

Fourier Analysis –  
IMA Summer Graduate Course on  
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at UMD, College Park

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# The Fourier transform

## Definition

The *Fourier transform* of  $f \in L_m^1(\mathbb{R})$  is the function  $F$  defined as

$$F(\gamma) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \gamma} dx, \quad \gamma \in \widehat{\mathbb{R}} = \mathbb{R}.$$

Notationally, we write the pairing between  $f$  and  $F$  as:

$$\widehat{f} = F.$$

The space of Fourier transforms of  $L_m^1(\mathbb{R})$  functions is

$$A(\widehat{\mathbb{R}}) = \{F : \widehat{\mathbb{R}} \rightarrow \mathbb{C} : \exists f \in L_m^1(\mathbb{R}) \text{ such that } \widehat{f} = F\}.$$

**OPEN PROBLEM** Give an intrinsic characterization of  $A(\widehat{\mathbb{R}})$ .

# The Fourier transform inversion formula

Let  $f \in L_m^1(\mathbb{R})$ . The *Fourier transform inversion formula* is

$$f(x) = \int_{\widehat{\mathbb{R}}} F(\gamma) e^{2\pi i x \gamma} d\gamma.$$

$\check{F} = f$  denotes this inversion.

There is a formal intuitive derivation of the Fourier transform inversion formula using a form of the uncertainty principle. JB-HAA

# Jordan pointwise inversion formula

## Theorem

Let  $f \in L^1_m(\mathbb{R})$ . Assume that  $f \in BV([x_0 - \epsilon, x_0 + \epsilon])$ , for some  $x_0 \in \mathbb{R}$  and  $\epsilon > 0$ . Then,

$$\frac{f(x_0+) + f(x_0-)}{2} = \lim_{M \rightarrow \infty} \int_{-M}^M \hat{f}(\gamma) e^{2\pi i x \gamma} d\gamma.$$

## Example

$f \in L^1_m(\mathbb{R})$  does not imply  $\hat{f} \in L^1_m(\widehat{\mathbb{R}})$ . In fact, if

$$f(x) = H(x) e^{-2\pi i r x},$$

where  $r > 0$  and  $H$  is the *Heaviside function*, i.e.,  $H = \mathbb{1}_{[0, \infty)}$ , then

$$\hat{f}(\gamma) = \frac{1}{2\pi(r + i\gamma)} \notin L^1_m(\widehat{\mathbb{R}}).$$

## Theorem

a. Let  $f_1, f_2 \in L_m^1(\mathbb{R})$ , and assume  $c_1, c_2 \in \mathbb{C}$ . Then,

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad (c_1 f_1 + c_2 f_2)^\wedge(\gamma) = c_1 \hat{f}_1(\gamma) + c_2 \hat{f}_2(\gamma).$$

b. Let  $f \in L_m^1(\mathbb{R})$  and assume  $F = \hat{f} \in L_m^1(\widehat{\mathbb{R}})$ . Then,

$$\forall x \in \mathbb{R}, \quad \widehat{F}(x) = f(-x).$$

c. Let  $f \in L_m^1(\mathbb{R})$ . Then,

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \widehat{\widehat{f}}(\gamma) = \overline{\widehat{f}(\gamma)}.$$

# Translation and dilation

For a fixed  $\gamma \in \mathbb{R}$ , we set

$$e_\gamma(x) = e^{2\pi i x \gamma}.$$

For  $t \in \mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{C}$ , the *translation operator*,  $\tau_t$ , is defined as

$$\tau_t(f)(x) = f(x - t),$$

and, for a fixed  $\lambda \in \mathbb{R} \setminus \{0\}$  and for a given function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , the *dilation operator* is defined by the dilation formula,

$$f_\lambda(x) = \lambda f(\lambda x).$$

## Example

Dilation and the Poisson function

If  $f(x) = e^{-2\pi r|x|}$ ,  $r > 0$ , then

$$\hat{f}(\gamma) = \frac{1}{r} P_{1/r}(\gamma) = \frac{1}{r} \frac{1}{\pi(1 + \gamma^2/r^2)} \in L_m^1(\widehat{\mathbb{R}}),$$

where  $P(\gamma) = 1/(\pi(1 + \gamma^2))$  is the *Poisson function*.

## Theorem

Let  $f \in L^1_m(\mathbb{R})$ , and fix  $t \in \mathbb{R}$ ,  $\xi \in \widehat{\mathbb{R}}$ , and  $\lambda \in \widehat{\mathbb{R}} \setminus \{0\}$ . Then,

a.

$$(e_\xi f)^\wedge(\gamma) = \tau_\xi(\hat{f})(\gamma),$$

b.

$$(\tau_t(f))^\wedge(\gamma) = e_{-t}(\gamma)\hat{f}(\gamma),$$

c.

$$(f_\lambda)^\wedge(\gamma) = \frac{\lambda}{|\lambda|} \hat{f}\left(\frac{\gamma}{\lambda}\right).$$

Rotations play a major role in Fourier analysis on  $\mathbb{R}^d$ .

# Formal properties of the Fourier transform

## Theorem

a.  $f$  is real if and only if  $\overline{F(\gamma)} = F(-\gamma)$ . In this case,

$$F(\gamma) = \int_{\mathbb{R}} f(x) \cos(2\pi x\gamma) dx - i \int_{\mathbb{R}} f(x) \sin(2\pi x\gamma) dx,$$

$$f(x) = 2 \operatorname{Re} \int_0^{\infty} F(\gamma) e^{2\pi i x \gamma} d\gamma.$$

b.  $f$  is real and even if and only if  $F$  is real and even,. In this case,

$$F(\gamma) = 2 \int_0^{\infty} f(x) \cos(2\pi x\gamma) dx, \quad f(x) = 2 \int_0^{\infty} F(\gamma) \cos(2\pi x\gamma) d\gamma.$$

c.  $f$  is real and odd if and only if  $F$  is odd and imaginary. In this case,

$$F(\gamma) = -2i \int_0^{\infty} f(x) \sin(2\pi x\gamma) dx, \quad f(x) = 2i \int_0^{\infty} F(\gamma) \sin(2\pi x\gamma) d\gamma.$$



# Fourier transform of the Gaussian

## Example

Let  $f(x) = e^{-\pi r x^2}$ ,  $r > 0$ . We could calculate  $\hat{f}$  by means of contour integrals, but we choose Feller's real approach.

$$(\hat{f})'(\gamma) = -2\pi i \int_{\mathbb{R}} t e^{-\pi r x^2} e^{-2\pi i x \gamma} dx.$$

Noting that

$$\frac{d}{dx}(e^{-\pi r x^2}) = -2\pi r x e^{-\pi r x^2},$$

integration by parts gives

$$(\hat{f})'(\xi) = -2\pi i \int \frac{-1}{2\pi r} (e^{-\pi r x^2})' e^{-2\pi i x \xi} dx = \frac{-2\pi \gamma}{r} \hat{f}(\gamma).$$

Thus,  $\hat{f}$  is a solution of  $F'(\gamma) = -\frac{2\pi \gamma}{r} F(\gamma)$ ; and so

$$\hat{f}(\gamma) = F(\gamma) = C e^{-\pi \gamma^2 / r}.$$

# Fourier transform of the Gaussian, continued

## Example

Taking  $\gamma = 0$  and using the definition of the Fourier transform,

$C = \int_{\mathbb{R}} e^{-\pi r x^2} dx$ . In order to calculate  $C$  we first evaluate

$$a = \int_0^{\infty} e^{-u^2} du.$$

$$a^2 = \int_0^{\infty} e^{-s^2} ds \int_0^{\infty} e^{-t^2} dt = \int_0^{\infty} \int_0^{\infty} e^{-(s^2+t^2)} ds dt$$

$$= \int_0^{\pi/2} \int_0^{\infty} e^{-r^2} r dr d\theta = \frac{\pi}{4} \int_0^{\infty} e^{-u} du = \frac{\pi}{4}.$$

Thus,  $\int_{\mathbb{R}} e^{-u^2} du = \sqrt{\pi}$ . Consequently,

$$C = \int_{\mathbb{R}} e^{-\pi r x^2} dx = \frac{1}{\sqrt{\pi r}} \int_{\mathbb{R}} e^{-u^2} du = \frac{1}{\sqrt{r}}.$$

Therefore,  $\hat{f}(\gamma) = \frac{1}{\sqrt{r}} e^{-\pi \gamma^2 / r}$ .

