Frame theory from signal processing and back again – a sampling

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Frames

- Let H be a separable Hilbert space, e.g., $H = L^2(\mathbb{R}^d)$, $\mathbb{R}^d$, or $\mathbb{C}^d$.
- $F = \{x_n\} \subseteq H$ is a *frame* for H if

$$\exists\, A, B > 0 \text{ such that } \forall\, x \in H, \quad A\|x\|^2 \leq \sum |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

Theorem

If $F = \{x_n\} \subseteq H$ is a frame for H then

$$\forall x \in H, \quad x = \sum \langle x, S^{-1}x_n\rangle x_n = \sum \langle x, x_n\rangle S^{-1}x_n,$$

where $S : H \to H$, $x \mapsto \sum \langle x, x_n\rangle x_n$ is well-defined.

- Frames are a natural tool for dealing with numerical stability, overcompleteness, noise reduction, and robust representation problems.

Outline

1. The narrow-band ambiguity function (with Robert L. Benedetto and Joseph Woodworth)
   - NWC Applications: Waveform design and radar
2. Ambiguity functions for vector-valued data (with Travis D. Andrews and Jeffrey J. Donatelli)
   - NWC Applications: Multi-sensor environments and MIMO
   - Set-up and problem
   - Group frame multiplications
3. Graph uncertainty principles (with Paul Koprowski)
   - NWC Applications: Non-linear spectral methods for dimension reduction and classification

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Outline continued

4. Balayage and STFT frame inequalities (with E. Au-Yeung)
   - NWC Applications: Non-uniform sampling and super-resolution imaging
5. Quantum detection (with Andrew Kebo)
   - NWC Applications: Finite frames and probabilistic frames in terms of quantum detection and POVMs
6. Reactive sensing (with Michael Dellomo)
   - NWC Applications: Engine diagnosis with disabled sensors by mean of multiplicative frames, whose factors account for parameter intensity and sensor sensitivity

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- THE NARROW BAND AMBIGUITY FUNCTION

Ambiguity function and STFT

- Woodward's (1953) *narrow band cross-correlation ambiguity function* of v, w defined on $\mathbb{R}^d$:

$$A(v,w)(t,\gamma) = \int v(s+t)\overline{w(s)}e^{-2\pi i s\cdot\gamma}ds.$$

- The *STFT* of v : $V_w v(t,\gamma) = \int v(x)\overline{w(x-t)}e^{-2\pi i x\cdot\gamma}dx.$
- $A(v,w)(t,\gamma) = e^{2\pi i t\cdot\gamma}V_w v(t,\gamma).$
- The *narrow band ambiguity function* $A(v)$ of v :

$$A(v)(t,\gamma) = A(v,v)(t,\gamma) = \int v(s+t)\overline{v(s)}e^{-2\pi i s\cdot\gamma}ds$$

The discrete periodic ambiguity function

- Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$.
- The *discrete periodic ambiguity function*,

$$A(u) : \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C},$$

of u is

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1} u[m+k]\overline{u[k]}e^{-2\pi ikn/N}.$$

CAZAC sequences

- $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$ is
  Constant Amplitude Zero Autocorrelation (CAZAC) if

  $$\forall m \in \mathbb{Z}/N\mathbb{Z}, \quad |u[m]| = 1, \ \text{(CA)}$$

  and

  $$\forall m \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}, \quad A(u)(m, 0) = 0. \ \text{(ZAC)}$$

- Are there only finitely many non-equivalent CAZAC sequences?
  - "Yes" for N prime and "No" for $N = MK^2$,
  - Generally unknown for N square free and not prime.

Björck CAZAC sequences

Let p be a prime number, and $(\frac{k}{p})$ the *Legendre symbol.*
A *Björck CAZAC sequence* of length p is the function $b_p : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ defined as

$$b_p[k] = e^{i\theta_p(k)}, \quad k = 0, 1, \dots, p-1,$$

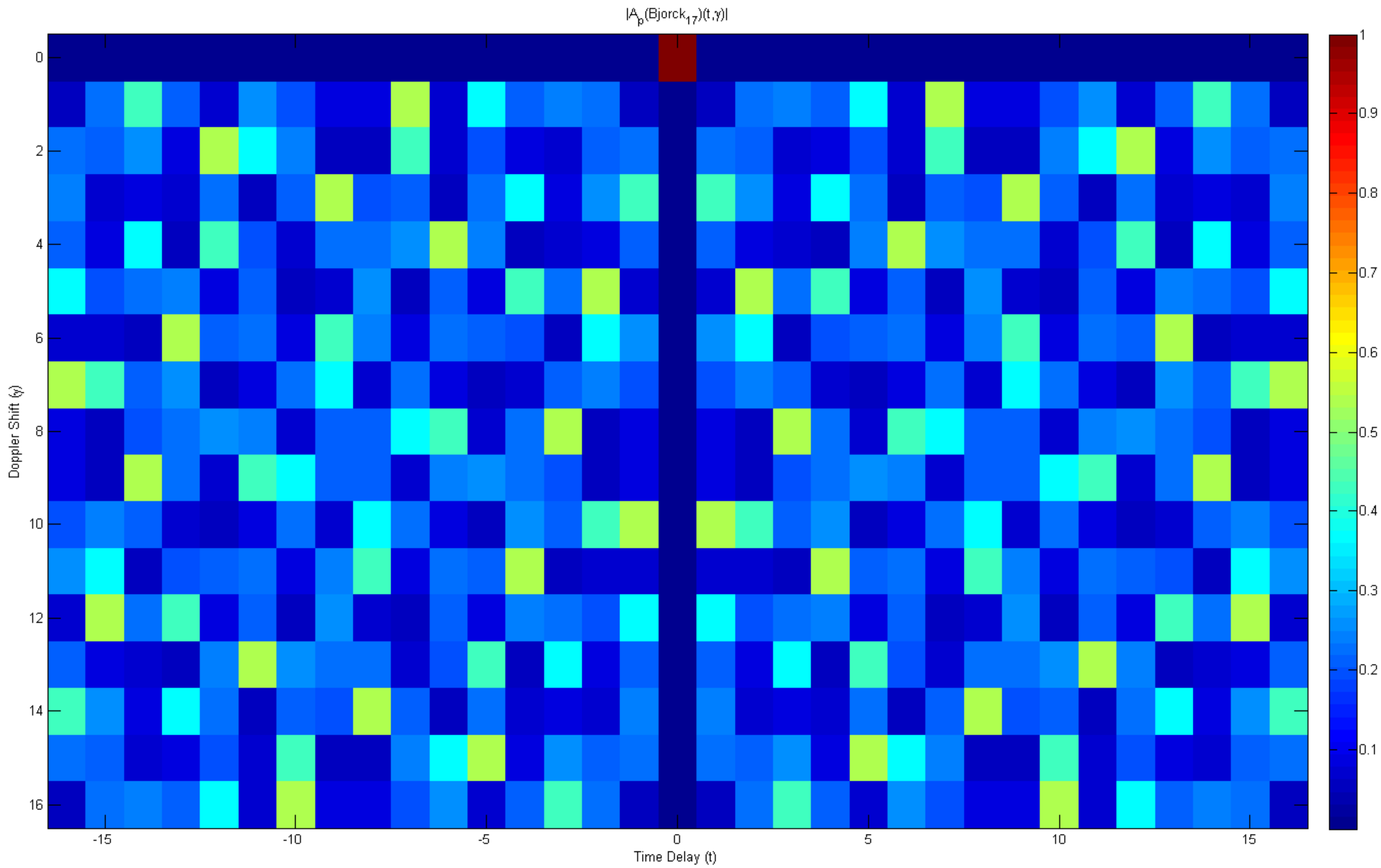
where, for $p = 1 \pmod 4$,

$$\theta_p(k) = \arccos\left(\frac{1}{1+\sqrt{p}}\right)\left(\frac{k}{p}\right),$$

