)$. (This calculation is legitimate when we consider $S(F)'$ as the distributional derivative of the function F defined 2Ω-periodically on $\mathbb{R}$, cf., *Definition 3.1.9*.)

b. Evaluate the Fourier coefficients of $F_m - \frac{1}{2} F_k$.

3.4. Using (3.1.7), prove that the function G defined in part b of the proof of *Theorem 3.1.6* is bounded in some interval centered at the origin.

3.5. Compute the Fourier series of

$$F(\gamma) = \frac{\pi - \gamma}{2}, \ \gamma \in [-\pi, \pi),$$

defined 2π-periodically on $\hat{\mathbb{R}}$, cf., *Example 3.3.6a* and *Exercise 3.29*.

3.6. Prove that the inclusion, $L^2(\mathbb{T}_{2\Omega}) \subseteq L^1(\mathbb{T}_{2\Omega})$, is proper.

3.7. a. Compute the Fourier series of the following functions defined 2π-periodically.

   i. $F(\gamma) = \sin \gamma, \ \gamma \in [0, 2\pi)$.
   ii. $F(\gamma) = \sin \gamma, \ \gamma \in [0, \pi)$, and extended evenly to $(-\pi, \pi)$.

   b. Compute the Fourier series of $F(\gamma) = \sin \gamma, \ \gamma \in [0, \pi)$, considered as a π-periodic function. Compare this result with part a.

3.8. Compute

$$\frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \left| \sum_{n=-100}^{100} e^{\pi i n \gamma / \Omega} \right|^2 d\gamma.$$

3.9. Prove (3.4.17), which states the *continuity of the inner product*.

3.10. Prove the divergence of the series

$$\sum_{|n| \geq 2} \frac{1}{|n| \log |n|}.$$

This fact played a role in *Example 3.3.4*.

3.11. a. Prove that

$$\sum_{|n| \leq N} e^{inx} = \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$$

for $x \notin 2\pi\mathbb{Z}$. For $x = -\pi\gamma/\Omega$, this establishes (3.4.5).

b. Prove that

$$\sum_{|n| \le N} \left(1 - \frac{|n|}{N+1}\right) e^{inx} = \frac{1}{N+1} \left(\frac{\sin(N+1)\frac{x}{2}}{\sin\frac{x}{2}}\right)^2$$

for $x \notin 2\pi\mathbb{Z}$, and complete the proof of (3.4.7).

3.12.   a. Let $P(\gamma) = \sum_{|n| \le N} c_n e^{-\pi in\gamma/\Omega}$ be a trigonometric polynomial and let $F \in L^1(\mathbb{T}_{2\Omega})$. Compute $F * P$. Is $F * P$ a trigonometric polynomial?

      b. Let $F \in L^1(\mathbb{T}_{2\Omega})$ and $G \in L^\infty(\mathbb{T}_{2\Omega})$. Prove that

$$\|F * G\|_{L^\infty(\mathbb{T}_{2\Omega})} \le \|F\|_{L^1(\mathbb{T}_{2\Omega})} \|G\|_{L^\infty(\mathbb{T}_{2\Omega})}.$$

      c. Let $F \in L^2(\mathbb{T}_{2\Omega})$ and $G \in L^1(\mathbb{T}_{2\Omega})$. Prove that

$$\|F * G\|_{L^2(\mathbb{T}_{2\Omega})} \le \|F\|_{L^2(\mathbb{T}_{2\Omega})} \|G\|_{L^1(\mathbb{T}_{2\Omega})}.$$

3.13.   a. Abel's *partial summation formula* is

$$(E3.1) \qquad \sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p,$$

where $\{a_n, b_n : n = 0, 1, \cdots\} \subseteq \mathbb{C}$, $0 \le p \le q$, $A_{-1} \equiv 0$, and $A_n \equiv \sum_{j=0}^{n} a_j$. Prove (E3.1).

      b. Prove that if $p, q \in \mathbb{Z}$, $p < q$, and $x \ne 2\pi k$ for any $k \in \mathbb{Z}$, then

$$\left|\sum_{n=p}^{q} e^{inx}\right| \le \frac{1}{|\sin\frac{x}{2}|};$$

and so

$$\forall \gamma \ne 2\Omega k, \quad \left|\sum_{n=p}^{q} e^{-\pi in\gamma/\Omega}\right| \le \frac{1}{|\sin\frac{\pi\gamma}{2\Omega}|}.$$

      c. Let $f[p] \ge f[p+1] \ge \ldots \ge f[q] \ge 0$. Prove that

$$\forall \gamma \ne 2\Omega k, \quad \left|\sum_{n=p}^{q} f[n] e^{-i\pi n\gamma/\Omega}\right| \le \frac{f[p]}{|\sin\frac{\pi\gamma}{2\Omega}|}.$$

[Hint. Use the partial summation formula (E3.1).]

3.14. Let $\Omega > 0$ and let $\alpha \in (0, \Omega)$. Let $F = 1$ on $\mathbb{T}_{2\Omega}$ and define $F_{(\alpha)}$ on $\mathbb{T}_{2\Omega}$ by setting $F_{(\alpha)} = 1_{[-\alpha, \alpha)}$ on $[-\Omega, \Omega)$.

   a. Compute the Fourier coefficients of F and of $G = F - cF_{(\alpha)}$ for $c \in \mathbb{C}$.

   b. Does there exist $c \neq 0$ such that for all $\alpha \in (0, \Omega)$,

   $$\|F\|_{L^2(\mathbb{T}_{2\Omega})} = \|G\|_{L^2(\mathbb{T}_{2\Omega})}?$$

   c. Let $k \in \mathbb{N}$ and let $\alpha = \Omega/k$. For which values of n can we assert that $\widehat{G}[n] = \widehat{F}[n] = 0$.

      The point of part a is that even a small frequency perturbation of a signal causes almost all of the "temporal" data on $\mathbb{Z}$ to change. This is precisely the uncertainty principle phenomenon discussed in *Remark 1.1.4*. This lack of time *and* frequency localization in Fourier analysis can be circumvented to some extent in wavelet theory, e.g., Daubechies' book [Dau92].

3.15. Prove *Theorem 3.3.7*. The right side of (3.3.9) in *Theorem 3.3.7* is defined for $G \in L^\infty(\mathbb{T}_{2\Omega})$. Why does *Theorem 3.3.7* generally fail in this case?