# Analytic properties of Fourier transforms

## Theorem

Let  $f \in L^1_m(\mathbb{R})$ .

a.  $\forall \gamma \in \widehat{\mathbb{R}}, |\hat{f}(\gamma)| \leq \|f\|_1$ .

b.  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\forall \gamma$  and  $\forall \zeta$ , for which  $|\zeta| < \delta$ , we have  $|\hat{f}(\gamma + \zeta) - \hat{f}(\gamma)| < \epsilon$ , i.e.,  $\hat{f}$  is uniformly continuous.

## Proof.

b. Note that  $|\hat{f}(\gamma + \zeta) - \hat{f}(\gamma)| \leq \int_{\mathbb{R}} |f(x)| |e^{-2\pi i x \zeta} - 1| dx$ .

Let  $g_\zeta(x) = |f(x)| |e^{-2\pi i x \zeta} - 1|$ . Since  $\lim_{\zeta \rightarrow 0} g_\zeta(x) = 0$  for all  $x \in \mathbb{R}$ , and since  $|g_\zeta(x)| \leq 2|f(x)|$ , LDC implies

$$\lim_{\zeta \rightarrow 0} \int_{\mathbb{R}} g_\zeta(x) dx = 0, \quad \text{independent of } \gamma.$$

Thus,

$$\forall \epsilon > 0, \exists \zeta_0 > 0, \forall \zeta \in (-\zeta_0, \zeta_0) \text{ and } \forall \gamma \in \widehat{\mathbb{R}}, \quad |\hat{f}(\gamma + \zeta) - \hat{f}(\gamma)| < \epsilon.$$

This is the desired uniform continuity. ■

# Analytic properties of Fourier transforms, continued

The following result for  $L_m^1(\mathbb{R})$  has essentially the same proof as the *Riemann-Lebesgue lemma for  $L_m^1(\mathbb{T})$* , see ahead.

## Theorem

*Riemann–Lebesgue lemma.* Assume  $f \in L_m^1(\mathbb{R})$ . Then,

$$\lim_{|\gamma| \rightarrow \infty} \hat{f}(\gamma) = 0.$$

## Example

$C_0(\widehat{\mathbb{R}}) \setminus A(\widehat{\mathbb{R}}) \neq \emptyset$ . Define

$$F(\gamma) = \begin{cases} \frac{1}{\log(\gamma)}, & \text{if } \gamma > e, \\ \frac{\gamma}{e}, & \text{if } 0 \leq \gamma \leq e, \end{cases}$$

on  $[0, \infty)$  and as  $-F(-\gamma)$  on  $(-\infty, 0]$ .

Also,  $A(\widehat{\mathbb{R}})$  is a set of first category in  $C_0(\widehat{\mathbb{R}})$ . Even more, a Baire category argument can also be used to show the existence of  $F \in C_c(\widehat{\mathbb{R}})$  for which  $F \notin A(\widehat{\mathbb{R}})$ . JB-HAA.

## Theorem

*Differentiation of Fourier transforms.*

a. Assume that  $f^{(n)}$ ,  $n \geq 1$ , exists everywhere and that

$$f(\pm\infty) = \dots = f^{(n-1)}(\pm\infty) = 0.$$

Then,

$$(f^{(n)})^\wedge(\gamma) = (2\pi i\gamma)^n \hat{f}(\gamma).$$

b. Assume that  $x^n f(x) \in L_m^1(\mathbb{R})$ , for some  $n \geq 1$ . Then,  $x^k f(x) \in L_m^1(\mathbb{R})$ ,  $k = 1, \dots, n-1$ ,  $(\hat{f})', \dots, (\hat{f})^{(n)}$  exist everywhere, and

$$\forall k = 0, \dots, n, \quad \left( (-2\pi i \cdot)^k f(\cdot) \right)^\wedge(\gamma) = \hat{f}^{(k)}(\gamma).$$

Smooth function and fast Fourier decay, and vice-versa.

Absolute continuity has spectacular and essential role. JB-HAA

# Convolution and Fourier transforms

The *convolution*  $f * g$  of  $f, g \in L^1_m(\mathbb{R})$  is

$$f * g(x) = \int_{\mathbb{R}} f(t)g(x-t) dt = \int_{\mathbb{R}} f(x-t)g(t) dt.$$

## Theorem

Let  $f, g \in L^1_m(\mathbb{R})$ . Then,  $f * g \in L^1_m(\mathbb{R})$  and

$$(f * g)^\wedge(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma).$$

For the proof,  $f * g \in L^1_m(\mathbb{R})$  and Fubini–Tonelli give

$$\begin{aligned}(f * g)^\wedge(\gamma) &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t)g(t)e^{-2\pi i(x-t)\gamma} e^{-2\pi it\gamma} dt dx \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x-t)e^{-2\pi i(x-t)\gamma} dx \right) g(t)e^{-2\pi it\gamma} dt \\ &= \int_{\mathbb{R}} \hat{f}(\gamma)g(t)e^{-2\pi it\gamma} dt = \hat{f}(\gamma)\hat{g}(\gamma).\end{aligned}$$

This innocent proposition is actually a *raison d'être* for transform methods, generally, and for the Fourier transform. JB-HAA

# Approximate identity

## Definition

An *approximate identity* is a family  $\{K_{(\lambda)} : \lambda > 0\} \subseteq L_m^1(\mathbb{R})$  of functions with the properties:

- i.  $\forall \lambda > 0, \int_{\mathbb{R}} K_{(\lambda)}(x) dx = 1,$
- ii.  $\exists M > 0$  such that  $\forall \lambda > 0, \|K_{(\lambda)}\|_1 \leq M,$
- iii.  $\forall \delta > 0, \lim_{\lambda \rightarrow \infty} \int_{|x| \geq \delta} |K_{(\lambda)}(x)| dx = 0,$

## Theorem

Let  $K \in L_m^1(\mathbb{R})$  have the property that  $\int_{\mathbb{R}} K(x) dx = 1$ . Then, the family,  $\{K_\lambda : K_\lambda(x) = \lambda K(\lambda x), \lambda > 0\} \subseteq L_m^1(\mathbb{R})$ , of dilations of  $K$  is an approximate identity.

Approximate identities in  $L_m^1(\mathbb{R}) \subseteq M_b(\mathbb{R})$  approximate  $\delta \in M_b(\mathbb{R})$ , where  $\delta(f) = f(0)$ .  $\delta$  is the multiplicative (under convolution) unit in the Banach algebra  $M_b(\mathbb{R})$ . The Banach algebra  $L_m^1(\mathbb{R})$  has no multiplicative unit.

# Examples of approximate identities

## Example

a. The *Fejér function*  $W$  is

$$W(x) = \frac{1}{2\pi} \left( \frac{\sin(x/2)}{x/2} \right)^2.$$

$W$  is non-negative and  $\int_{\mathbb{R}} W(x) dx = 1$ . Thus, the *Fejér kernel*  $\{W_\lambda : \lambda > 0\} \subseteq L^1_m(\mathbb{R})$  is an approximate identity.

b. The *Dirichlet function*  $D$  is

$$D(x) = \frac{\sin(x)}{\pi x}.$$

Although  $\int_{\mathbb{R}} D(t) dt = 1$ , we have  $D \notin L^1_m(\mathbb{R})$ . Thus, the *Dirichlet kernel*  $\{D_\lambda : \lambda > 0\}$  is not an approximate identity.

c. The *Poisson kernel*  $P_\lambda$  and the *Gauss kernel*  $G_\lambda$ , defined by  $P(x) = \frac{1}{\pi(1+x^2)}$  and  $G(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ , are approximate identities.