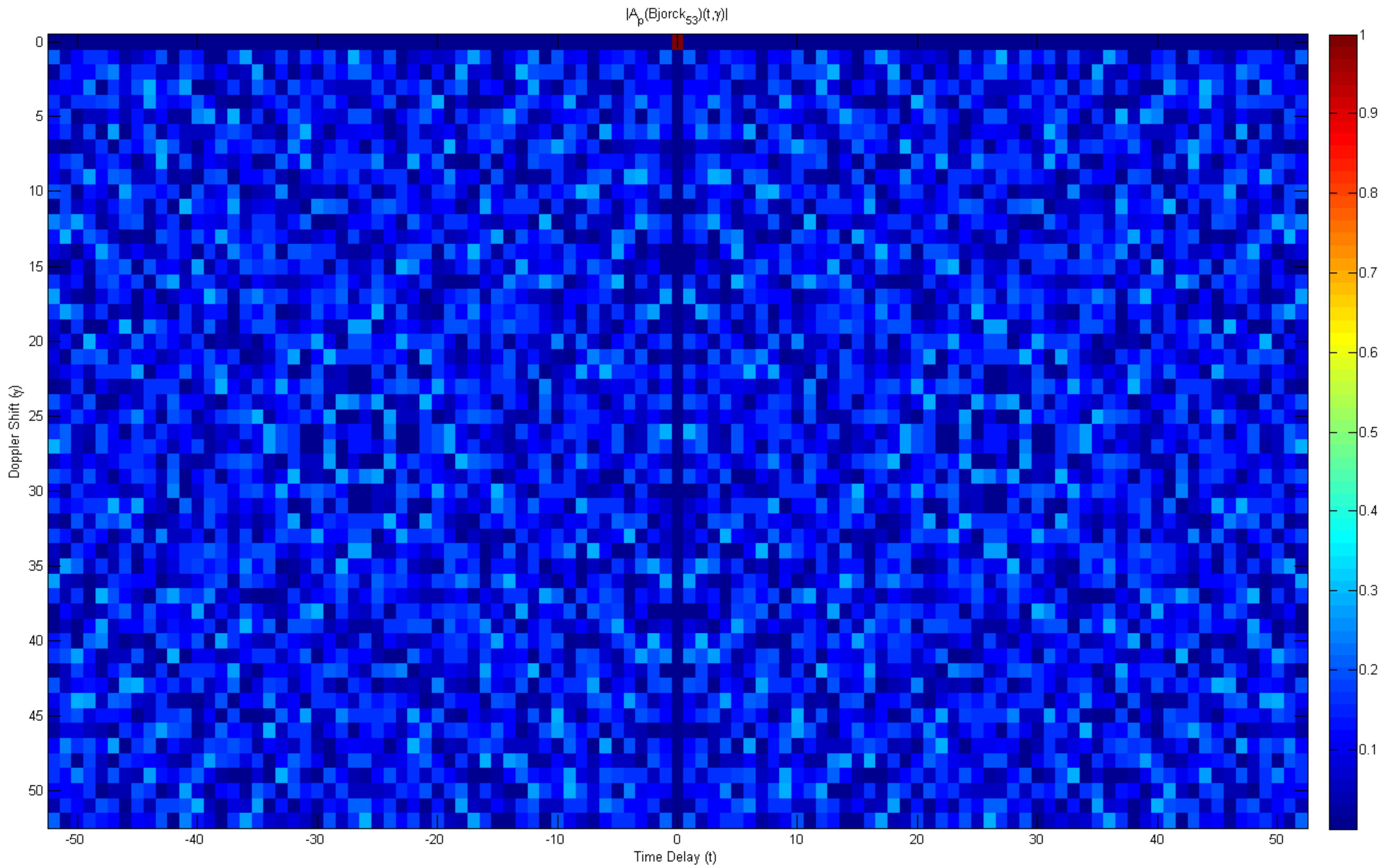
and, for $p = 3 \pmod 4$,

$$\theta_p(k) = \frac{1}{2}\arccos\left(\frac{1-p}{1+p}\right)\left[(1-\delta_k)\left(\frac{k}{p}\right) + \delta_k\right].$$

δ_k is the Kronecker delta symbol.



Absolute value of Bjorck code of length 17



Absolute value of Bjorck code of length 53

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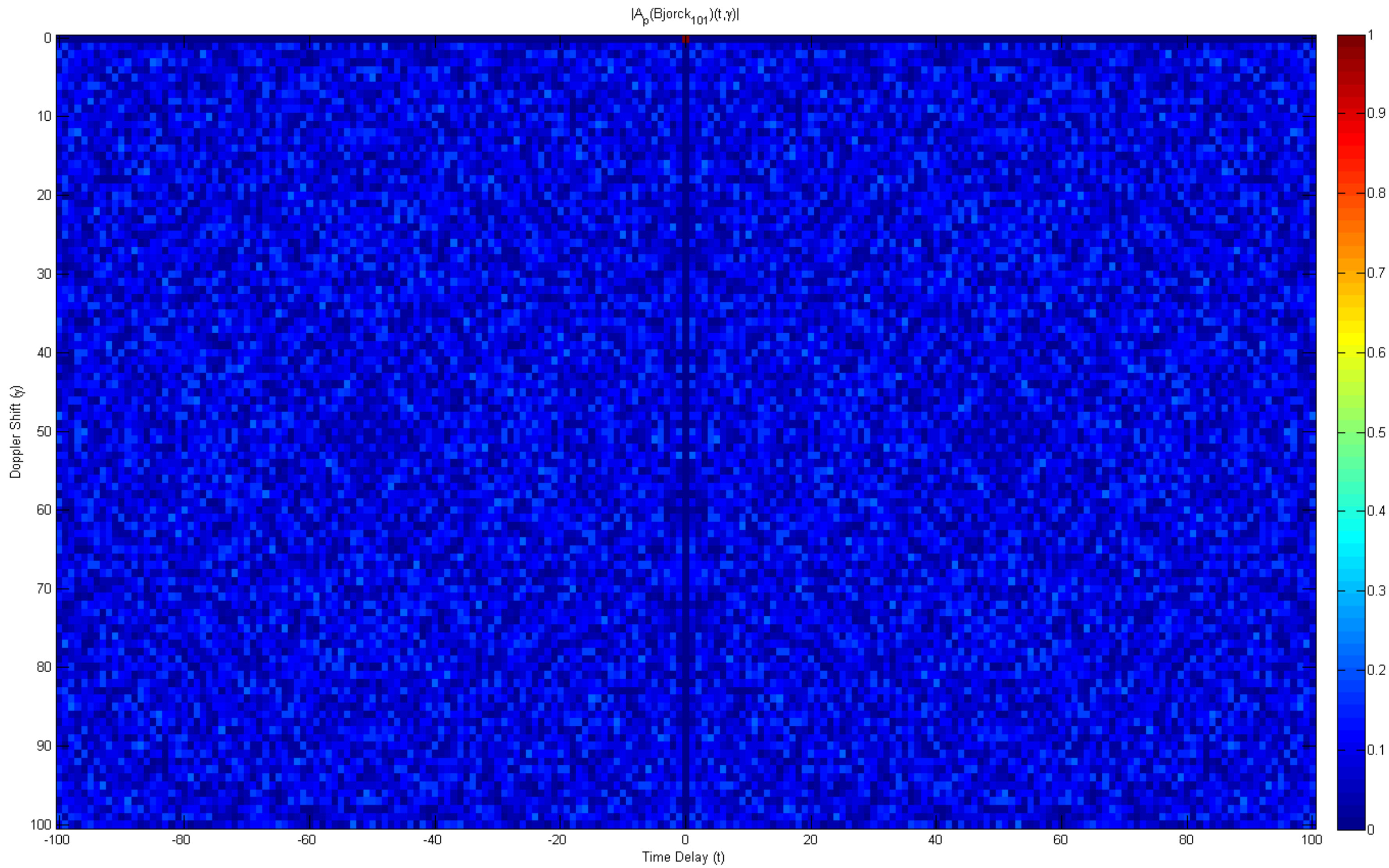
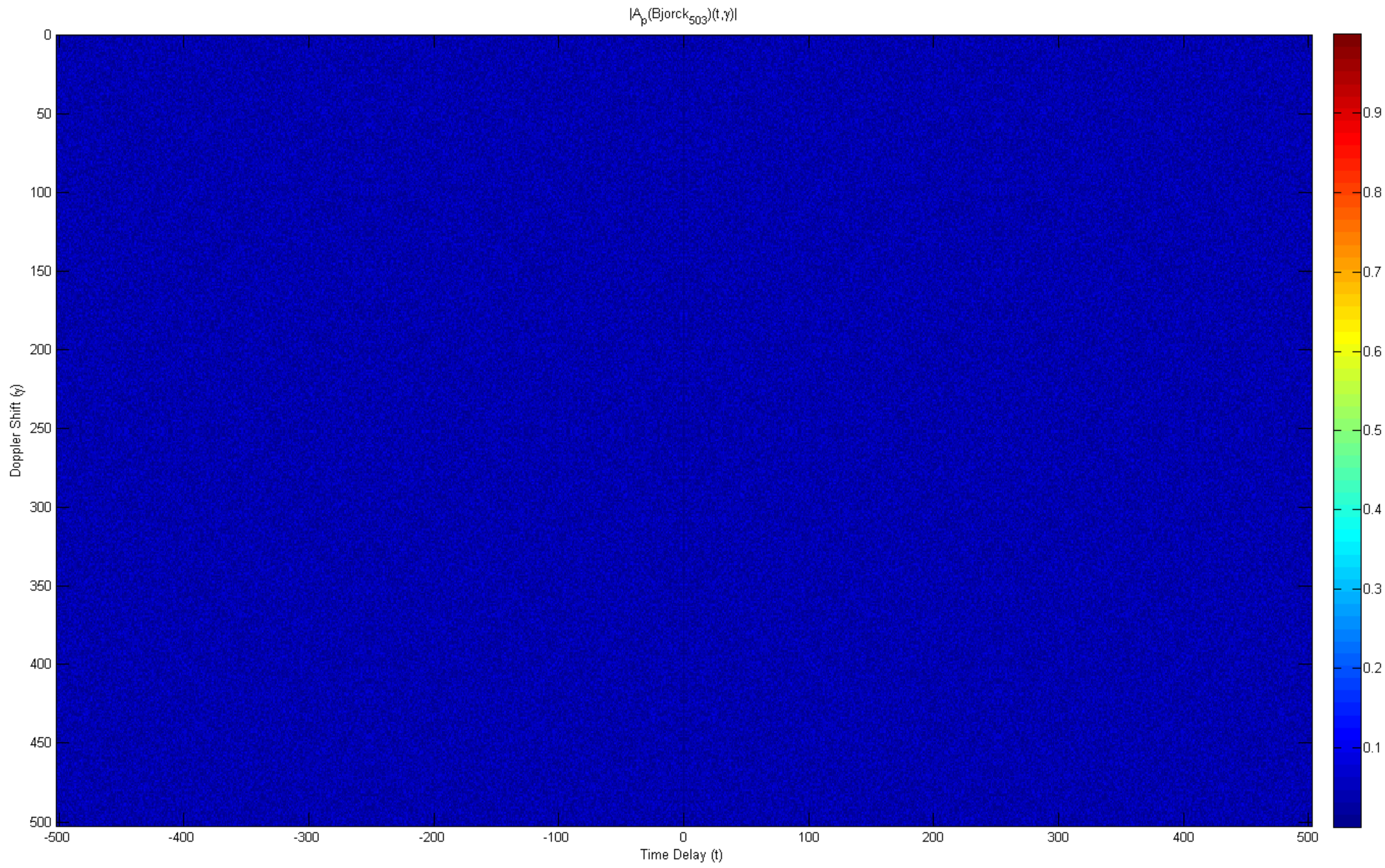