3.16. Prove *Theorem 3.4.4b*. [Hint. Let $\epsilon > 0$ and let $G \in C(\mathbb{T}_{2\Omega})$ have the property that $\|F - G\|_{L^1(\mathbb{T}_{2\Omega})} < \epsilon/(2C)$, where $\|K_{(\lambda)}\|_{L^1(\mathbb{T}_{2\Omega})} \leq C$, $C \geq 1$. Then, by (3.4.1), you can show that

$$\|F - F * K_{(\lambda)}\|_{L^1(\mathbb{T}_{2\Omega})} < \frac{\epsilon}{2} + \|G - G * K_{(\lambda)}\|_{L^\infty(\mathbb{T}_{2\Omega})}$$
$$+ C\|G - F\|_{L^1(\mathbb{T}_{2\Omega})}.$$

The result follows from *Theorem 3.4.4a* and by taking a $\overline{\lim}$.]

3.17. In this excercise we shall approximate Fourier transforms on $\mathbb{R}$ by DFTs (*Definition 3.8.1*). Consider intervals $[a, b] \subseteq \mathbb{R}$ and $[\alpha, \beta] \subseteq \widehat{\mathbb{R}}$, and denote their lengths by $T = b - a$ and $2\Omega = \beta - \alpha$. Suppose $2T\Omega \equiv N$ is a positive integer, and define

$$t_m = a + m\Delta t \quad \text{and} \quad \gamma_n = \alpha + n\Delta\gamma,$$

where $\Delta t \equiv 1/(2\Omega)$, $\Delta\gamma \equiv 1/T$, and $m, n = 0, 1, \dots, N - 1$. Let f be a continuous function on $[a, b]$, set $g[m] \equiv f(t_m)e^{-2\pi i m\alpha/(2\Omega)}$, and let G be the DFT of g. Show that

$$\forall n = 0, 1, \dots, N - 1, \quad \int_a^b f(t)e^{-2\pi i t\gamma_n} \, dt \approx \frac{1}{2\Omega}e^{-2\pi i a\gamma_n} G[n].$$

[Hint. Use a Riemann sum approximation of the left side.]

We can effectively use this exercise to compute Fourier transforms, e.g., *Exercises 3.18* and *3.19*. The symbol "$\approx$" means that the right side approximates the left side. This imprecise but meaningful statement can be quantified in terms of the Poisson Summation Formula (*Theorem 3.10.8*), e.g., [AG89], [BSS88], [BrHe95].

3.18. We computed the Fourier transform of the Gaussian $g(t) = \frac{1}{\sqrt{\pi}}e^{-t^2}$ in *Example 1.3.3*.

    a. Using the MATLAB fft function and the approach in *Exercise 3.17*, verify numerically that $\hat{g}(\gamma) = e^{-(\pi\gamma)^2}$. [Hint. To begin, let $[a, b] = [-32, 32]$, $[\alpha, \beta] = [-32, 32]$, $\Delta t = 1/64$, and $\Delta\gamma = 1/64$, and consider the vector

$$t = -32 : (1/64) : 32 - (1/64).]$$

    b. With regard to PSF (*Theorem 3.10.8*), and in MATLAB terminology, compare *sum* $(f. * f)$ and *sum* $(fhat. * fhat)$, where $f \equiv \exp(-t.*t)$ and $fhat \equiv sqrt(pi)*\exp(-(pi*t).\char94 2)$.

3.19.   a. Using the MATLAB fft function and the approach in *Exercise 3.17*, graph the following Fourier transform pairs:

$$d_\lambda \quad \longleftrightarrow \quad \mathbf{1}_{[-\frac{\lambda}{2\pi}, \frac{\lambda}{2\pi}]},$$

$$w_\lambda \quad \longleftrightarrow \quad \max(1 - \frac{|2\pi\gamma|}{\lambda}, 0),$$

$$p_\lambda \quad \longleftrightarrow \quad e^{-2\pi|\gamma|/\lambda},$$

$$g_\lambda \quad \longleftrightarrow \quad e^{-(\pi\gamma/\lambda)^2},$$

where $\{d_\lambda\}$, $\{w_\lambda\}$, $\{p_\lambda\}$, and $\{g_\lambda\}$ are the Dirichlet, Fejér, Poisson, and Gauss kernels, respectively.

   b. Graph the de la Vallée-Poussin kernel $\{v_\lambda\}$ which was defined in *Exercise 1.43*, cf., *Exercise 1.50* and *Example 3.5.3b*.

   c. For each of the pairs in part *a*, as well as the de la Vallée-Poussin kernel, compare the behavior of the function with its transform as $\lambda > 0$ increases. Compare the manner and speed with which each transform converges to 1 as $\lambda > 0$ increases.

3.20. The following is a recursive MATLAB program (m-file) which calculates the two step FFT algorithm of *Theorem 3.9.1*:

```
function y = myfft(x)
n = length(x);
if  n == 1
      y = 1;
else
      xo = x(2 : 2 : n); xe = x(1 : 2 : n − 1);
      w = exp((−2*pi*i/n)*(0 : n − 1));
      xohat = myfft(x0);
      y = [xehat xehat] + w.*[xohat xohat];
end
```

   a. Make an ascii file called *myfft.m* in some directory, say *c:\ mymfiles*. Give the MATLAB command

$$path(`c : \backslash mymfiles', path);$$

   b. Make a vector f of random numbers, e.g., set $f \equiv \mathrm{rand}(1, 256)$. Compute the DFT of f by means of *myfft* as well as by the MATLAB *fft* function. Compare the results.

3.21. Let $f, g \in \ell^1(\mathbb{Z})$ have "compact support", i.e., there are integers A, B, C, D for which $f[n] = 0$ exceptfor $A \leq n \leq B$ and $g[n] = 0$ except for $C \leq n \leq D$. Recall the definition of $f * g$ in *Definition 3.5.1*. Part *a* is a *fast convolution algorithm*.

"rand" italicized?
_____

see "sum" in
Exercise 3.18b

a. Let $N \geq B-A+D-C+1$, and let F and G be the DFTs of f and g on $\mathbb{Z}_N$, respectively. Prove that if $A+C \leq n \leq B+D$ then

$$f * g[n] = \mathcal{F}_N^{-1}(FG)[n - A - C],$$

and that $f * g[n] = 0$ elsewhere.

b. Prove that a direct implementation of the definition of convolution requires $(B - A + 1)(D - C + 1)$ multiplications.

c. Using the FFT algorithm of *Section 3.9*, show that the method of part *a* requires $3N \log_2 N + N$ multiplications.