# Approximate identity theorem and uniqueness

## Theorem

Let  $f \in L^1_m(\mathbb{R})$ .

a. If  $\{K_{(\lambda)} : \lambda > 0\} \subseteq L^1_m(\mathbb{R})$  is an approximate identity, then

$$\lim_{\lambda \rightarrow \infty} \|f - f * K_{(\lambda)}\|_1 = 0.$$

b. We have

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}} \left| f(x) - \int_{-\lambda/2\pi}^{\lambda/2\pi} \left(1 - \frac{2\pi|\gamma|}{\lambda}\right) \hat{f}(\gamma) e^{2\pi i x \gamma} d\xi \right| dx = 0.$$

c. If  $\hat{f} = 0$  on  $\widehat{\mathbb{R}}$ , then  $f$  is the 0 function.

## Theorem

Let  $f \in L^\infty_m(\mathbb{R})$  be continuous on  $\mathbb{R}$ . Then,

$$\forall x \in \mathbb{R}, \quad \lim_{\lambda \rightarrow \infty} f * K_{(\lambda)}(x) = f(x).$$

# Approximate identity theorem and uniqueness, cont.

## Proof.

Part *c.* follows from *b.*, and *b.* follows from *a.* For *a.*, compute

$$\begin{aligned}\|f - f * K_{(\lambda)}\|_1 &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K_{(\lambda)}(t) f(x) dt - \int_{\mathbb{R}} K_{(\lambda)}(t) f(x-t) dt \right| dx \\ &\leq \int_{\mathbb{R}} |K_{(\lambda)}(t)| \left( \int_{\mathbb{R}} |f(x) - f(x-t)| dx \right) dt.\end{aligned}$$

Let  $\epsilon > 0$ .  $\exists \delta > 0$  such that, for  $\|K_{(\lambda)}\|_1 \leq M$ ,

$$\forall |t| < \delta, \quad \int_{\mathbb{R}} |f(x) - f(x-t)| dx < \frac{\epsilon}{M}, \text{ and so}$$

$$\begin{aligned}\|f - f * K_{(\lambda)}\|_1 &\leq 2\|f\|_1 \int_{|t| \geq \delta} |K_{(\lambda)}(t)| dt + \frac{\epsilon}{M} \int_{|t| \leq \delta} |K_{(\lambda)}(t)| dt \\ &\leq 2\|f\|_1 \int_{|t| \geq \delta} |K_{(\lambda)}(t)| dt + \epsilon.\end{aligned}$$

Definition of a.i.,  $\epsilon > 0$  arbitrary, and  $\overline{\lim}$  give result. ■

# Inversion formula for $L_m^1(\mathbb{R}) \cap A(\mathbb{R})$

If  $f \in L_m^1(\mathbb{R})$  and  $\hat{f} \in L_m^1(\widehat{\mathbb{R}})$ , we can use the Approximate Identity Theorem to prove the following pointwise inversion theorem. What we explicitly mean in its statement is that if  $f \in L_m^1(\mathbb{R})$  and  $\hat{f} \in L_m^1(\widehat{\mathbb{R}})$ , then the inversion formula is true *m-a.e.*; and that if  $f$  is continuous then it is true for all  $x \in \mathbb{R}$ . Compare the proof in JB-HAA, pages 38–39.

## Theorem

Let  $f \in L_m^1(\mathbb{R}) \cap A(\mathbb{R})$ . Then,

$$\forall x \in \mathbb{R}, \quad f(x) = \int_{\widehat{\mathbb{R}}} \hat{f}(\gamma) e^{2\pi i x \gamma} d\gamma.$$

a. *m-a.e.* proofs such as the following lead to convergence in larger sets, so called *Lebesgue sets*. JB-HAA

b. The assumptions on the approximate identity in the proof are easily satisfied, e.g., by  $K_{(\lambda)} = W_\lambda$ .

# Proof of inversion formula for $L_m^1(\mathbb{R}) \cap A(\mathbb{R})$

## Proof.

The statement of the theorem follows from two observations. First, if  $\{K_{(\lambda)} : \lambda > 0\} \subseteq L_m^1(\mathbb{R})$  is an approximate identity, then there exists a subsequence  $\{\lambda_n : n = 1, \dots\}$  such that

$$\lim_{n \rightarrow \infty} f * K_{(\lambda_n)} = f \quad m\text{-a.e.}$$

This fact is a consequence of the Aproximate Identity Theorem. Second, assume that  $\hat{f} \in L_m^1(\hat{\mathbb{R}})$ , that  $(K_{(\lambda)})^\wedge \in L_m^1(\hat{\mathbb{R}})$ , and

$$\forall x \in \mathbb{R}, \quad K_{(\lambda)}(x) = \int_{\hat{\mathbb{R}}} (K_{(\lambda)})^\wedge(\gamma) e^{2\pi i x \gamma} d\gamma.$$

Then,

$$\lim_{\lambda \rightarrow \infty} \left\| \int_{\hat{\mathbb{R}}} \hat{f}(\xi) e^{2\pi i x \gamma} d\gamma - f * K_{(\lambda)}(x) \right\|_{\infty} = 0.$$



# The $L_m^2(\mathbb{R})$ theory of Fourier transforms

We have defined the Fourier transform of  $f \in L_m^1(\mathbb{R})$ . Now our goal is to define it for  $f \in L_m^2(\mathbb{R})$ . Clearly,  $L_m^2(\mathbb{R}) \not\subseteq L_m^1(\mathbb{R})$ , and so we cannot use the integral formula, that defines  $\widehat{f}$  of  $f \in L_m^1(\mathbb{R})$ , since the function under the integral sign may not be integrable.

## Theorem

*Plancherel theorem.*  $\exists$  unique linear bijection  $\mathcal{F} : L_m^2(\mathbb{R}) \longrightarrow L_m^2(\widehat{\mathbb{R}}) :$

a.  $\forall f \in L_m^1(\mathbb{R}) \cap L_m^2(\mathbb{R})$  and  $\forall \gamma \in \widehat{\mathbb{R}}, \widehat{f}(\gamma) = \mathcal{F}(f)(\gamma);$

b.  $\forall f \in L_m^2(\mathbb{R}), \|f\|_2 = \|\mathcal{F}(f)\|_2.$

a. Because of the translation invariance of Lebesgue measure (and the time-invariance required for most physical experiments), it is natural to extend Fourier analysis to all groups, not only  $\mathbb{R}$ . Limitations and difficulties arise immediately even though there are invariant measures on locally compact groups, i.e., Haar measures. For some of these groups, the  $L^2$  theory is still state of the art!

b. Compute  $\mathcal{F}(f)$ .

c. Compare  $L^1$  and  $L^2$  theories – remember the  $A(\widehat{\mathbb{R}})$  open problem!

# A lemma for the $L_m^2(\mathbb{R})$ theory

## Lemma

Let  $f \in C_c(\mathbb{R})$ . Then,  $\hat{f} \in L_m^2(\widehat{\mathbb{R}})$  and  $\|f\|_2 = \|\hat{f}\|_2$ .

## Proof.

Let  $\tilde{f}(t) = \overline{f(-t)}$ .  $\hat{f} \in A(\widehat{\mathbb{R}})$ , since  $C_c(\mathbb{R}) \subseteq L_m^1(\mathbb{R})$ . Define  $g = f * \tilde{f}$ .  $g$  is continuous,  $g \in L_m^1(\mathbb{R}) \cap L_m^\infty(\mathbb{R})$ , and  $g(0) = \|f\|_2^2$ .