Absolute value of Bjorck code of length 101



Absolute value of Bjorck code of length 503

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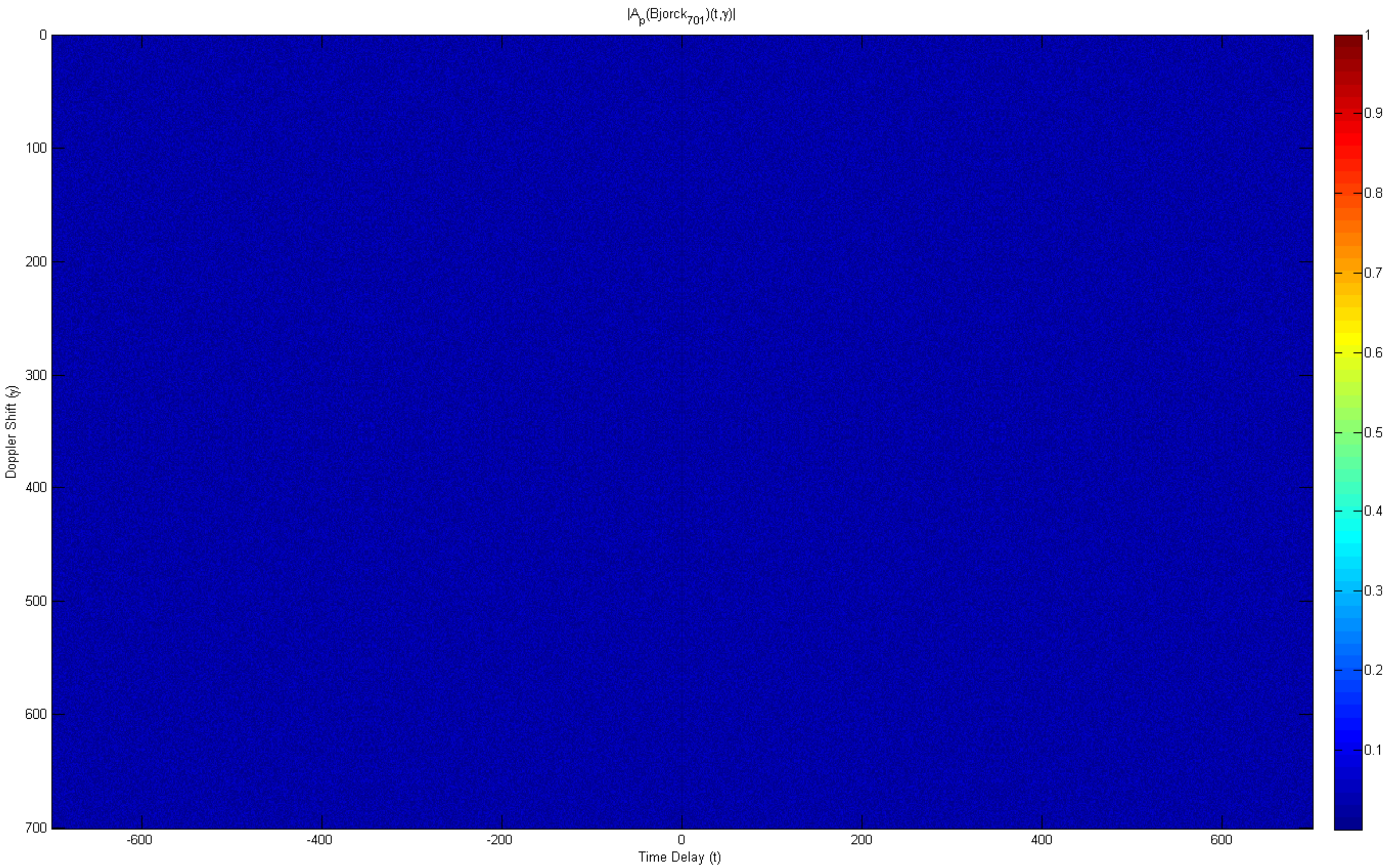