3.22. The *z-transform* F of $f : \mathbb{Z} \longrightarrow \mathbb{C}$ is the function $F(f) : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$F(f)(z) = \sum f[n]z^{-n}.$$

Of course, $F(f)$ may not exist for some $z \in \mathbb{C}$. Note that there is an intrinsic relationship between Fourier series and z-transforms; in fact, we have $\widehat{f}(\gamma) = F(f)(e^{2\pi i \gamma})$, $\gamma \in \widehat{\mathbb{R}}$.

Let $a, \omega \in \mathbb{C}$ and define $z_k = a\omega^k$ for $C \leq k \leq D$. Consider the sequences f, g, and h, where $f[n] = 0$ except for $A \leq n \leq B$, $h[n] = 0$ except for $C - B \leq n \leq D - A$ where it is defined as $h[n] = \omega^{n^2/2}$, and $g[n] = 0$ except for $A \leq n \leq B$ where it is defined as $g[n] = f[n]a^{-n}\overline{h[n]}$. Prove that the z-transform F of f can be written as

$$F(f)(z_k) = \overline{h[k]}(g * h)[k].$$

This is the *chirp z-algorithm*.

3.23. a. Using MATLAB, implement the chirp z-algorithm of *Exercise 3.22* to approximate $\int e^{-t^2} e^{-2\pi i t\gamma} dt$, cf., *Example 2.10.5*. [Hint. Approximate the Fourier transform at the points $\gamma \equiv \gamma_n \equiv n\Delta\gamma$, where $\Delta\gamma \equiv .01$ and $-100 \leq n \leq 100$. Use the values $f[m] \equiv e^{-(t_m)^2}$ of the Gaussian, where $t_m \equiv m\Delta t$, $\Delta t \equiv .01$, and $-300 \leq m \leq 300$. Show that the desired integral is approximately

$$F(f)(z_k)\Delta t,$$

where $f[m] = 0$ except for $-300 \leq m \leq 300$, and where $z_k \equiv (e^{-2\pi i \Delta t \Delta \gamma})^k$. Finally, invoke the fast convolution algorithm (with 1024 point FFTs) from *Exercise 3.21* to perform the convolution used in the chirp z-algorithm of *Exercise 3.22*.]

b. Compare the number of multiplications required for the chirp z-transform in part *a* with the number of multiplications required in the computation of the same integral in *Exercise 3.18*. Note that the chirp z-transform calculation provides points γ_n on a finer mesh than the FFT approach of *Exercise 3.18*, and requires only 65% of the number of multiplications.

3.24. Evaluate

$$\sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{1}{n^2}.$$

[Hint. Consider *Exercise 3.16*.]

3.25.   a. Let $F \in L^1(\mathbb{T})$ and $G \in L^\infty(\mathbb{T})$, and let $f = \{f[n]\}$ and $g = \{g[n]\}$ be their sequences of Fourier coefficients. Prove that

$$\lim_{n \to \infty} \int_{\mathbb{T}} F(\gamma)G(n\gamma)\, d\gamma = f[0]g[0].$$

The Riemann-Lebesgue Lemma (*Theorem 3.1.5*) is the special case of this result for $G(\gamma) = e^{2\pi i \gamma}$.

b. Let $F \in L^1(\mathbb{T})$ be real valued. Using the notation from *Example 3.1.4* and the result from part *a*, prove that

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} \frac{b_{kn}}{n} = \lim_{k \to \infty} \sum_{n=1}^{\infty} (-1)^n \frac{a_{k(2n+1)}}{2n+1} = 0,$$

cf., [Lux62].

3.26.   a. Let $F \in AC(\mathbb{T}_{2\Omega})$, and let $f = \{f[n]\}$ be the sequence of Fourier coefficients of F. Prove that $F' \in L^1(\mathbb{T}_{2\Omega})$, $(F')^\vee[n] = -\frac{\pi i n}{\Omega} f[n]$ for each $n \in \mathbb{Z}$, and $\lim_{|n| \to \infty} nf[n] = 0$.

Thus, if $F \in C(\mathbb{T}_{2\Omega}) \cap BV(\mathbb{T}_{2\Omega})$ has Fourier coefficients $f[n]$, $n \in \mathbb{Z}$, and if

$$\varlimsup_{|n| \to \infty} |n f[n]| > 0$$

then $F \notin AC(\mathbb{T}_{2\Omega})$.

b. Prove (3.5.3). [Hint. For $F \in AC(\mathbb{T}_{2\Omega})$, let $G = F' \in L^1(\mathbb{T}_{2\Omega})$ so that $G^\vee[0] = 0$ and $F(\gamma) = \int_0^\gamma G(\lambda)\, d\lambda + F(0)$ on $[0, 2\Omega]$. Apply part a, Hölder's Inequality, and Parseval's formula to the right side of (3.5.2).]

scan for "F" or "f"
I think it is "f"
in "formula".

c. Let $G \in L^1(\mathbb{T}_{2\Omega})$ and suppose $G^\vee[0] = 0$. Prove

$$\int_0^{2\Omega} \int_0^\gamma G(\lambda)\, d\lambda d\gamma = -\int_0^{2\Omega} \gamma G(\gamma)\, d\gamma.$$

d. Let $F \in BV(\mathbb{T}_{2\Omega})$, and let $f = \{f[n]\}$ be the sequence of Fourier coefficients of F. Prove that

$$\exists M > 0 \quad \text{such that} \quad \forall n \in \mathbb{Z}, \quad |n f[n]| \le M.$$

M can be taken as the *variation* of F, i.e., $M = \inf\{\sum |F(\gamma_j) - (\gamma_{j-1})|\}$, where the infimum is taken over all finite partitions $\{\gamma_j\}$ of $[0, 2\Omega]$, cf., the definition of bounded variation in *Definition 1.1.5*.

Part d can be proved by a calculation for step functions and an approximation, but we also recommend an ingenious calculation due to M. Taibleson (*Fourier coefficients of functions of bounded variation*, Proc. Amer. Math. Soc. 18(1967), 766), e.g., [Ben76, page 120].