Fubini and the translation invariance of Lebesgue measure imply

$$\forall \gamma \in \widehat{\mathbb{R}}, \quad \hat{g}(\gamma) = |\hat{f}(\gamma)|^2.$$

Since  $g$  is continuous, we deduce that

$$g(0) = \lim_{\lambda \rightarrow \infty} \int_{-\lambda/2\pi}^{\lambda/2\pi} \left(1 - \frac{2\pi|\gamma|}{\lambda}\right) |\hat{f}(\gamma)|^2 d\gamma.$$

Finally, Levi–Lebesgue (LDC) allows us to assert that  $\hat{f} \in L_m^2(\widehat{\mathbb{R}})$  and

$$\|\hat{f}\|_2^2 = g(0) = \|f\|_2^2.$$

# Proof of the Plancherel theorem

i. We define the action of  $\mathcal{F}$  on  $C_c(\mathbb{R})$  by  $\mathcal{F}(f) = \hat{f}$ . The Lemma implies  $\mathcal{F}(f) \in L_m^2(\widehat{\mathbb{R}})$  for  $f \in C_c(\mathbb{R})$ ,

ii. We prove that  $\mathcal{F}(C_c(\mathbb{R})) \subseteq A(\widehat{\mathbb{R}}) \cap L_m^2(\widehat{\mathbb{R}})$  is a dense subspace of  $L_m^2(\widehat{\mathbb{R}})$ . Indeed, let  $g \in L_m^2(\widehat{\mathbb{R}})$  and suppose that

$$\forall f \in C_c(\mathbb{R}), \quad \int_{\mathbb{R}} \hat{f}(\gamma) \overline{g(\gamma)} d\gamma = 0. \quad (1)$$

If  $f \in C_c(\mathbb{R})$ , then  $\tau_u(f) \in C_c(\mathbb{R})$ , and so

$$\forall f \in C_c(\mathbb{R}) \text{ and } \forall u \in \mathbb{R}, \quad \int_{\mathbb{R}} \hat{f}(\gamma) \overline{g(\gamma)} e^{-2\pi i u \gamma} d\gamma = 0.$$

By Hölder,  $\hat{f}\overline{g} \in L_m^1(\widehat{\mathbb{R}})$ , and so  $\hat{f}\overline{g} = 0$   $m$ -a.e. for each  $f \in C_c(\mathbb{R})$ . Also,

$$\forall f \in C_c(\mathbb{R}) \text{ and } \forall \gamma \in \widehat{\mathbb{R}}, \quad e^{2\pi i x \gamma} f(x) \in C_c(\mathbb{R}).$$

Thus,  $\mathcal{F}(C_c(\mathbb{R}))$  is translation invariant, i.e.,

$$\forall f \in C_c(\mathbb{R}) \text{ and } \forall \gamma \in \widehat{\mathbb{R}}, \quad \tau_u(\hat{f}) \in \mathcal{F}(C_c(\mathbb{R})).$$

From this we *claim* that,  $\forall \gamma_0 \in \widehat{\mathbb{R}}, \exists f \in C_c(\mathbb{R})$  for which  $|\hat{f}| > 0$  on some interval centered about  $\gamma_0$ .

# Proof of the Plancherel theorem, continued

To verify this *claim*, suppose  $\exists \gamma_0$  such that  $\forall f \in C_c(\mathbb{R})$  and interval  $I$  centered at  $\gamma_0$ ,  $\hat{f}$  has a zero in  $I$ .  $\therefore \hat{f}(\gamma_0) = 0$  for each  $f \in C_c(\mathbb{R})$ . By the translation invariance of  $\mathcal{F}(C_c(\mathbb{R}))$ ,  $\tau_\eta(\hat{f}) \in \mathcal{F}(C_c(\mathbb{R}))$  for each  $\eta \in \widehat{\mathbb{R}}$ , and so

$$\forall f \in C_c(\mathbb{R}) \text{ and } \forall \eta \in \widehat{\mathbb{R}}, \quad \tau_\eta(\hat{f})(\gamma_0) = 0,$$

i.e.,  $\hat{f} = 0$  on  $\widehat{\mathbb{R}}$  for each  $f \in C_c(\mathbb{R})$ . This contradicts the uniqueness theorem, and the *claim* is proved.

$\therefore$  by the *claim*, (1) implies  $g = 0$  *m-a.e.*

$\therefore$  by the Hahn-Banach theorem and since  $L_m^2(\widehat{\mathbb{R}})$  is self-dual, we have  $\overline{\mathcal{F}(C_c(\mathbb{R}))} = L_m^2(\widehat{\mathbb{R}})$ .

*iii.* We have shown that  $\mathcal{F}$  is a continuous linear injection  $C_c(\mathbb{R}) \rightarrow L_m^2(\widehat{\mathbb{R}})$ , when  $C_c(\mathbb{R})$  has the  $L_m^2(\mathbb{R})$  norm, and so  $\mathcal{F}$  has a unique linear injective extension to  $L_m^2(\mathbb{R})$ . Also,  $\mathcal{F}(C_c(\mathbb{R}))$  is closed and dense in  $L_m^2(\widehat{\mathbb{R}})$  by the Lemma and by part *ii.* Thus,  $\mathcal{F}$  is also surjective.

*a.* follows since  $\overline{C_c(\mathbb{R})} = L_m^1(\mathbb{R})$ , when equipped with  $L_m^1(\mathbb{R})$  norm; and *b.* follows from the continuity of  $\mathcal{F}$ .



## Theorem

*Parseval formula.* Let  $f, g \in L_m^2(\mathbb{R})$ . Then, the following formulas hold:

$$\int_{\mathbb{R}} f(x)\overline{g(x)} dx = \int_{\widehat{\mathbb{R}}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)} d\gamma$$

and

$$\int_{\mathbb{R}} f(x)g(x) dx = \int_{\widehat{\mathbb{R}}} \hat{f}(\gamma)\hat{g}(-\gamma) d\gamma.$$

## Proof.

The first formula follows from the Plancherel theorem and the fact that

$$4f\overline{g} = |f + g|^2 - |f - g|^2 + i|f + ig|^2 - i|f - ig|^2.$$



# Parseval and beyond

- a. Parseval was a French engineer, who gave his formula in 1799 (published in 1805).
- b. Weak solutions in physics, and test functions for *unseen* solutions.
- c. Creative formulas. From Pythagoras to defining the length of curves by calculus to Hilbert space. JB-HAA
- d. Duality formalism. Little spaces and their big dual spaces.
- e. Distribution theory. To define the Fourier transform on large spaces of objects (distributions), originally arising in applications, in terms of their precise definition on small spaces such as  $C_c(\mathbb{R})$ .

# Hilbert transform

a. The *Hilbert transform*  $\mathcal{H}(f)$  of  $f : \mathbb{R} \rightarrow \mathbb{C}$  is the convolution,

$$\mathcal{H}(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{|t-x| \geq \epsilon} \frac{f(x)}{t-x} dx.$$

The Hilbert transform opens the door to a profound area of harmonic analysis associated with the theory, relevance, and importance of *singular integrals*, e.g., the books of Stein and of Garcia-Cueva and Rubio de Francia, cf., Neri's SLN for a magnificent introduction.

b.  $\mathcal{H} \in \mathcal{L}(L_m^2(\mathbb{R}))$ ,  $\mathcal{H}$  is unitary on  $L_m^2(\mathbb{R})$ ,  $\mathcal{H} \circ \mathcal{H} = -Id$  on  $L_m^2(\mathbb{R})$ , and

$$\mathcal{H} = \mathcal{F}^{-1} \sigma(\mathcal{H}) \mathcal{F}, \text{ where } \sigma(\mathcal{H})(\gamma) = -i \operatorname{sgn}(\gamma).$$

c. Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  satisfy  $\operatorname{supp}(f) \subseteq [0, \infty)$ , and define the unilateral *Laplace transform* of  $f$  as  $\mathcal{L}(f)(t) = \int_0^\infty f(x) e^{-tx} dx$ . A formal calculation, which is valid under mild hypotheses, shows that

$$\forall t > 0, \quad \mathcal{L}(\mathcal{L}(f))(t) = -\pi \mathcal{H}(f)(-t).$$

d. See JB-HAA, Problem 2.57, for a role of  $\mathcal{H}$  in signal processing, in particular, wavelet auditory modeling, as related to the Paley-Wiener logarithmic integral theorem.