Absolute value of Bjorck code of length 701

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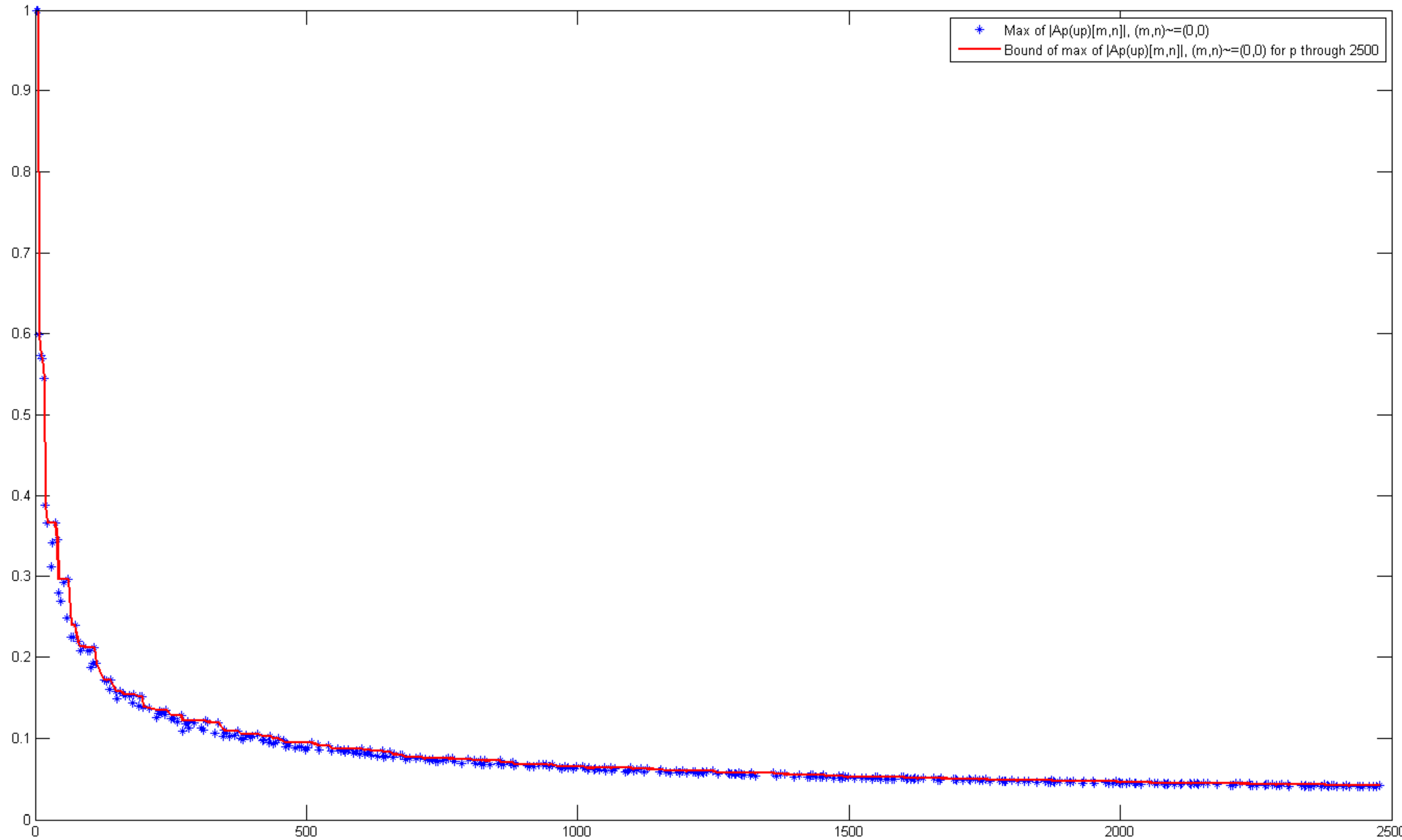

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Björck CAZAC discrete periodic ambiguity function

Let $A(b_p)$ be the Björck CAZAC discrete periodic ambiguity function defined on $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Theorem (J. and R. Benedetto and J. Woodworth)

$$|A(b_p)(m,n)| \leq \frac{2}{\sqrt{p}} + \frac{4}{p}$$

for all $(m,n) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \setminus (0,0)$.

- The proof is at the level of Weil's proof of the Riemann hypothesis for finite fields and depends on Weil's exponential sum bound.
- Elementary construction/coding and intricate combinatorial/geometrical patterns.

Gabor matrices

Optimal Compressive Sensing Technique, Theme, and Method

- Technique - Best Bounds: Use the number theoretic Riemann hypothesis for finite fields on special CAZAC (Constant Amplitude Zero Auto-Correlation) sequences.
- Theme: Incoherence extends time-frequency duality to exploit sparsity within uncertainty principle bounds.
- The Method, which is computationally efficient, follows.

Method: Sparsity and the Role of Best Bounds Theorem

Sparsity is the key for transformation-based image compression, e.g., JPEG 2000.

- Sparsity improves speed of execution, e.g., fast MRI data acquisition by non-uniform sampling.
- Sparsity reduces time of transmission and storage requirements.
- Sparsity yields efficient endmember or anomaly detection, without reconstruction expense, with fewer measurements.
- Sparsity allows for the reduction of resources needed in sensing.

Best bounds theorem $\Rightarrow$ maximal incoherence $\Rightarrow$ optimal hypothesis for OMP and BP.

Method: Matrix Equations and Coherence

Given F, an $m \times n$ full rank matrix, $n > m$.

- $F = [e_1|e_2|\dots|e_n]$, where $e_k = (e_k(1), \dots, e_k(m))^\top$ and $\|e_k\|_2 = 1$.
- $\{e_k : k = 1, \dots, n\} \subseteq \mathbb{C}^m$ is a FUNTF for $\mathbb{C}^m$.
- For $x \in \mathbb{C}^n$, set $\|x\|_0 = \text{card}\{j : x(j) \neq 0\}$, $x = (x(1), \dots, x(n))$.

Given $b \in \mathbb{C}^m$. Solve the *sparsity optimization* problem:

$$(P_0) \quad \min_x \|x\|_0 \quad \text{subject to} \quad Fx = b.$$

- The *coherence* $\mu(F)$ of F is

$$\mu(F) = \max_{k \neq l} |\langle e_k, e_l \rangle|.$$

Theorem

Assume $Fx = b$ and $\|x\|_0 < (1 + \mu(F)^{-1})/2$.

(a) Uniqueness and sparsity. If $Fy = b$ and $y \neq x$, then $\|x\|_0 < \|y\|_0$.

(b) Greedy algorithms. An orthogonal matching pursuit (OMP) run with threshold parameter $\epsilon = 0$ finds x exactly.

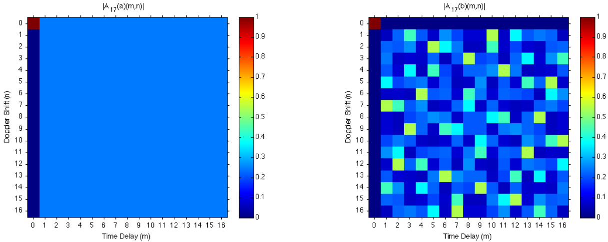
Given $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$. Let $\tau_j g(l) = g(l-j)$ and $e_k(l) = e^{2\pi ikl/N}$, $l = 0, 1, \ldots, N-1$. The $N \times N^2$ *Gabor Matrix* G for g is

$$G = [G_0|G_1|\ldots|G_{N-1}], \quad G_j \ N \times N,$$

$$G_j = [e_0\tau_{j-N}g|e_1\tau_{j-N}g|\ldots|e_{N-1}\tau_{j-N}g],$$

$$(g)^j_k = e_k\tau_{j-N}g = (e_k(0)\tau_{j-N}g(0), \ldots, e_k(l)\tau_{j-N}g(l), \ldots, e_k(N-1)\tau_{j-N}g(N-1))^\top.$$

- $\{(g)^j_k\}$ is a tight frame for $\mathbb{C}^N$.
- g CA $\Rightarrow \mu(G) = \max\{|A(g)(m,n)| : (m,n) \neq (0,0)\}$.
- $g = \mu_p$, Björck, $p = N \Rightarrow \mu(G) \leq 5/\sqrt{p}$.
- $g = \mu_p$, Björck, $b \in \mathbb{C}^N$, $Gx = b$, and $\|x\|_0 < (1 + \mu(G)^{-1})/2 \Rightarrow x$ is sparsest solution of $Gx = b$ and OMP constructs x.



(a) (b)

Figure : Absolute value of the ambiguity functions of the Alltop and Björck sequences with $N = 17$.

Problems and remarks

- For given CAZACs u_p of prime length p, estimate minimal local behavior $|A(u_p)|$. For example, with b_p we know that the lower bounds of $|A(b_p)|$ can be much smaller than $1/\sqrt{p}$, making them more useful in a host of mathematical problems, cf. Welch bound.
- Even more, construct all CAZACs of prime length p.
- Optimally small coherence of b_p allows for computing sparse solutions of Gabor matrix equations by greedy algorithms such as OMP.

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- AMBIGUITY FUNCTIONS FOR VECT0R-VALUED DATA

FUNTF

- A set $F = \{e_j\}_{j \in J} \subseteq \mathbb{F}^d$ is a *frame* for $\mathbb{F}^d$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, if

$$\exists\, A, B > 0 \quad \text{such that} \quad \forall\, x \in \mathbb{F}^d, \quad A\|x\|^2 \leq \sum_{j \in J} |\langle x, e_j \rangle|^2 \leq B\|x\|^2.$$

- F *tight* if $A = B$. A finite unit-norm tight frame F is a FUNTF.
- N row vectors from any fixed $N \times d$ submatrix of the $N \times N$ DFT matrix, $\frac{1}{\sqrt{d}}(e^{2\pi i mn/N})$, is a FUNTF for $\mathbb{C}^d$.
- If F is a FUNTF for $\mathbb{F}^d$, then

$$\forall x \in \mathbb{F}^d, \quad x = \frac{d}{N} \sum_{j=1}^{N} \langle x, e_j \rangle e_j.$$

- Frames: redundant representation, compensate for hardware errors, inexpensive, numerical stability, minimize effects of noise.