3.27. Let $S(\gamma) = 1/(1 + \sin 2\pi\gamma)$. Then $1/S \ge 0$ on $\mathbb{T}$ has a zero at $\gamma = -\frac{1}{4} + n$. Prove that $S \notin L^1(\mathbb{T})$.

3.28. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \ge 1$ with complex coefficients. By the Fundamental Theorem of Algebra there is $\alpha \in \mathbb{C}$ such that $P(\alpha) = 0$. The following facts are well-known.

i. $P(x)$ has the unique representation

$$P(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n),$$

where $\{\alpha_1,\ldots,\alpha_n\} \subseteq \mathbb{C}$ is the set of zeros of P;

ii. the zeros and coefficients of P are related by the formulas,

$$\begin{aligned}
a_{n-1} &= -\sum_{j=1}^{n} \alpha_j \\
a_{n-2} &= \sum_{i<j} \alpha_i \alpha_j \\
a_{n-3} &= -\sum_{i<j<k} \alpha_i \alpha_j \alpha_k \\
&\vdots \\
a_0 &= \pm\alpha_1 \alpha_2 \ldots \alpha_n.
\end{aligned}$$

Fill in the details of the following brilliant persuasive rationale by Euler to establish that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. The zeros of

$$P(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

are $\{n\pi : n \in \mathbb{Z}\backslash\{0\}\}$. Arguing by analogy, Euler obtains

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n\pi)^2}\right),$$

cf., *Exercise 1.36*; and, hence,

$$\frac{1}{3!} = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2},$$

by parts *i* and *ii*.

3.29.   a. Prove that

$$\lim_{N\to\infty} \sum_{0<|n|\le N} \frac{e^{-\pi i n\gamma/\Omega}}{n}.$$

exists for each $\gamma \in (0, 2\Omega)$. [Hint. See *Example 3.3.6a*.]

b. Let $f[n] \geq f[n + 1]$, for $n \geq 1$, and let $\lim_{n \to \infty} f[n] = 0$. Prove that

$$\sum_{n=1}^{\infty} f[k] e^{-\pi i n \gamma / \Omega}$$

exists for each $\gamma \in (0, 2\Omega)$, and that the convergence is uniform on any closed subset of $(0, 2\Omega)$.

c. Let $f[n] \geq f[n+1]$, for $n \geq 1$, and assume $nf[n] \leq C$. Prove that

$$\sup_N |\sum_{n=1}^{N} f[n] \sin \frac{\pi n \gamma}{\Omega}| < \infty,$$

and that the series converges to a continuous function on $(0, 2\Omega)$, cf., *Exercise 3.43*.

The relation between this exercise and pseudo-measures supported by arithmetic progressions is found in [Ben71, Section 4.3].

3.30.   a. Prove Abel's original "Abelian theorem": *if $\sum_{n=1}^{\infty} a_n x^n$ converges on $[0, 1)$ and $\sum_{n=1}^{\infty} a_n = S$, then*

$$(E3.2) \qquad \lim_{x \to 1-} \sum_{n=1}^{\infty} a_n x^n = S.$$

[Hint. Use (E3.1).]

The first Tauberian Theorem, by Tauber in 1897, came as a response to the problem of finding some sort of converse to Abel's result. Tauber proved that *if $f(x) \equiv \sum_{n=1}^{\infty} a_n x^n$ converges on $[0, 1), f(x-) \equiv S$, and $\lim_{n \to \infty} na_n = 0$, then $\sum_{n=1}^{\infty} a_n = S$.* The boundedness condition "$\lim_{n \to \infty} na_n = 0$" is the "Tauberian condition" required to effect the converse. This is a special case of the Tauberian condition "$\varphi \in L^{\infty}(\mathbb{R})$" used in Wiener's Tauberian Theorem (*Theorem 2.9.12*), e.g., [Ben75, Section 2.3].

3.31. Prove that

$$(E3.3) \qquad \forall \gamma \in (0, 2\pi), \quad -\log(2 \sin \frac{\gamma}{2}) = \sum_{n=1}^{\infty} \frac{\cos n \gamma}{n}.$$

[Hint. By a power series expansion we have

$$\operatorname{Re}\log\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty}\frac{r^n\cos n\gamma}{n},$$

where $z = re^{i\gamma}$ and $r \in [0,1)$. Compute

$$\lim_{r\to 1-}\sum_{n=1}^{\infty}\frac{r^n\cos n\gamma}{n} = -\log(2\sin\frac{\gamma}{2})$$

on $(0, 2\pi)$. Since the right side of (E3.3) exists on $(0, 2\pi)$ by *Exercise 3.29b*, we can apply *Exercise 3.30* to obtain (E3.3).]

3.32. Let $\Delta(ABC)$ be a triangle with vertices A, B, C; and let a and c be points on the segments BC and BA, respectively, so that

$$< (cCB) =< (cCA) \quad \text{and} \quad < (aAB) =< (aAC).$$

Is this a good "angle" symbol?

Assume $|Aa| = |Cc|$. Prove that $\Delta(ABC)$ is an isosceles triangle. [Hint. Use the law of sines.] This is a difficult exercise.

3.33. Prove that $\sum_{n=1}^{\infty}\cos(2\pi n\gamma)$ diverges for all $\gamma \in [0,1)$, and that $\sum_{n=1}^{\infty}\sin(2\pi n\gamma)$ diverges for all $\gamma \in (0,1)$.

3.34. Analogous to the definition of the Hilbert transform in *Definition 2.5.11*, we define the Hilbert transform of the sequence $f = \{f[n]\}$ as the sequence $\mathcal{H}f$, where

$$(\mathcal{H}f)[n] = \frac{1}{\pi}\sum_{\substack{m\in\mathbb{Z}\\m\neq n}}\frac{f[m]}{n-m}.$$

Prove that if $f \in \ell^2(\mathbb{Z})\backslash\{0\}$ is real valued then

$$\|\mathcal{H}f\|_{\ell^2(\mathbb{Z})} < \|f\|_{\ell^2(\mathbb{Z})}.$$

This is the analogue on $\mathbb{Z}$ of *Theorem 2.5.12a*.