## Definition

a. Let  $F \in L^1_{loc}(\widehat{\mathbb{R}})$  be  $2\Omega$ -periodic. The *Fourier series* of  $F$  is

$$S(F)(\gamma) = \sum_{n \in \mathbb{Z}} f[n] e^{-2\pi i n \gamma / (2\Omega)},$$

with *Fourier coefficients*

$$\forall n \in \mathbb{Z}, \quad f[n] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{2\pi i n \gamma / (2\Omega)} d\gamma.$$

b. If  $f \in \ell^1(\mathbb{Z})$ , then the Fourier series of  $F$  is well-defined, and  $F$  is the *Fourier transform* of  $f$ .

**GOAL** Find conditions on  $F$  so that  $S(F) = F$ . In fact, if  $S(F) = F$ , then the integral formula for  $f$  can be formally verified.

JB-HAA, Section 3.2, for the history of Fourier series.

If  $\Omega > 0$  and  $F \in L^1_{loc}(\widehat{\mathbb{R}})$  is  $2\Omega$ -periodic, we write  $F \in L^1(\mathbb{T}_{2\Omega})$ , where  $\mathbb{T}_{2\Omega} = \widehat{\mathbb{R}}/(2\Omega\mathbb{Z})$ . The  $L^1$ -norm of  $F \in L^1(\mathbb{T}_{2\Omega})$  is

$$\|F\|_{L^1(\mathbb{T}_{2\Omega})} = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)| \, d\gamma.$$

If  $F$  is a  $2\Omega$ -periodic, Lebesgue measurable function, and  $F^2 \in L^1(\mathbb{T}_{2\Omega})$ , we write  $F \in L^2(\mathbb{T}_{2\Omega})$ . The  $L^2$ -norm of  $F \in L^2(\mathbb{T}_{2\Omega})$  is

$$\|F\|_{L^2(\mathbb{T}_{2\Omega})} = \left( \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} |F(\gamma)|^2 \, d\gamma \right)^{1/2}.$$

$L^2(\mathbb{T}_{2\Omega}) \subseteq L^1(\mathbb{T}_{2\Omega})$ , and  $\|F\|_{L^1(\mathbb{T}_{2\Omega})} \leq \|F\|_{L^2(\mathbb{T}_{2\Omega})}$ ,  $F \in L^2(\mathbb{T}_{2\Omega})$ .

## Definition

The *Fourier transform* of  $F \in L^1(\mathbb{T}_{2\Omega})$  is  $f = \{f[n] : n \in \mathbb{Z}\}$ , where

$$\forall n \in \mathbb{Z}, \quad f[n] = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{-2\pi i n \gamma / (2\Omega)} \, d\gamma.$$

# The great books (that I know)

- Werner W. Rogosinski, *Fourier Series* (Ger.), 1930.
- Norbert Wiener, *The Fourier Integral and Certain of its Applications*, 1933.
- Raymond E. A. C. Paley and Norbert Wiener, *Fourier Transforms in the Complex Domain*, 1934.
- Antoni Zygmund, *Trigonometric Series*, 1935, 1959 (2 volumes).
- E. C. Titchmarsh, *Theory Fourier Integrals*, 1937 (2nd ed. 1948).
- Solomon Bochner and K. Chandrasekharan, *Fourier Transforms*, 1949.
- G. H. Hardy and Werner W. Rogosinski, *Fourier Series*, 1956.
- Nina Bari, *A Treatise on Trigonometric Series*, 1961 (2 volumes).
- Walter Rudin, *Fourier Analysis on Groups*, 1962.
- Jean-Pierre Kahane and Raphaël Salem, *Perfect Sets and Trigonometric Series* (Fr.), 1963.
- Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, Volumes 1 (1963) and 2 (1970).

# The great books (that I know), continued

- Torsten Carleman, L'Intégrale de Fourier, 1967.
- Robert E. Edwards, Fourier Series, 1967 (2 volumes)..
- Yitzhak Katznelson, Harmonic Analysis, 1968.
- William F. Donoghue, Distributions and Fourier Transforms, 1969.
- Jean-Pierre Kahane, Absolutely Convergent Fourier Series, 1970.
- Elias M. Stein and Guido Weiss, Fourier Analysis on  $\mathbb{R}^d$ , 1971.
- Harry Dym and Henry P. McKean, Fourier Series and Integrals, Volumes 1 (1972) and 2 (1976).
- Henry Helson, Harmonic Analysis, 1983.
- Thomas W. Körner, Fourier Analysis, 1988.
- Wavelet and Time-Frequency Analysis literature initiated by Yves Meyer, Ingrid Daubechies, Stéphane Mallat, Hans Feichtinger, Karlheinz Gröchenig., late 1980s until the present.
- Hugh L. Montgomery, Early Fourier Analysis, 2014.

# Riemann-Lebesgue lemma

## Theorem

If  $F \in L^1(\mathbb{T}_{2\Omega})$ , then  $\lim_{|n| \rightarrow \infty} f[n] = 0$ , where  $f = \{f[n]\}$  is the sequence of Fourier coefficients of  $F$ , i.e.,  $\hat{f} = F$ .

## Proof.

a. Assume  $F \in C^1(\mathbb{T}_{2\Omega})$ .  $\therefore G = F' \in L^1(\mathbb{T}_{2\Omega})$ ,  $\int_{-\Omega}^{\Omega} G = 0$ , and

$$\forall \gamma \in [-\Omega, \Omega], \quad F(\gamma) = \int_{-\Omega}^{\gamma} G + F(-\Omega).$$

Definition of  $f[n]$  and integration by parts gives

$$\forall n \neq 0, \quad |f[n]| \leq \frac{\Omega}{\pi |n|} \|G\|_{L^1(\mathbb{T}_{2\Omega})}.$$

b.  $C^1$  approximations to  $F \in L^1(\mathbb{T}_{2\Omega})$ , e.g., using an a.i., plus a  $\overline{\lim}$  argument does it!



# Dirichlet theorem

We shall use the Riemann–Lebesgue lemma to verify Johann Peter Gustav Lejeune Dirichlet's (1805–1859) fundamental theorem, which provides sufficient conditions on a function  $F \in L^1(\mathbb{T}_{2\Omega})$  so that  $S(F)(\gamma_0) = F(\gamma_0)$  for a given  $\gamma_0$ . The following ingenious proof is due to Paul R. Chernoff. The Dirichlet theorem for Fourier series naturally preceded the analogous inversion theorem for Fourier transforms, as formulated and proved by Jordan for  $\mathbb{R}$ , and stated above.

## Theorem

If  $F \in L^1(\mathbb{T}_{2\Omega})$  and  $F$  is differentiable at  $\gamma_0$ , then  $S(F)(\gamma_0) = F(\gamma_0)$  in the sense that

$$\lim_{M, N \rightarrow \infty} \sum_{n=-M}^N c_n e^{-2\pi i n \gamma_0 / (2\Omega)} = F(\gamma_0),$$

where  $c = \{c_n : n \in \mathbb{Z}\}$  is the sequence of Fourier coefficients of  $F$ .

# Proof of Dirichlet theorem

*i.* Without loss of generality, assume  $\gamma_0 = 0$  and  $F(\gamma_0) = 0$ . In fact, if  $F(\gamma_0) \neq 0$ , then consider the function  $F - F(\gamma_0)$  instead of  $F$ , which is also an element of  $L^1(\mathbb{T}_{2\Omega})$ , and then translate this function to the origin.