DFT FUNTFs

Let $N \geq d$ and form an $N \times d$ matrix using any d columns of the $N \times N$ DFT matrix $(e^{2\pi ijk/N})_{j,k=0}^{N-1}$. The rows of this $N \times d$ matrix, up to multiplication by $\frac{1}{\sqrt{d}}$, form a FUNTF for $\mathbb{C}^d$.

$$N = 8, d = 5 \qquad \frac{1}{\sqrt{5}}\begin{bmatrix} * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \\ * & * & \cdot & \cdot & * & * & * & \cdot \end{bmatrix}$$

$$x_m = \frac{1}{5}(e^{2\pi i\frac{m}{8}}, e^{2\pi im\frac{2}{8}}, e^{2\pi im\frac{5}{8}}, e^{2\pi im\frac{6}{8}}, e^{2\pi im\frac{7}{8}})$$

$$m = 1, \ldots, 8.$$

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DFT Frames

Definition

Let $N \geq d$ and let $s : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$ be injective. The rows $\{E_m\}_{m=0}^{N-1}$ of the $N \times d$ matrix

$$\left(e^{2\pi i m s(n)/N}\right)_{m,n}$$

form an equal-norm tight frame for $\mathbb{C}^d$ which we call a *DFT frame*.

Vector-valued DFT

Definition

Let $\{E_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$. Given $u : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^d$, we define the *vector-valued discrete Fourier transform* of u by

$$\forall\ n \in \mathbb{Z}_N, \quad F(u)(n) = \widehat{u}(n) = \sum_{m=0}^{N-1} u(m) * E_{-mn},$$

where $*$ is pointwise (coordinatewise) multiplication. We have that

$$F : \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}) \to \ell^2(\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z})$$

is a linear operator.

Vector-valued Fourier inversion

Theorem (Andrews, Benedetto, Donatelli)

The vector valued Fourier transform is invertible if and only if s, the function defining the DFT frame, has the property that

$$\forall n \in \mathbb{Z}/d\mathbb{Z}, \quad (s(n), N) = 1.$$

The inverse is given by

$$\forall\ m \in \mathbb{Z}/N\mathbb{Z}, \quad u(m) = F^{-1}\widehat{u}(m) = \frac{1}{N}\sum_{n=0}^{N-1} \widehat{u}(n) * E_{mn}.$$

In this case we also have that $F^*F = FF^* = NI$ *where* I *is the identity operator.*

Modeling for multi-sensor environments

- Multi-sensor environments and vector sensor and MIMO capabilities and modeling.
- Vector-valued DFTs
- Discrete time data vector $u(k)$ for a d-element array,

$$k \longmapsto u(k) = (u_0(k), \ldots, u_{d-1}(k)) \in \mathbb{C}^d.$$

We can have $\mathbb{R}^N \to GL(d, \mathbb{C})$, or even more general.

Preliminary multiplication problem

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d$.
- If $d = 1$ and $e_n = e^{2\pi in/N}$, then

$$A(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k)e_{nk}\rangle.$$

Preliminary multiplication problem

To characterize sequences $\{\varphi_k\} \subseteq \mathbb{C}^d$ and compatible multiplications $*$ and $\bullet$ so that

$$A^1(u)(m,n) = \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \in \mathbb{C}$$

is a meaningful and well-defined *ambiguity function.* This formula is clearly motivated by the STFT.

$A^1(u)$ for DFT frames

- Given $u : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}^d, d \leq N$.
- Let $\{\varphi_k\}_{k=0}^{N-1}$ be a DFT frame for $\mathbb{C}^d$, let $*$ be componentwise multiplication in $\mathbb{C}^d$ with a factor of $\sqrt{d}$, and let $\bullet = +$ in $\mathbb{Z}/N\mathbb{Z}$.

In this case $A^1(u)$ is well-defined by

$$
\begin{aligned}
A^1(u)(m,n) &= \frac{1}{N}\sum_{k=0}^{N-1}\langle u(m+k), u(k) * \varphi_{n\bullet k}\rangle \\
&= \frac{d}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\langle \varphi_j, u(k)\rangle\langle u(m+k), \varphi_{j+nk}\rangle.
\end{aligned}
$$

$A^1(u)$ for cross product frames

- Take $* : \mathbb{C}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ to be the cross product on $\mathbb{C}^3$ and let $\{i, j, k\}$ be the standard basis.
- $i * j = k$, $j * i = -k$, $k * i = j$, $i * k = -j$, $j * k = i$, $k * j = -i$, $i * i = j * j = k * k = 0$. $\{0, i, j, k, -i, -j, -k, \}$ is a tight frame for $\mathbb{C}^3$ with frame constant 2. Let

$$\varphi_0 = 0, \varphi_1 = i, \varphi_2 = j, \varphi_3 = k, \varphi_4 = -i, \varphi_5 = -j, \varphi_6 = -k.$$

- The index operation corresponding to the frame multiplication is the non-abelian operation $\bullet : \mathbb{Z}_7 \times \mathbb{Z}_7 \longrightarrow \mathbb{Z}_7$, where $1 \bullet 2 = 3, 2 \bullet 1 = 6, 3 \bullet 1 = 2, 1 \bullet 3 = 5, 2 \bullet 3 = 1, 3 \bullet 2 = 4$, etc.
- We can write the cross product as

$$u \times v = u * v = \frac{1}{2^2} \sum_{s=1}^{6} \sum_{t=1}^{6} \langle u, \varphi_s \rangle \langle v, \varphi_t \rangle \varphi_{s \bullet t}.$$

- Consequently, $A^1(u)$ is well-defined.

Generalize to quaternion groups, order 8 and beyond.

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Frame multiplication

Definition (Frame multiplication)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{C}$, and let $\Phi = \{\varphi_j\}_{j \in J}$ be a frame for $\mathcal{H}$. Assume $\bullet : J \times J \to J$ is a binary operation. The mapping $\bullet$ is a *frame multiplication* for Φ if it extends to a bilinear product $*$ on all of $\mathcal{H}$.

- The mapping $\bullet$ is a frame multiplication for Φ if and only if there exists a bilinear product $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that

$$\forall j, k \in J, \quad \varphi_j * \varphi_k = \varphi_{j \bullet k}.$$

- There are frames with no frame multiplications.

Harmonic frames

- Slepian (1968) - *group codes*.
- Forney (1991) - *geometrically uniform* signal space codes.
- Bölcskei and Eldar (2003) - *geometrically uniform* frames.
- Han and Larson (2000) - *frame bases and group representations*.
- Zimmermann (1999), Pfander (1999), Casazza and Kovacević (2003), Strohmer and Heath (2003), Vale and Waldron (2005), Hirn (2010), Chien and Waldron (2011) - *harmonic frames*.
- Han (2007), Vale and Waldron (2010) - *group frames, symmetry groups*.

Harmonic frames

- $(\mathcal{G}, \bullet) = \{g_1, \ldots, g_N\}$ abelian group with $\widehat{G} = \{\gamma_1, \ldots, \gamma_N\}$.
- $N \times N$ matrix with (j, k) entry $\gamma_k(g_j)$ is *character table* of $\mathcal{G}$.
- $K \subseteq \{1, \ldots, N\}, |K| = d \leq N$, and columns $k_1, \ldots, k_d$.

Definition

Given $U \in \mathcal{U}(\mathbb{C}^d)$. The *harmonic frame* $\Phi = \Phi_{\mathcal{G},K,U}$ for $\mathbb{C}^d$ is

$$\Phi = \{U\left((\gamma_{k_1}(g_j), \ldots, \gamma_{k_d}(g_j))\right) : j = 1, \ldots, N\}.$$

Given $\mathcal{G}, K$, and $U = I$. Φ is the $DFT - FUNTF$ on $\mathcal{G}$ for $\mathbb{C}^d$. Take $\mathcal{G} = \mathbb{Z}/N\mathbb{Z}$ for usual $DFT - FUNTF$ for $\mathbb{C}^d$.

Group frames

Definition

Let $(\mathcal{G}, \bullet)$ be a finite group, and let $\mathcal{H}$ be a finite dimensional Hilbert space. A finite tight frame $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ for $\mathcal{H}$ is a *group frame* if there exists

$$\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H}),$$

a unitary representation of $\mathcal{G}$, such that

$$\forall g, h \in \mathcal{G}, \quad \pi(g)\varphi_h = \varphi_{g \bullet h}.$$

Harmonic frames are group frames.