[Hint. First show that if m, n are fixed and unequal and if $j \neq m, n$, then

$$\sum_{j}\frac{1}{(j-n)(j-m)} = \frac{2}{(m-n)^2}.$$

Use this fact to compute
(*E*3.4)

$$\|\mathcal{H}f\|^2_{\ell^2(\mathbb{Z})} = \sum_n f[n]^2 \sum_{j \neq n} \frac{1}{(j-n)^2} + \sum_n \sum_{m \neq n} f[n]f[m]\frac{2}{(m-n)^2}$$

for f vanishing off some finite set. Since $2f[n]f[m] \leq f[n]^2 + f[m]^2$, we can invoke (3.3.8) to bound the right side of (E3.4) by $\pi^2 \sum f[n]^2$.] This clever, elementary proof is due to [Graf94], cf., [HLP52, pages 206ff.,212,226,235] for other proofs, [OS75, Chapter 7] for signal processing applications, and [Ivi85, pages 129 ff.] for applications in analytic number theory.

3.35. Let $\{z_1, \ldots, z_n\} \subseteq \mathbb{C}$. Prove there is $S \subseteq \{1, \ldots, n\}$ such that

(*E*3.5) $$\sum_{j=1}^n |z_j| \leq 4\sqrt{2}|\sum_{j \in S} z_j|.$$

[Hint. Begin by dividing $\mathbb{C}$ into four "diagonal" quadrants, e.g., [Ben76, page 217]. In particular, if $F(\gamma) = \sum f[n]e^{-\pi i t_n \gamma/\Omega}$, where $f = \{f[n]\}$ is a finite sequence and $\{t_n\} \subseteq \mathbb{R}$, then

$$\|F\|_{L^\infty(\widehat{\mathbb{R}})} \leq \|f\|_{\ell^1(\mathbb{Z})} \leq 4\sqrt{2} \inf_{\gamma \in \widehat{\mathbb{R}}} |\sum_{n \in S} f[n]e^{-\pi i t_n \gamma/\Omega}|,$$

for some finite sequence $S \subseteq \mathbb{Z}$, cf., *Definition 3.1.8*.

Inequalities such as (E3.5) are used in measure theory and to prove versions of Schur's Lemma, e.g., [Ben76, Section 6.2]. The constant $4\sqrt{2}$ can be replaced by π, which is best possible, e.g., [Ben76, page 172] for references to more advanced material.

3.36. Assume $\sum |c_n|^2 < \infty$ and

$$\lim_{N \to \infty} \sum_{|n| \leq N} c_n e^{-2\pi i n \gamma} = 0 \quad \text{a.e.}$$

Prove that $c_n = 0$ for all n, cf., the discussion in *Section 3.2.6*.

3.37. Let $0 \leq a < b < 1$. Prove that $[a, b]$ is not a U-set. In fact, if $E \subseteq \mathbb{T}$ is measurable and $|E| > 0$ then E is not a U-set, cf., the discussion in *Section 3.2.4*.

3.38.   a. Prove that

$$\left(\int_0^1 \frac{dt}{1+t^2} \right)^2$$

(E3.6)

$$= \int_0^{\pi/4} \frac{\log(1+\cos 2\theta)}{\cos 2\theta} \, d\theta = \frac{3}{4} \int_0^1 \frac{\log(1+t)}{t} \, dt,$$

e.g., [Ebe83].

b. We know the value of the left side of (E3.6). Expanding $t^{-1} \log(1+t)$ in a Maclaurin series, evaluate

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}.$$

c. Using part b, evaluate $\zeta(2)$ using the fact that

$$\zeta(2) - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{1}{2} \zeta(2).$$

3.39. Prove that π is irrational, cf., [Ben77] for history and perspective on this material. [Hint. Assume $\pi^2 = a/b$, where $a, b \in \mathbb{N}$. Define $f_n(x) = \frac{x^n(1-x)^n}{n!}$ on $[0,1]$ for each $n \geq 1$; and set

$$F_n(x) = b^n \{ \pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \pi^{2n-4} f_n^{(4)}(x) - \dots$$
$$+ (-1)^j \pi^{2n-2j} f_n^{(2j)}(x) \dots (-1)^n f_n^{(2n)}(x) \}.$$

Compute

$$\frac{d}{dx} (F_n'(x) \sin \pi x - \pi F_n(x) \cos \pi x) = \pi^2 a^n f_n(x) \sin \pi x,$$

and deduce that

$$\forall n \geq 1, \quad \pi \int_0^1 f_n(x) \sin \pi x \, dx \in \mathbb{Z}.$$

Obtain a contradiction for large n since $0 < f_n(x) < 1/n!$ and $\sin \pi x > 0$ on $(0,1)$.]

One can also show, more simply than for the case of π, that $e \notin \mathbb{Q}$. In fact, e and π are not only irrational, but are *transcendental*, i.e., they are not zeros of any polynomial P having rational coefficients. It is not known whether or not $\pi + e$ or π^e is irrational. On the other hand, e^π is transcendental.

3.40. A sequence $\{r_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$ is *equidistributed modulo* 1 or, equivalently, *uniformly distributed modulo* 1 if the sequence $\{(r_n)\} \subseteq [0, 1)$ of fractional parts of the r_n are uniformly distributed in the sense that

$$\lim_{N \to \infty} \frac{\nu(I, N)}{N} = |I|,$$

for every interval $I \subseteq (0, 1)$, where $\nu(I, N)$ is the number of elements from $\{(r_1), \dots, (r_N)\}$ contained in I. ((r) is defined as $r - [r]$, where $[r]$ is the largest integer less than or equal to r.)

The Weyl Equidistribution Theorem (*Remark 1.9.2*) asserts that $\{r_n\}$ is equidistributed modulo 1 if and only if

$$\forall n \in \mathbb{Z} \backslash \{0\}, \quad \lim_{N \to \infty} \frac{e^{2\pi i n r_1} + e^{2\pi i n r_2} + \cdots + e^{2\pi i n r_N}}{N} = 0,$$

e.g., [KK64], [Cha68, pages 84–90]. As a corollary, prove Kronecker's Theorem in one-dimension: *if* $r \in \mathbb{R} \backslash \mathbb{Q}$ *then* $\{(nr) : n \in \mathbb{N}\}$ *is dense* in $[0, 1)$. [Hint. Use *Exercise 3.13b*.]