*ii.* Since  $F(0) = 0$  and  $F'(0)$  exists, we can verify that

$$G(\gamma) = \frac{F(\gamma)}{e^{-2\pi i\gamma/(2\Omega)} - 1}$$

is bounded in some interval centered at the origin. To see this note that

$$G(\gamma) = \frac{F(\gamma)}{\gamma} \frac{1}{\sum_{j=1}^{\infty} (-2\pi i/(2\Omega))^j (1/j!) \gamma^j},$$

and, hence,  $G(\gamma)$  is close to  $-\Omega F'(0)/(\pi i)$  in a neighborhood of the origin.

# Proof of Dirichlet theorem, continued

The boundedness near the origin, coupled with integrability of  $F$  on  $\mathbb{T}_{2\Omega}$ , yields the integrability of  $G$  on  $\mathbb{T}_{2\Omega}$ . Therefore, since  $F(\gamma) = G(\gamma)(e^{-2\pi i\gamma/(2\Omega)} - 1)$ , we compute  $c_n = d_{n+1} - d_n$ , where  $d = \{d_n : n \in \mathbb{Z}\}$  is the sequence of Fourier coefficients of  $G$ . Thus, the partial sum  $\sum_{n=-M}^N c_n e^{-2\pi i n \gamma_0 / (2\Omega)}$  is the telescoping series

$$\sum_{n=-M}^N (d_{n+1} - d_n) = d_{N+1} - d_{-M}.$$

Consequently, we can apply the Riemann–Lebesgue lemma to the sequence of Fourier coefficients of  $G$  to obtain

$$\lim_{M, N \rightarrow \infty} \sum_{n=-M}^N c_n e^{-\pi i n \gamma / \Omega} = 0.$$

- With regard to Dirichlet's theorem, one can assert that *if  $F \in BV_{loc}(\mathbb{R})$ ,  $F$  is  $2\Omega$ -periodic, and  $F$  is continuous on a closed subinterval  $I \subseteq \mathbb{T}_{2\Omega}$ , then*

$$\sum_{n=-N}^N \hat{F}[n] e^{-2\pi i n x / (2\Omega)}$$

*converges uniformly to  $F$  on  $I$ .*

- If  $f \in BV(\mathbb{R})$ , then  $f \in L^1_{loc}(\mathbb{R})$  and  $f' \in M_b(\mathbb{R})$  distributionally, and every  $\mu \in M_b(\mathbb{R})$  can be written this way. This is the Riesz representation theorem!
- BV in  $\mathbb{R}^d$  is really exciting, e.g., JB-WC's Integration and Modern Analysis, Chapter 8 for starters.

# Up, up, and away

1. Poisson summation and classical sampling, periodization, computation of Fourier transforms using the DFT, the FFT, and DFT frames.
2. Haar measure on LCGs and harmonic analysis on LCAGs  $G$ .
3. Uncertainty principle theory
4. Fourier analysis of  $M_b(G)$ , Herglotz-Bochner theorem, Levy continuity theorem, Cohen idempotent theorem, homomorphism theory Ideal structure of Banach algebras including  $L^1(G)$  and  $M_b(G)$ .
5. Distribution theory and Wiener's Generalized Harmonic Analysis.
6. Radial and geometric Fourier analysis.
7. Wavelet theory.
8. Applied and number theoretic harmonic analysis.
9.  $\dots \infty$ .

# The Poisson summation formula

## Theorem

a. Formally, we have

$$T \sum_{n \in \mathbb{Z}} f(t + nT) = \sum_{n \in \mathbb{Z}} \hat{f}(n/T) e^{2\pi i n t / T}.$$

b. Let  $T > 0$ . Then  $T \sum_{n \in \mathbb{Z}} \delta_{nT} \in \mathcal{S}'(\mathbb{R}) \cap \mathcal{M}(\mathbb{R})$ , and

$$\left( T \sum_{n \in \mathbb{Z}} \delta_{nT} \right)^\wedge = \sum_{n \in \mathbb{Z}} \delta_{n/T}$$

distributionally.

- $\exists f \in L^1(\mathbb{R})$ , such that  $f$  is continuous on  $\mathbb{R}$ ,  $f(n) = 0$  all  $n \in \mathbb{Z}$ ,  $\hat{f}(n) = 0$  for every  $n \neq 0$ , and  $\hat{f}(0) = 1$ , (Katznelson) - fantastic.
- *a* requires hypotheses, e.g.,  $f \in \mathcal{S}(\mathbb{R})$  suffices. *b* follows from *a*, see JB-HAA, Section 3.10. See JFAA 3 (1997), 505-523 for intricacies.

# The classical sampling theorem

- The *Paley-Wiener* space:

$$PW_{\Omega} = \{f \in L^2(\mathbb{R}) : \text{supp} \hat{f} \subseteq [-\Omega, \Omega]\}.$$

- The *PSF* is equivalent to the classical sampling theorem.
- The classical sampling theorem goes back to Cauchy (1840s).

## Theorem

Let  $T, \Omega > 0$  satisfy the condition that  $0 < 2T\Omega \leq 1$ , and let  $s$  be an element of the Paley-Wiener space  $PW_{1/(2T)}$  satisfying the conditions that  $\hat{s} \equiv S = 1$  on  $[-\Omega, \Omega]$  and  $S \in L^{\infty}(\widehat{\mathbb{R}})$ . Then

$$\forall f \in PW_{\Omega}, \quad f = T \sum_{n \in \mathbb{Z}} f(nT) \tau_{nT} s,$$

where the convergence is in the  $L^2(\mathbb{R})$  norm and uniformly in  $\mathbb{R}$ . One possible sampling function  $s$  is

$$s(t) = \frac{\sin(2\pi\Omega t)}{\pi t}.$$

# Wavelets and sampling functions

- Let  $2T\Omega = 1$  and  $s_\Omega(t) = d_{2\pi\Omega}(t) = \sin(2\pi\Omega t)/\pi t$ . Recall that the inverse Fourier transform of  $\mathbb{1}_{[-\Omega,\Omega]}$  is  $s_\Omega$ .
- Set  $\varphi(t) = s_\Omega(t)/\sqrt{2\Omega}$ , and let  $V_0 = \overline{\text{span}}\{\tau_{nT}\varphi\}$ . Defining spaces such as  $V_0$  leads to the fundamental idea of a *multi-resolution analysis MRA* in wavelet theory. Let

$$\psi(t) = (1/\sqrt{2\Omega})(s_{2\Omega}(t) - s_\Omega(t)).$$

$\psi$  is the *Shannon wavelet* or *Littlewood-Paley wavelet*.

- Let

$$\psi_{m,n}(t) = 2^{m/2}\psi(2^m t - n).$$

Then  $\psi_{m,n}$  is a *dyadic wavelet ONB* for  $L^2(\mathbb{R})$ .

- The classical sampling theorem had a significant impact on various topics in mathematics, including number theory and interpolation theory, long before Shannon's application of it in communications.
- Today we are in the sampling age.



# Potpourri and tittillation for PSF and sampling

1. The formula,

$$\frac{1}{T} \sum_{n \in \mathbb{Z}} \delta_{n/T}(\gamma) = \sum_{n \in \mathbb{Z}} e^{-2\pi i n T \gamma},$$

is false, meaningful, and a beautiful source of exercises.

2. Classical sampling theorem and relations to locally compact Abelian groups.
3. Euler-MacLaurin formula:  $T \sum_0^\infty f(nT) = \int_0^\infty f(t) dt + \text{error}$ .
4. Jacobi formula:  $\theta(t) = \sum e^{-\pi n^2 t}$ .
5.  $\forall t > 0, \theta(t) = \frac{1}{\sqrt{t}} \theta\left(\frac{1}{t}\right)$ , leading to basic analytic continuation formulas in analytic number theory.
6. Diffusion equations.
7. Statistical mechanics.
8. Automorphic forms and elliptic functions.
9. Deligne's proof of the Ramanujan conjecture.
10. The Selberg trace formula can be considered to be a version of the *PSF* in a number theoretic, non-abelian setting.