Abelian results

Theorem (Abelian frame multiplications – 1)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g\in\mathcal{G}}$ be a tight frame for $\mathcal{H}$. Then $\bullet$ defines a frame multiplication for Φ if and only if Φ is a group frame.

Abelian results

Theorem (Abelian frame multiplications – 2)

Let $(\mathcal{G}, \bullet)$ be a finite abelian group, and let $\Phi = \{\varphi_g\}_{g \in \mathcal{G}}$ be a tight frame for $\mathbb{C}^d$. If $\bullet$ defines a frame multiplication for Φ, then Φ is unitarily equivalent to a harmonic frame and there exists $U \in \mathcal{U}(\mathbb{C}^d)$ and $c > 0$ such that

$$cU\left(\varphi_g * \varphi_h\right) = cU\left(\varphi_g\right) cU\left(\varphi_h\right),$$

where the product on the right is vector pointwise multiplication and $*$ is defined by $(\mathcal{G}, \bullet)$, i.e., $\varphi_g * \varphi_h := \varphi_{g \bullet h}$.

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Remarks

- There is an analogous characterization of frame multiplication for non-abelian groups (T. Andrews).
- Consequently, vector-valued ambiguity functions $A^d(u)$ exist for functions u on a finite dimensional Hilbert space $\mathcal{H}$ if frame multiplication is well-defined for a given tight frame for $\mathcal{H}$ and a given finite group $\mathcal{G}$.
- It remains to extend the theory to infinite Hilbert spaces and groups.
- It also remains to extend the theory to the non-group case, e.g., our cross product example.

3

- GRAPH UNCERTAINTY PRINCIPLES

Uncertainty principles – 1

The Heisenberg uncertainty principle inequality is

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 4\pi \, \|t\, f(t)\|_{L^2(\mathbb{R})} \left\|\gamma\, \hat{f}(\gamma)\right\|_{L^2(\widehat{\mathbb{R}})}.$$

Additively, we have

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|^2_{L^2(\mathbb{R})} \leq 2\pi \left(\|t\, f(t)\|^2_{L^2(\mathbb{R})} + \left\|\gamma\, \hat{f}(\gamma)\right\|^2_{L^2(\widehat{\mathbb{R}})} \right).$$

Equivalently, for $f \in \mathcal{S}(\mathbb{R})$,

$$\|f\|^2_{L^2(\mathbb{R})} \leq \left\|\widehat{f'}\right\|^2_{L^2(\widehat{\mathbb{R}})} + \|f'\|^2_{L^2(\mathbb{R})}.$$

We shall extend this inequality to graphs.

Uncertainty principles – 2

- In signal processing, uncertainty principles dictate the trade off between high spectral and high temporal accuracy, establishing limits on the extent to which the "instantaneous frequency" of a signal can be measured (Gabor, 1946)
- Weighted, Euclidean, LCAG, non-L^2 uncertainty principles, proved by Fourier weighted norm inequalities, e.g., Plancherel, generalizations of Hardy's inequality, e.g., integration by parts, and Hölder (alas).
- DFT: Chebatorov, Grünbaum, Donoho and Stark,Tao.
- Generalize the latter to graphs.

Graph theory – background

- Problem: propose, prove, and understand uncertainty principle inequalities for graphs, see A. Agaskar and Y. M. Lu on *A spectral graph uncertainty principle*
- Generally: There is no obvious solution because of the loss on general graphs of the cyclic structure associated with the DFT.
- Locally: Radar/Lidar data analysis at NWC uses non-linear spectral kernel methods, with *essential* graph theoretic components for dimension reduction and remote sensing.

Graph theory – definition

Definition

A *graph* is $G = \{V, \mathbf{E} \subseteq V \times V, w\}$ consisting of a set V called vertices, a set $\mathbf{E}$ called edges, and a weight function

$$w : V \times V \longrightarrow [0, \infty).$$

Write $V = \{v_j\}_{j=0}^{N-1}$ and keep the ordering fixed, but arbitrary.

Graph theory – assumptions

- For any $(v_i, v_j) \in V \times V$ we have

$$w(v_i, v_j) = \begin{cases} 0 & \text{if } (v_i, v_j) \in \mathbf{E}^c \\ c > 0 & \text{if } (v_i, v_j) \in \mathbf{E}. \end{cases}$$

- G is undirected, i.e., $w(v_i, v_j) = w(v_j, v_i)$.
- $w(v_i, v_i) = 0$, i.e., G has no loops.
- G is connected, i.e., for any $(v_i, v_j) \in V \times V$, there exists a sequence, $\{v_k\} \subseteq V$, $k = 0, \ldots, d \leq N - 1$, such that

$$(v_i, v_0), (v_0, v_1), \ldots, (v_d, v_j) \in \mathbf{E}.$$

- G is *unit weighted* if w takes only the values 0 and 1.

Graph Laplacian

- $N \times N$ symmetric *adjacency matrix*, A, for G :

$$A = (A_{ij}) = (w(v_i, v_j)).$$

- The *degree matrix*, D, is the $N \times N$ diagonal matrix,

$$D = diag\left(\sum_{j=0}^{N-1} A_{0j}, \sum_{j=0}^{N-1} A_{1j}, \cdots, \sum_{j=0}^{N-1} A_{(N-1)j}\right).$$

- The *graph Laplacian*,

$$L = D - A,$$

  is the $N \times N$ symmetric, positive semi-definite matrix, with real ordered eigenvalues $0 = \lambda_0 \leq \ldots \leq \lambda_{N-1}$ and orthonormal eigenbasis, $\{\chi_j\}_{j=0}^{N-1}$, for $\mathbb{R}^N$.

Graph Fourier transform

- Formally, the Fourier transform $\hat{f}$ at γ of f defined on $\mathbb{R}$ is the inner product of f with the complex exponentials, that are the eigenfunctions of the Laplacian operator $\frac{d^2}{dt^2}$ on $\mathbb{R}$.
- Thus, define the *graph Fourier transform*, $\widehat{f}$, of $f \in \ell^2(G)$ in the graph Laplacian eigenbasis:

$$\widehat{f}[j] = \langle \chi_j, f \rangle, \quad j = 0, \ldots, N-1.$$

If

$$\chi = [\chi_0, \chi_1, ..., \chi_{N-1}],$$

then $\widehat{f} = \chi^* f$, and, since χ is unitary, we have the *inversion formula*:

$$f = \chi\chi^* f = \chi\hat{f}.$$

Difference operator for graphs

The *difference operator*,

$$D_r : \ell^2(G) \longrightarrow \mathbb{R}^{|\mathbf{E}|},$$

with coordinate values representing the change in f over each edge, is defined by

$$(D_r f)[k] = (f[j] - f[i]) \, (w(e_k))^{1/2} \,,$$

where $e_k = (v_j, v_i)$ and $j < i$.

- D_r can be defined by the *incidence matrix* of G.
- If G is a unit weighted circulant graph, then D_r is the intuitive difference operator of Lammers and Maeser.

Difference uncertainty principle for graphs

Theorem

Let G be a connected and undirected graph. Then,

$$\forall f \in \ell^2(G), \quad 0 < \tilde{\lambda}_0 \|f\|^2 \leq \|D_r f\|^2 + \left\|D_r \hat{f}\right\|^2 \leq \tilde{\lambda}_{N-1} \|f\|^2,$$

where

$$\Delta = \mathrm{diag}\{\lambda_0, \ldots, \lambda_{N-1}\}$$

and where $0 < \tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \ldots \leq \tilde{\lambda}_{N-1}$ are the eigenvalues of $L + \Delta$. The bounds are sharp.

Frame difference uncertainty principle for graphs

$\{e_j\}_{j=0}^{N-1} \subseteq \mathbb{C}^d$ is a *frame* for $\mathbb{C}^d$ if

$$\exists\, 0 < A \leq B \text{ such that } \forall f \in \mathbb{C}^d, \quad 0 < A\,\|f\|^2 \leq \sum_{j=0}^{N-1} |\langle f, e_j\rangle|^2 \leq B\,\|f\|^2 .$$

- If $A = B = 1$ then the frame is a *Parseval frame*.
- Define the $d \times N$ matrix $E = [e_0, e_1, \ldots, e_{N-1}]$, where $\{e_j\}_{j=0}^{N-1}$ is a Parseval frame for $\mathbb{C}^d$. Then $EE^* = I_{d\times d}$.