3.41. The d-dimensional version of Kronecker's Theorem was stated in *Section 3.2.10*. Prove that it is equivalent to the following assertion. *Let* $\{\gamma_1, \dots, \gamma_d\} \subseteq \mathbb{R}$ *be linearly independent over the rationals, let* $\gamma_0 = 0$, *and let* $c_0, c_1, \dots, c_d \in \mathbb{C}$; *then*

$$\sup_{t \in \mathbb{R}} \left| \sum_{j=0}^{d} c_j e^{-2\pi i t \gamma_j} \right| = \sum_{j=0}^{d} |c_j|.$$

3.42. Let $F \in L^1(\mathbb{T}_{2\Omega})$. Prove the following results.

a. If $\lim_{\lambda \to 0} \frac{1}{|\lambda|^2} \|\tau_\lambda F - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0$ then $F = 0$ a.e.

b. If $\lim_{\lambda \to 0} \frac{1}{|\lambda|} \|\tau_\lambda F - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0$ then F is a constant a.e.

c. If there exist $c, C > 0$ such that

$$\forall |\lambda| \leq c, \quad \|\tau_\lambda F - F\|_{L^1(\mathbb{T}_{2\Omega})} \leq C|\lambda|$$

then $F \in BV(\mathbb{T}_{2\Omega})$.

d. $\lim_{\lambda \to 0} \|\tau_\lambda F - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0$.

Part *a* is a consequence of part *b*; part *b* follows from a refinement of FTC, e.g., [Ben76, pages 141-142] and *Remark 1.7.9* on the Lebesgue set; part *c* is due to Hardy and Littlewood (1928), e.g., [Ben76, pages 124-126]; and part *d* is an elementary and fundamental fact from *Appendix A*. The converse of part *c* results from a straightforward calculation using the classical form of the Jordan Decomposition Theorem (*Remark 1.7.4*).

3.43.　a. Let $\Omega > 0$ and let $f \in c_0(\mathbb{Z})$ have the property that $\{nf[n] : n \in \mathbb{N}\}$ decreases monotonically to 0 after a certain point. Prove that $\sum_{n=1}^{\infty} f[n] \sin \frac{\pi n \gamma}{\Omega}$ converges uniformly on $\widehat{\mathbb{R}}$, cf., *Example 3.3.4*, *Exercise 3.29*, and [Zyg59, Volume I, pages 182–183].

　　b. Let $f[n] = 1/(n \log n)$, $n \geq 3$. By part *a*,

$$F(\gamma) = \sum_{n=3}^{\infty} \frac{1}{n \log n} \sin \left(\frac{\pi n \gamma}{\Omega} \right)$$

converges uniformly on $\widehat{\mathbb{R}}$. Prove that $F \in C(\mathbb{T}_{2\Omega}) \backslash A(\mathbb{T}_{2\Omega})$, cf., *Example 1.4.4* and *Example 3.3.4a*.

3.44. Verify the inclusions and inequalities (3.1.8)–(3.1.11). Prove that the inclusions are proper, e.g., *Exercise 3.43*.

3.45. Use the Uniform Boundedness Principle (*Theorem B.8*), in a manner similar to *Example 3.4.9c*, to prove that there are continuous functions whose Fourier series diverge at a point, cf., the discussion in *Section 3.2.8*. This proof, due to Lebesgue (1905), is short

and nonconstructive. [Hint. Define the linear functionals

$$L_N : C(\mathbb{T}) \longrightarrow \mathbb{C}$$
$$F \longrightarrow S_N(F)(0).$$

Let $F_N \in C(\mathbb{T})$ equal sgn D_N except on small intervals about the discontinuities of sgn D_N; further, construct F_N so that $|F_N| \leq 1$. Then

$$\|L_N\| \geq |L_N(F_N)| = |\int_{\mathbb{T}} D_N(\gamma)F_N(\gamma)\, d\gamma|,$$

and the right side is close to $\|D_N\|_{L^1(\mathbb{T})}$. Thus, $\{\|L_N\|\}$ is unbounded, and the result is obtained analogous to *Example 3.4.9c*.]

Fejér's *construction* in 1911 of continuous functions whose Fourier series diverge at a point makes *implicit* use of the Uniform Boundedness Principle. His proof provides divergence at much larger sets of measure 0 than single points, e.g., [Rog59, pages 75-77], [Zyg59, Volume I, Chapter VIII.1], cf., the theorem of Kahane and Katznelson quoted in *Section 3.2.8*. On the other hand, Fejér's type of example has unbounded partial sums. Using Riesz products, Zygmund (1948) constructed $F \in L^\infty(\mathbb{T})$ for which $\{S_N(F)\}$ is *uniformly bounded* and $S(F)$ diverges on uncountable dense sets of measure 0, e.g., [Zyg59, Volume I, page 302].

3.46. If $F \in L^1(\mathbb{T})$, define

$$M(F)(\gamma) = \sup_N \{|S_N(F)(\gamma)|\}.$$

The main part of Carleson's proof of the Lusin Conjecture (*Section 3.2.8*) is his theorem that

(E3.7) $\exists C$ such that $\forall F \in L^2(\mathbb{T}), \quad \|M(F)\|_{L^2(\mathbb{T})} \leq C\|F\|_{L^2(\mathbb{T})}.$

Using (E3.7) prove the Lusin Conjecture:

$$\forall F \in L^2(\mathbb{T}), \quad \lim_{N \to \infty} S_N(F) = F \quad \text{a.e.}$$

With regard to (E3.7), compare Zygmund's result quoted in *Exercise 3.45*.

3.47. Let $F \in L^1(\mathbb{T})$ and $G \in L^\infty(\mathbb{T})$ have Fourier coefficients $f = \{f[n]\} \in A(\mathbb{Z})$ and $g = \{g[n]\} \in A'(\mathbb{Z})$, respectively. Compute $(FG)^\vee$ and $(F * G)^\vee$ in terms of f and g.

3.48. a. Let $F \in A(\mathbb{T})$ be nonnegative. With regard to *Remark 3.6.5d* on the Fejér-Riesz Theorem, verify whether or not there is a sequence $\{B_n\}$ of trigonometric polynomials on $\mathbb{T}$ for which

$$\lim_{n \to \infty} \|F - |B_N|^2\|_{A(\mathbb{T})} = 0,$$

cf., the Szegö Factorization Theorem stated in *Section 3.7*.

b. Let $F \in C(\mathbb{T})$, and suppose the sequence $f = \{f[n]\}$ of Fourier coefficients of F is nonnegative, i.e., $f[n] \geq 0$ for each n. Prove that $F \in A(\mathbb{T})$.

[Hint. f is a positive distribution of $\mathbb{Z}$ so that F is continuous and positive definite on $\mathbb{T}$.]