# The discrete Fourier transform (DFT)

Given  $\mathbb{Z}/N\mathbb{Z}$ . The *discrete Fourier transform DFT* of  $f: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$  as  $F: \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ , where

$$\forall n \in \mathbb{Z}/N\mathbb{Z}, \quad F[n] = \sum_{m \in \mathbb{Z}/N\mathbb{Z}} f[m] e^{-2\pi i m n / N}.$$

The *Fourier inversion theorem and formula* for the DFT is elementary:

## Theorem

Given  $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{C}$  with DFT  $F$ . Then,

$$\forall m \in \mathbb{Z}/n\mathbb{Z} \quad f[m] = \frac{1}{N} \sum_{n \in \mathbb{Z}/N\mathbb{Z}} F[n] e^{2\pi i m n / N}.$$

## Proof.

Substitute the definition of the DFT into the right side of the claim, and use the fact that  $\sum_{n=0}^{N-1} e^{2\pi i n / N} = 0$ . ■

# Computation of Fourier transforms

## Theorem

Let  $T, \Omega > 0$  satisfy the property that  $2T\Omega = 1$ , let  $N \geq 2$  be an even integer, and let  $f \in PW_\Omega \cap L^1(\mathbb{R})$ . (In particular,  $f$  can be considered as a continuous function on  $\mathbb{R}$ .) Consider the dilation  $f_T(t) = Tf(Tf)$  as a continuous function on  $\mathbb{R}$ , as well as a function on  $\mathbb{Z}$  defined by  $m \mapsto f_T[m]$ , where  $f_T[m] \equiv f_T(m)$ . Assume that  $f_T \in \ell^1(\mathbb{Z})$ . Set  $W_N \equiv e^{-2\pi i/N}$ . Then,

$$\forall n \in [-N/2, N/2], \quad \hat{f}\left(\frac{2\Omega n}{N}\right) = \hat{f}\left(\frac{n}{NT}\right) = \sum_{m=0}^{N-1} (f_T)_N^0[m] W_N^{mn},$$

where

$$(f_T)_N^0[m] \equiv T \sum_{k \in \mathbb{Z}} f((m + kN)T).$$

- The FT in terms of the DFT with error manageable coefficients!

# The proof of the computation of FTs theorem

By the Classical Sampling Theorem, we have

$$f = T \sum f(kT) \tau_{kT} d_{2\pi\Omega}$$

in  $L^2(\mathbb{R})$  and uniformly on  $\mathbb{R}$ . By the continuity of the Fourier transform mapping,

$$L^2(\mathbb{R}) \longrightarrow L^2(\widehat{\mathbb{R}}),$$

we have

$$\hat{f} = T \sum_k (f(kT) \tau_{kT} d_{2\pi\Omega})^\wedge(\gamma) = T \sum_k (f(kT) e^{-2\pi i(kT)\gamma} \mathbb{1}_{[\Omega, \Omega]}(\gamma))$$

in  $L^2(\widehat{\mathbb{R}})$ . However,  $\hat{f}$  is continuous on  $\mathbb{R}$  and the right side converges absolutely to a continuous function since  $f_T \in \ell^1(\mathbb{Z})$ . Therefore, the equation is valid for every  $\gamma \in \widehat{\mathbb{R}}$ .

# The proof of the computation of FTs theorem, cont.

Letting  $\gamma = 2\Omega n/N$  and using the fact that  $2T\Omega = 1$ , we obtain

$$\hat{f}\left(\frac{2\Omega n}{N}\right) = T \sum_k f(kT) e^{-2\pi i k n/N} \mathbb{1}_{[\Omega, \Omega]} \left(\frac{2\Omega n}{N}\right).$$

If  $|n| \leq N/2$ , then this equation gives  $\mathbb{1}_{[\Omega, \Omega]}(2\Omega n/N) = 1$ , and so

$$\forall |n| \leq \frac{N}{2}, \quad \hat{f}\left(\frac{2\Omega n}{N}\right) = \sum_k T f(kT) e^{-2\pi i k n/N}.$$

By the absolute convergence, we rearrange summation. Thus,

$$\begin{aligned} \forall |n| \leq \frac{N}{2}, \quad \hat{f}\left(\frac{2\Omega n}{N}\right) &= \sum_k \sum_{m=0}^{N-1} T f((kN + m)T) e^{-2\pi i (kN+m)n/N} \\ &= \sum_{m=0}^{N-1} \sum_k T f((kN + m)T) e^{-2\pi i k N n/N} e^{-2\pi i m n/N} \\ &= \sum_{m=0}^{N-1} (f_T)_N^0[m] e^{-2\pi i m n/N}. \end{aligned}$$

# Comments on the $L^1(\mathbb{T}_{2\Omega})$ theory of Fourier series

- $A(\mathbb{T}_{2\Omega})$  and  $A(\mathbb{Z})$  are as complicated as  $A(\widehat{\mathbb{R}})$ .
- $A(\mathbb{T}_{2\Omega}) \subseteq C(\mathbb{T}_{2\Omega}) \subseteq L^\infty(\mathbb{T}_{2\Omega}) \subseteq L^2(\mathbb{T}_{2\Omega}) \subseteq L^1(\mathbb{T}_{2\Omega})$ .
- $\ell^2(\mathbb{Z}) \subseteq A(\mathbb{Z})$ .
- R-L implies: if  $F \in L^1(\mathbb{T}_{2\Omega})$ , then  $\lim_{n \rightarrow \pm\infty} f[n] = 0$ .
- If  $T(\gamma) = \sum c_n e^{-2\pi i n \gamma / 2\Omega}$  and  $c_n \rightarrow 0$ , it is not necessarily true that  $T$  is the Fourier series of  $F \in L^1(\mathbb{T}_{2\Omega})$ . (Riemann sets of uniqueness, Menshov and strict multiplicity, continuous pseudo measures.) Set

$$T(\gamma) = \sum_{n=3}^{\infty} \frac{\sin(\pi n x / \Omega)}{\log(n)}.$$

•

$$\sum_{n=2}^{\infty} \frac{\sin(\pi n x / \Omega)}{n \log(n)}$$

converges uniformly on  $\widehat{\mathbb{R}}$  to  $F \in C(\mathbb{T}_{2\Omega}) \setminus A(\mathbb{T}_{2\Omega})$ .

# Convergence of Fourier series

- If  $F \in L^1(\mathbb{T}_{2\Omega})$  and  $S_N(F)(\gamma) = \sum_{n=-N}^N f[n]e^{-2\pi i n \gamma / (2\Omega)}$ , does

$$\lim_{N \rightarrow \infty} \|S_N(F) - F\|_{L^1(\mathbb{T}_{2\Omega})} = 0? \quad \text{NO.}$$

- $\{F_n : n = 1, \dots\} \subseteq L^1(\mathbb{T}_{2\Omega})$  converges to  $F \in L^1(\mathbb{T}_{2\Omega})$  weakly, i.e.,

$$\forall G \in L^\infty(\mathbb{T}_{2\Omega}), \quad \lim_{n \rightarrow \infty} \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} (F_n(\gamma) - F(\gamma))G(\gamma) d\gamma = 0,$$

if and only if

$$\lim_{n \rightarrow \infty} \int_A (F_n(\gamma) - F(\gamma)) d\gamma = 0$$

for every Lebesgue measurable set  $A \subseteq \mathbb{T}_{2\Omega}$ .

- If weak convergence, then norm convergence (first bullet) is true for  $S_N(F) = F_n$  if  $\{F_n : n \in \mathbb{N}\}$  converges to  $F$  in measure.
- Dieudonné-Grothendieck for open  $A$  and measures.