Theorem

Let G be a connected and undirected graph. Then, for every $d \times N$ Parseval frame E,

$$\sum_{j=0}^{d-1} \tilde{\lambda}_j \leq \|D_r \chi^* E^*\|_{fr}^2 + \|D_r E^*\|_{fr}^2 \leq \sum_{j=N-d}^{N-1} \tilde{\lambda}_j .$$

The bounds are sharp.

Complete graph

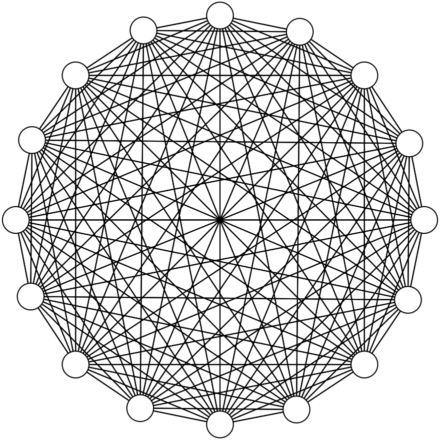


Figure : A unit weighted complete graph with 16 vertices.

Feasibility region

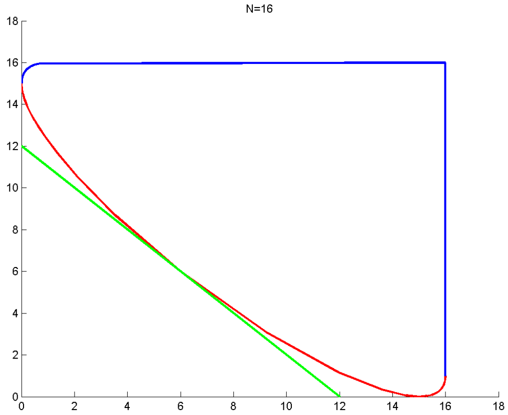
The difference operator *feasibility region FR* is

$$FR = \{(x,y) : \exists f \in \ell^2(G),\ \|f\| = 1,\ \text{such that}\ \|D_r f\|^2 = x \text{ and } \left\|D_r \widehat{f}\right\|^2 = y\}.$$

Theorem

a. FR is a closed subset subset of $[0, \lambda_{N-1}] \times [0, \lambda_{N-1}]$, where λ_{N-1} is the maximum eigenvalue of the Laplacian L.
b. $(\frac{1}{N}\sum_{j=0}^{N-1} \lambda_j, 0)$ and $(0, L_{0,0})$ are the only points of FR on the axes.
c. FR is in the half plane defined by $x + y \geq \tilde{\lambda}_0 > 0$ with equality if and only if $\hat{f}$ is in the eigenspace associated with $\tilde{\lambda}_0$.
d. If $N \geq 3$, then FR is a convex region.

Feasibility region



Uncertainty principle problem and comparison

- Lammers and Maeser, Grünbaum, Agaskar and Lu.
- The Agaskar and Lu problem.
- Critical comparison between the graph theoretical feasibility region and the comparable Bell Labs uncertainty principle region.

4

- BALAYAGE AND STFT FRAME INEQUALITIES

Fourier frames, goal, and a litany of names

Definition

$E = \{x_n\} \subseteq \mathbb{R}^d, \Lambda \subseteq \widehat{\mathbb{R}}^d$. E is a *Fourier frame* for $L^2(\Lambda)$ if

$\exists A, B > 0, \forall F \in L^2(\Lambda)$,

$$A \, ||F||^2_{L^2(\Lambda)} \leq \sum_n | < F(\gamma), e^{-2\pi i x_n \cdot \gamma} > |^2 \leq B \, ||F||^2_{L^2(\Lambda)}.$$

- *Goal* Formulate a general theory of Fourier frames and non-uniform sampling formulas parametrized by the space $M(\mathbb{R}^d)$ of bounded Radon measures.
- *Motivation* Beurling theory (1959-1960).
- *Names* Riemann-Weber, Dini, G.D. Birkhoff, Paley-Wiener, Levinson, Duffin-Schaeffer, Beurling-Malliavin, Beurling, H.J. Landau, Jaffard, Seip, Ortega-Certà–Seip.

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Balayage

- Let $M(G)$ be the algebra of bounded Radon measures on the LCAG G.
- Balayage in potential theory was introduced by Christoffel (early 1870s) and Poincaré (1890).

Definition

(Beurling) Balayage is possible for $(E, \Lambda) \subseteq G \times \widehat{G}$, a LCAG pair, if

$$\forall \mu \in M(G), \exists \nu \in M(E) \text{ such that } \hat{\mu} = \hat{\nu} \text{ on } \Lambda.$$

We write balayage (E, Λ).

- The set, Λ, of group characters is the analogue of the original role of Λ in balayage as a collection of potential theoretic kernels.
- Kahane formulated balayage for the harmonic analysis of restriction algebras.

Spectral synthesis

Definition

(Wiener, Beurling) Closed $\Lambda \subseteq \widehat{G}$ is a set of *spectral synthesis* (S-set) if

$\forall \mu \in M(G), \forall f \in C_b(G),$

supp$(\widehat{f}) \subseteq \Lambda$ and $\hat{\mu} = 0$ on $\Lambda \Longrightarrow \int_G f \, d\mu = 0.$

$(\forall T \in A'(\widehat{G}), \forall \phi \in A(\widehat{G}),$ supp$(T) \subseteq \Lambda$ and $\phi = 0$ on $\Lambda \Rightarrow T(\phi) = 0.)$

- Ideal structure of $L^1(G)$ - the Nullstellensatz of harmonic analysis
- $T \in D'(\widehat{\mathbb{R}}^d), \phi \in C_c^\infty(\widehat{\mathbb{R}}^d)$, and $\phi = 0$ on supp$(T) \Rightarrow T(\phi) = 0$, with same result for $M(\widehat{\mathbb{R}}^d)$ and $C_0(\widehat{\mathbb{R}}^d)$.
- $S^2 \subseteq \widehat{\mathbb{R}}^3$ is not an S-set (L. Schwartz), and every non-discrete $\widehat{G}$ has non-S-sets (Malliavin).
- Polyhedra are S-sets. The $\frac{1}{3}$-Cantor set is an S-set with non-S-subsets.

Strict multiplicity

Definition

$\Gamma \subseteq \widehat{G}$ is a set of *strict multiplicity* if

$$\exists\ \mu \in M(\Gamma)\backslash\{0\} \text{ such that } \check{\mu} \text{ vanishes at infinity in } G.$$

- Riemann and sets of uniqueness in the wide sense.
- Menchov (1916): $\exists$ closed $\Gamma \subseteq \widehat{\mathbb{R}}/\mathbb{Z}$ and $\mu \in M(\Gamma)\backslash\{0\}$,
$$|\Gamma| = 0 \text{ and } \check{\mu}(n) = O((\log|n|)^{-1/2}), |n| \to \infty.$$
- 20th century history to study rate of decrease: Bary (1927), Littlewood (1936), Salem (1942, 1950), Ivašev-Mucatov (1957), Beurling.

Assumption

$\forall\ \gamma \in \Lambda$ and $\forall\ N(\gamma)$, compact neighborhood, $\Lambda \cap N(\gamma)$ is a set of strict multiplicity.

A theorem of Beurling

Definition

E $= \{x_n\} \subseteq \mathbb{R}^d$ is *separated* if

$$\exists\, r > 0,\ \forall m, n,\ m \neq n \Rightarrow ||x_m - x_n|| \geq r.$$

Theorem

Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a compact S-set, symmetric about $0 \in \widehat{\mathbb{R}}^d$, and let E $\subseteq \mathbb{R}^d$ be separated. If balayage (E,Λ), then

E is a Fourier frame for $L^2(\Lambda)$.

- Equivalent formulation in terms of

$$PW_\Lambda = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}(\hat{f}) \subseteq \Lambda\}.$$

- $\forall F \in L^2(\Lambda), \qquad F = \sum_{x \in \mathrm{E}} < F, S^{-1}(e_x) >_\Lambda e_x$ in $L^2(\Lambda)$.
- For $\mathbb{R}^d$ and other generality beyond Beurling's theorem in $\mathbb{R}$, the result above was formulated by Hui-Chuan Wu and JB (1998), see Landau (1967).

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Balayage and a non-uniform Gabor frame theorem

Let $\mathcal{S}_0(\mathbb{R}^d)$ be the *Feichtinger algebra*.