There are also several classical ways of proving part *b*.

3.49. A fundamental problem in harmonic analysis is to quantify properties such as support, smoothness, convergence, and decay of f and its approximants in terms of the behavior of $\hat{f}$; and many of our results can be put in this context, e.g., the Riemann-Lebesgue Lemma, *Exercise 1.13*, *Exercise 3.26*, and *Example 3.5.3* where we proved that $C^1(\mathbb{T}) \subseteq A(\mathbb{T})$. In this regard, prove that if $1 \leq m < \infty$ then

$$\forall F \in C^m(\mathbb{T}), \quad \lim_{N \to \infty} \|F - S_N(F)\|_{L^\infty(\mathbb{T})} = 0.$$

Estimate $\|F - S_N(F)\|_{L^\infty(\mathbb{T})}$ in terms of N and m.

3.50. Prove the assertions in *Definition 3.1.9d*.

3.51. a. Let $L \in \mathcal{L}(L^2(\mathbb{T}))$, i.e., $L : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ is a continuous linear function (*Definition B.6*). If $F \in L^2(\mathbb{T})$, prove that

$$L(F)(\gamma) = \sum (L_d F^\vee)[n] e^{-2\pi i n \gamma}$$

where

$$(L_d F^\vee)[n] = \sum_m \ell_{mn} F^\vee[m]$$

and

$$\ell_{mn} = \langle L e_{-m}, e_{-n} \rangle.$$

Thus, the action (operation) of L on $L^2(\mathbb{T})$ is *equivalent* to the action of the matrix $L_d \equiv \{\ell_{mn} : m, n \in \mathbb{Z}\}$ on $\ell^2(\mathbb{Z})$.

b. Now, further suppose that L is *translation invariant*, i.e., $L(\tau_\gamma F) = \tau_\gamma(LF)$ for all $\gamma \in \mathbb{T}$ and all $F \in L^2(\mathbb{T})$, *Exercise 2.58*. Prove that L_d is a diagonal matrix.

Parts *a* and *b* allow us to conclude that $\{e_{-m}\}$ *diagonalizes all continuous translation invariant linear functions on $L^2(\mathbb{T})$*; and it is also clear that *each e_{-m} is an eigenfunction of the derivative operator*. This fact accounts for the classical success of Fourier analysis in dealing wilth linear partial differential equations, since it provides exactly the advantage of doing matrix calculations in a basis that diagonalizes the matrix. In more recent times, parts *a* and *b* are the starting point for the harmonic analysis of singular integral operators, e.g., the Calderón- Zygmund Theory [Ste70], as well as LTI systems in engineering, e.g., *Definition 2.6.5*.

3.52. Let $\{r_j : j = 0, \pm 1, \dots, \pm N\} \subseteq \mathbb{C}$ satisfy the condition $r_j = \bar{r}_{-j}$ for each j, and define the $(N+1) \times (N+1)$ matrix $R = (r_{jk})$, where $j, k \geq 0$ and $r_{jk} \equiv r_{j-k}$. Also, define the functional L on the trigonometric polynomials $F(\gamma) = \sum_{n=-N}^{N} f[n] e^{-2\pi i n \gamma}$ by the rule

$$L(F) = \sum_{n=-N}^{N} f[n] \bar{r}_n.$$

a. Prove that if $R >> 0$ and $1 \leq j \leq N$ then $r_0 > |r_j|$.

b. Prove that for each polynomial $G(\gamma) = \sum_{n=0}^{N} g[n] e^{-2\pi i n \gamma}$,

$$L(|G|^2) = \sum r_{j-k}\, g[k]\overline{g[j]}.$$

c. We shall say that the functional L is *positive*, written $L \geq 0$, if $L(F) \geq 0$ and

$$L(F) = 0 \quad \text{implies} \quad F = 0$$

for all trigonometric polynomials $F(\gamma) = \sum_{n=-N}^{N} f[n] e^{-2\pi i n \gamma}$. Prove that $R \gg 0$ if and only if $L \geq 0$ cf., *Exercise 2.52a* for the analogous situation in terms of positive semidefinite matrices. [Hint. Part *b* is used in both directions, and the direction to prove $L \geq 0$ requires the Fejér-Riesz Theorem.]

3.53. Prove that $(\sin x)^{-2} < x^{-2} + 1$ on $(0, \pi/2]$. The proof is elementary, e.g., [Mon71, page 155–156]. This inequality should be compared with the inequality, $\frac{2}{\pi} x \leq \sin x$ on $[0, \frac{\pi}{2}]$, which is usually used in the proof of Jordan's Inequality (*Exercise 2.62*).

This inequality can be used to prove the following number theoretic "sieve theorem" due to Bombieri, e.g., [Mon71, Chapters 2 and 3]. *Let $\delta > 0$ and $R \subseteq \mathbb{R}$ have the property that the distance of $r - s$ to any integer is greater than or equal to δ for all unequal $r, s \in R$; then*

$$\delta \sum_{\gamma \in R} |F(\gamma)|^2 \leq \left(\delta N + \frac{2}{\sqrt{3}} + 3\delta \right) \sum |c_n|^2$$

for all trigonometric polynomials

$$F(\gamma) = \sum_{n=M+1}^{M+N} c_n e^{-2\pi i n \gamma}.$$

Because of the Parseval formula, it is interesting to note that if the elements of R are (relatively) equally spaced modulo 1 then $\delta \sum_{\gamma \in R} |F(\gamma)|^2$ is a Riemann sum approximating $\|F\|_{L^2(\mathbb{T})}^2$. The relation of such approximations to sampling theory is the subject of [Ben92b, pages 492–494].

3.54.   a. Prove that if $W \in L^1(\mathbb{T})$ is positive on $\mathbb{T}$, then $\log W \in L^1(\mathbb{T})$.

   b. Prove that if $W \in L^1(\mathbb{T})$ is nonnegative, then $\log W \in L^1(\mathbb{T})$
   if and only if

   $$\int_{\mathbb{T}} \log W(\gamma)\, d\gamma > -\infty.$$

3.55. We discussed the Littlewood Flatness Problem in *Remark 3.8.11b*.
   In this regard, prove that if $F(\gamma) = \sum_{n=0}^{N-1} f[n]e^{-2\pi in\gamma}$, then $|F|$
   and $|f|$ cannot simultaneously take constant values on $\mathbb{T}$ and
   $\{0, 1, \ldots, N-1\}$, respectively.