# $L^2(\mathbb{T}_{2\Omega})$ convergence and the Parseval formula

## Theorem

If  $f \in L^2(\mathbb{T}_{2\Omega})$ , then

$$\lim_{N \rightarrow \infty} \|f - S_N(f)\|_{L^2(\mathbb{T}_{2\Omega})} = 0.$$

Using this theorem we can prove

## Theorem

Let  $F, G \in L^2(\mathbb{T}_{2\Omega})$  with sequences  $c = \{c_n : n \in \mathbb{Z}\}$ ,  $d = \{d_n : n \in \mathbb{Z}\}$  of Fourier coefficients of  $F$  and  $G$ . Then,  $c, d \in \ell^2(\mathbb{Z})$  and

$$\frac{1}{2\Omega} \int_{\Omega} F(\gamma) \overline{G(\gamma)} d\gamma = \sum_{n \in \mathbb{Z}} c_n \overline{d_n};$$

and, in particular,

$$\frac{1}{2\Omega} \int_{\Omega} |F(\gamma)|^2 d\gamma = \sum_{n \in \mathbb{Z}} |c_n|^2.$$



# The Lusin conjecture and Carleson's theorem

# Haar measure

- An additive group  $G$  with a locally compact Hausdorff topology is a *locally compact group* if  $G \times G \rightarrow G, (x, y) \mapsto x - y$ , is continuous.
- For significant, classical treatises on locally compact groups, that were begun in the 1930s, see the works of Lev Semenovich Pontryagin (1908 - 1988) and André Weil (1906 - 1998).

## Theorem

If  $G$  is a locally compact group, then there is a Borel measure  $m_G$  on  $G$  such that

$$\forall B \in \mathcal{B}(G) \text{ and } \forall x \in G, \quad m_G(B) = m_G(B + x),$$

where  $B + x = \{y + x : y \in B\}$ . In this case  $m_G$  is a *right Haar measure* on  $G$ ; and, when  $B + x$  is replaced by  $x + B$ , then  $m_G$  is a *left Haar measure* on  $G$ . (Translation invariant like Lebesgue on  $\mathbb{R}$ .)

- If every right Haar measure is a left Haar measure on a locally compact group  $G$ , and vice-versa, then  $G$  is *unimodular*. Compact and LCAGs are unimodular.

# Markov–Kakutani fixed point theorem

We prove the existence of a Haar measure on  $G$  compact and Abelian using the Markov–Kakutani fixed point theorem by Varopoulos, see JB-WC IMA.

## Theorem

Let  $X$  be a Hausdorff topological vector space, take a compact and convex set  $K \subseteq X$ , and let  $\{T_\alpha\}$  be a family of continuous linear maps  $T_\alpha : X \rightarrow X$ , that satisfies

$$\forall \alpha, \quad T_\alpha(K) \subseteq K$$

and

$$\forall \alpha, \beta \quad T_\alpha \circ T_\beta = T_\beta \circ T_\alpha.$$

Then, there is  $k \in K$  such that

$$\forall \alpha, \quad T_\alpha(k) = k.$$

# Proof of the existence of Haar measure

## Proof.

Let  $M_1(G) = \{\mu \in M_b(G) : \|\mu\|_1 \leq 1\}$ . By the Banach–Alaoglu theorem,  $M_1(G)$  is weak \* compact in  $M_b(G)$ .

Let  $M_1^+(G) = \{\mu \in M_1(G) : \mu(1) = 1\}$ .

Note that  $\mu$  is positive if  $\mu \in M_1^+(G)$ ; to prove this we assume the opposite and obtain a contradiction using the fact that  $\|\mu\|_1 = \mu(1)$ .

If  $M_b(G)$  is taken with the weak \* topology, then the map  $M_b(G) \rightarrow \mathbb{C}$ ,  $\mu \mapsto \mu(1)$ , is continuous. Hence,  $\{\mu \in M_b(G) : \mu(1) = 1\}$  is weak \* closed. Thus,  $M_1^+(G)$  is weak \* compact. It is easy to check that  $M_1^+(G)$  is convex.

For  $x \in G$  and  $\mu \in M_b(G)$  we define the translation  $\tau_x(\mu)$  as

$$\tau_x(\mu)(f) = \int f(y - x) d\mu(y),$$

where  $f \in C(G)$ . ■

# Proof of the existence of Haar measure, cont.

## Proof.

Then, for each  $x \in G$  we define the map  $T_x : M_b(G) \rightarrow M_b(G)$ ,  $\mu \mapsto \tau_x(\mu)$ . Note that  $T_x$  is continuous with the weak \* topology on both domain and range, linear, and

$$\forall x, y \in G, \quad T_x \circ T_y = T_{x+y} = T_y \circ T_x,$$

since  $G$  is Abelian.

It is also elementary to check that, for each  $x \in G$ ,

$$T_x(M_1^+(G)) \subseteq M_1^+(G).$$

Therefore, by Markov–Kakutani, there is  $m_G \in M_1^+(G)$  such that  $\tau_x(m_G) = m_G$ , for all  $x \in G$ , the required translation invariance. Further,  $\|m_G\|_1 = 1$ ,  $m_G(1) = 1$ , and  $m_G$  is positive. ■

# History of Haar measure

- The question of existence of Haar measures goes back to Sophus Lie (1842-1899). Alfred Haar (1885-1933), of wavelet fame!, proved the existence of translation invariant measures on separable compact groups. As a matter of fact, Haar credits Adolf Hurwitz (1859-1919) for a remark, which is essential for proving the existence of a Haar measure on a Lie group. Existence of a Haar measure on a general locally compact group was first proved by Weil and, later the same year, by Henri Cartan.
- Besides the existence, it is natural to ask about the uniqueness of Haar measure on locally compact groups. This question was first answered by von Neumann (1903-1957) for compact groups, who later extended his own result to second countable locally compact groups (employing a different technique). Here we prove the uniqueness of Haar measure in the simple context of a LCAG. We follow the proof in Rudin's *Fourier Analysis on Groups* book, see also the books of Bourbaki, Loomis, Nachbin, Sally, and Weil. For a short proof in the non-Abelian case, which uses a notion of an approximate identity, we refer to Johnson (1976).

# Uniqueness of Haar measure

## Theorem

Let  $G$  be a LCAG. Let  $m_G^1$  and  $m_G^2$  be two Haar measures on  $G$ . Then, there exists  $C > 0$  such that  $m_G^1 = C m_G^2$ .

## Proof.

Let  $g_1 \in C_c^+(G)$  be chosen so that  $\int_G g_1 dm_G^1 = 1$ , and let  $C = \int_G g_1(-x) dm_G^2(x)$ . Then, for all  $g_2 \in C_c^+(G)$ , we have

$$\begin{aligned}\int_G g_2 dm_G^2 &= \int_G g_1(x_1) dm_G^1(x_1) \int_G g_2(x_2) dm_G^2(x_2) \\ &= \int_G \left( \int_G g_2(x_2) dm_G^2(x_2) \right) g_1(x_1) dm_G^1(x_1) \\ &= \int_G \left( \int_G g_2(x_1 + x_2) dm_G^2(x_2) \right) g_1(x_1) dm_G^1(x_1) \\ &= \int_G \int_G g_1(x_1) g_2(x_1 + x_2) dm_G^2(x_2) dm_G^1(x_1)\end{aligned}$$

# Proof of uniqueness, continued

Proof.

This equals

$$\begin{aligned} &= \int_G \int_G g_1(x_1) g_2(x_1 + x_2) dm_G^1(x_1) dm_G^2(x_2) \\ &= \int_G \int_G g_1(y_1 - y_2) g_2(y_1) dm_G^1(y_1) dm_G^2(y_2) \\ &= \int_G \int_G g_1(y_1 - y_2) g_2(y_1) dm_G^2(y_2) dm_G^1(y_1) \\ &= \int_G \left( \int_G g_1(y_1 - y_2) dm_G^2(y_2) \right) g_2(y_1) dm_G^1(y_1) \\ &= \left( \int_G g_1(-y_2) dm_G^2(y_2) \right) \int_G g_2(y_1) dm_G^1(y_1) = C \int_G g_2 dm_G^1. \end{aligned}$$