3.56. Consider the following formal calculation for $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$,
   where $f(t, u) \equiv f(t - u)$.

   $$\iint f(t, u)e^{-2\pi i(t\lambda + u\gamma)}\, dt\, du$$

   $$(E3.8) \qquad = \int \left(\int f(v)e^{-2\pi i(u+v)\lambda}e^{-2\pi iu\gamma}\, dv \right)\, du$$

   $$= \widehat{f}(\lambda) \int e^{-2\pi iu(\lambda + \gamma)}\, du = \widehat{f}(\lambda)\delta(\lambda + \gamma)$$

   $$= \begin{cases} \widehat{f}(\lambda), & \text{if } \lambda = -\gamma, \\ 0, & \text{if } \lambda \neq -\gamma. \end{cases}$$

   a. Recalling the definition of a Toeplitz matrix $R = (r_{jk})$, viz.,
   $r_{jk} = r_{j-k}$, prove that the two dimensional DFT of a Toeplitz
   matrix is a diagonal matrix, cf., *Exercise 3.51*.

   b. Provide the hypotheses and details to make (E3.8) into a
   theorem.

3.57. Let $\mu \in M(\mathbb{T})$ and assume $\lim_{n \to \infty} \mu^{\vee}[n] = c \in \mathbb{C}$. Prove that
   $\lim_{n \to -\infty} \mu^{\vee}[n] = c$. [Hint. By subtracting $c\delta$ from μ we can
   assume, without loss of generality, that $\lim_{n \to \infty} \mu^{\vee}[n] = 0$. By the
   Radon-Nikodym Theorem, let $\mu = F|\mu|$, where F is μ-measurable
   and $|F| \equiv 1$, e.g., [Ben76, Chapter 5]. If $G \equiv \overline{F}/F$ then

   $$(E3.9) \qquad \overline{\mu^{\vee}[-n]} = \int_{\mathbb{T}} e^{2\pi in\gamma}\, G(\gamma)\, d\mu(\gamma).$$

If G is a trigonometric polynomial the right side tends to 0 as $n \longrightarrow \infty$, and so the result is obtained by (E3.9). The result for $G = \overline{F}/F$ follows by the Weierstrass Approximation Theorem. (*Theorem 3.4.6* and *Remark 3.4.8*) and approximation properties of bounded μ-measurable functions (such as G) by continuous functions, cf., [Ben76, pages 201].]

3.58. Complete the proof of the Classical Sampling Theorem which uses the PSF, i.e., provide the mathematical justification for the formal steps in *Proof 1* of *Theorem 3.10.10*.

3.59. Let $s(t) = d_{2\pi\Omega}(t) = \frac{\sin 2\pi\Omega t}{\pi t}$ and assume equation (3.10.21),

$$f = T \sum f(nT)\tau_{nT}s,$$

is true for all $f \in PW_\Omega$. Prove that $2T\Omega \leq 1$, e.g., [Ben92b, page 450].

3.60. Let $F \in L^2(\mathbb{T})$, and let $f = \{f[n]\}$ be the sequence of Fourier coefficients of F. Assume

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \Big(\sum_{|k|=n}^{\infty} |f[k]|^2 \Big)^{1/2} < \infty.$$

Prove that $F \in A(\mathbb{T})$.

We have commented often on the difficulty of characterizing $A(\mathbb{T})$. The result of this exercise can be considered the first step in a fairly sophisticated classical line of thinking, e.g., [Ben75, pages 150–154].

3.61. We shall say that $L \in \mathcal{L}(L^2(\mathbb{T}))$ is *translation invariant* if

$$\forall F \in L^2(\mathbb{T}) \text{ and } \forall \gamma \in \mathbb{T}, \quad L(\tau_\gamma F) = \tau_\gamma(LF),$$

cf., *Definition 2.6.5.* Prove that the translation invariant continuous linear operators $L : L^2(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ are precisely of the form

$(E3.10)$ $\qquad\qquad \forall F \in L^2(\mathbb{T}), \quad LF = G * F,$

where $G \in A'(\mathbb{T})$, i.e., $g \equiv G^{\vee} \in \ell^{\infty}(\mathbb{Z})$; and also show that $\|g\|_{\ell^{\infty}(\mathbb{Z})} = \|L\|$, where $\|L\|$ is defined in *Definition B.6*. [Hint. If L is defined by (E3.10) then it is easy to see that $L \in \mathcal{L}(L^2(\mathbb{T}))$ and that L is translation invariant; and it is elementary to prove the norm equality. The converse is a consequence of *Exercise 3.51*.

In fact, if $L \in \mathcal{L}(L^2(\mathbb{T}))$ is translation invariant, then $(LF)^{\vee} = L_d f$, where L_d is the diagonal matrix of *Exercise 3.51* which has entries $g[n] \equiv \langle Le_{-n}, e_{-n} \rangle$ on the diagonal. If $g \equiv \{g[n]\}$ then $G^{\vee} \equiv g \in \ell^{\infty}(\mathbb{Z})$ since $|g[n]| \le \|L\|$, and $LF = G * F$ for this G.]

In *Example 2.6.6* we stated the L^1 analogue of the above result. In the L^1 setting, $A'(\mathbb{T})$ is replaced by $M(\mathbb{T})$. In $L^1(\mathbb{T})$, if L is also a projection, i.e., $L \circ L = L$ on $L^1(\mathbb{T})$, then the corresponding measure $\mu \in M(\mathbb{T})$ is an idempotent measure. Idempotent measures were discussed in *Remark 3.10.13*.

3.62. Let $T > 0$ and define $\mu = \sum e^{2\pi i n t/T}$ on $\mathcal{S}\mathbb{R}$) by the rule

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \mu(f) = \lim_{N \to \infty} \left(\sum_{|n| \le N} e^{2\pi i n t/T} \right)(f(t)).$$

Recall from *Chapter 2* that $g(f)$ denotes $\int g(t) f(t)\, dt$.

    a. Prove that $\mu \in \mathcal{S}'(\mathbb{R})$.

    b. Prove that $\mu \in M(\mathbb{T})$ and

$$T \sum \delta_{nT} = \sum e^{2\pi i n t/T}.$$

    ↓
$$\longrightarrow \mathcal{S}(\mathbb{R})$$

— Should be cap "N" not "N". I can't see which it is.