Theorem

Let $E = \{(s_n, \sigma_n)\} \subseteq \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a separated sequence; and let $\Lambda \subseteq \widehat{\mathbb{R}}^d \times \mathbb{R}^d$ be an S-set of strict multiplicity that is compact, convex, and symmetric about $0 \in \widehat{\mathbb{R}}^d \times \mathbb{R}^d$. Assume balayage is possible for (E, Λ). Given $g \in L^2(\mathbb{R}^d),$ such that $\|g\|_2 = 1$. Then

$$\exists\ A,\ B > 0,\ \text{such that } \forall f \in \mathcal{S}_0(\mathbb{R}^d), \text{ for which } \operatorname{supp}(\widehat{V_g f}) \subseteq \Lambda,$$

$$A\,\|f\|_2^2 \leq\ \sum\nolimits_{n=1}^{\infty} |V_g f(s_n, \sigma_n)|^2 \leq B\,\|f\|_2^2\,.$$

Balayage and a non-uniform Gabor frame theorem (continued)

Theorem

Consequently, the frame operator, $S = S_{g,E}$, is invertible in $L^2(\mathbb{R}^d)$–norm on the subspace of $\mathcal{S}_0(\mathcal{R}^d)$, whose elements f have the property, $supp\,(\widehat{V_g f}) \subseteq \Lambda$. Further, if $f \in \mathcal{S}_0(\mathbb{R}^d)$ and $\mathrm{supp}(\widehat{V_g f}) \subseteq \Lambda$, then

$$f = \sum_{n=1}^{\infty} \langle f, \tau_{s_n} e_{\sigma_n} g \rangle S_{g,E}^{-1}(\tau_{s_n} e_{\sigma_n} g),$$

where the series converges unconditionally in $L^2(\mathbb{R}^d)$.

E does not depend on g.

- QUANTUM DETECTION, POVMs, AND FRAME POTENTIAL ENERGY

Positive operator valued measure (POVM)

Quantum theory gives the probability that a measured outcome lies in a specified region. These probabilities are defined in terms of *POVMs*, motivated from *von Neumann measurements*.

Definition

Let $\mathcal{B}$ be a σ-algebra of sets of X, and let H be a separable Hilbert space. A *positive operator-valued measure* (POVM) is a function $\Pi : \mathcal{B} \to \mathcal{L}(H)$ such that:

$$a.\ \forall\, U \in \mathcal{B},\ \Pi(U) \text{ is a positive self-adjoint operator } H \to H,$$

$$b.\ \Pi(\emptyset) = 0 \text{ (zero operator)},$$

$$c.\ \forall \text{ disjoint } \{U_i\}_{i=1}^{\infty} \subseteq \mathcal{B}, \text{ and } \forall x, y \in H,$$

$$\left\langle \Pi\left(\bigcup_{i=1}^{\infty} U_i\right) x, y \right\rangle = \sum_{i=1}^{\infty} \langle \Pi(U_i)x, y\rangle\,,$$

$$d.\ \Pi(X) = I \text{ (identity operator)}.$$

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Probability of detection

Definition

Given a separable Hilbert space H, a measurable space $(\mathcal{B}, X)$, and a POVM Π. If the state of the system is $x \in H$ with $\|x\| = 1$, then the *probability that the measured outcome lies in a region* $U \in \mathcal{B}$ is

$$P_\Pi(U) = \langle x, \Pi(U)x \rangle.$$

This is what we mean by a *POVM measurement*.

Given a countable measure space $X \subseteq \mathbb{Z}$ and a sequence $\{x_i : \|x_i\| = 1\}_{i \in X} \subseteq H$ of possible states with probabilities $\rho_i, \sum_{i \in X} \rho_i = 1$, i.e., ρ_i is the probability the system is in the state x_i.

The *problem* is to determine the state of the system and this requires a measurement. The *probability of detection error* (average probability that the measurement is incorrect) is

$$P_e = 1 - \sum_{i \in X} \rho_i \langle x_i, \Pi(i)x_i \rangle.$$

Hence, we want to construct a POVM Π that minimizes P_e, and this is the quantum mechanical *quantum detection problem*.

1 - tight frames are POVMs

- Let H be a separable Hilbert space and let $X \subseteq \mathbb{Z}$. X is the set of *outcomes*, considered as the set of all possible values a dynamical quantity, e.g., energy, can attain.

Theorem

- *Assume* $\{e_i\}_{i \in X} \subseteq H$ is a 1-tight frame ($A = B = 1$) for H.
- *Define* a family $\{\Pi(w)\}_{w \subseteq X}$ of self-adjoint positive operators on H :

$$\forall x \in H, \quad \Pi(w)x = \sum_{i \in w} \langle x, e_i \rangle e_i.$$

- Then Π is a POVM.

- $a = c$ of a POVM are easy to check, and d follows since

$$\forall x \in H, \quad \Pi(X)x = \sum_{i \in X} \langle x, e_i \rangle e_i = x.$$

- The converse of the Theorem is also true.

Quantum detection and 1-tight frames

- Given *POVMs = 1-tight frames*, the detection error P_e becomes

$$\begin{aligned} P_e &= 1 - \sum_{i \in X} \rho_i \langle x_i, \Pi(i) x_i \rangle \\ &= 1 - \sum_{i \in X} \rho_i \langle x_i, \langle e_i, x_i \rangle e_i \rangle \\ &= 1 - \sum_{i \in X} \rho_i |\langle x_i, e_i \rangle|^2. \end{aligned}$$

Thus the quantum mechanical quantum detection problem can be viewed as *constructing* a 1-tight frame that minimizes the last term.

- Partial results in Physics by Helstrom, Kennedy (IEEE-IT 1974), Kennedy, Yuen, and Lax (IEEE-IT 1975), Hausladen,Wootters (J. Modern Optics 1994), Helstrom (J. Stat. Physics 1969), Peres, Wootters (Phys. Rev. Lett. 1991), Peres, Terno (J. Phys. 1998).

Frame optimization problem

Problem

Let H be a d-dimensional Hilbert space. Given a sequence $\{x_i\}_{i=1}^N \subseteq H$ of unit normed vectors and a sequence $\{\rho_i\}_{i=1}^N \subseteq \mathbb{R}$ of positive weights that sums to 1. The frame optimization problem is to construct a 1-tight frame $\{e_i\}_{i=1}^N$ that minimizes the quantity,

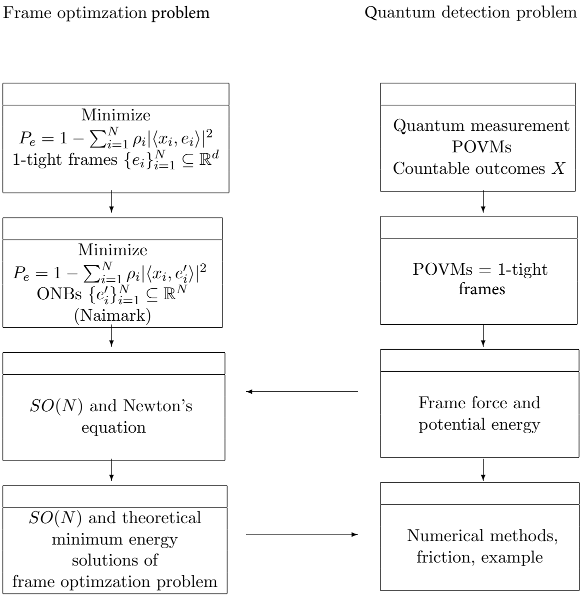
$$P_e(\{e_i\}_{i=1}^N) = 1 - \sum_{i=1}^N \rho_i |\langle x_i, e_i \rangle|^2,$$

taken over all N-element 1-tight frames.

• Such a tight frame exists by a compactness argument, which itself is difficult. Our goal is to quantify this existence, while keeping close to the quantum detection problem.

• There are more elementary related optimization problems, e.g., weighted MSE, that have been solved, but that are not related to quantum detection.

Outline of solution



Consequence of Naimark and its converse

Theorem

H d-dimensional Hilbert space, $\{x_i : \|x_i\| = 1\}_{i=1}^N \subseteq H$, and $\{\rho_i : \rho_i > 0\}_{i=1}^N$ summing to 1. H' N-dimensional Hilbert space such that H is a linear subspace of H', and $\{e_i\}_{i=1}^N$ a 1-tight frame for H that minimizes P_e over all N element 1-tight frames for H, i.e.,

$$P_e(\{e_i\}_{i=1}^N) = \inf \left\{ P_e(\{y_i\}_{i=1}^N) : \{y_i\}_{i=1}^N \text{ 1-tight frame for } H \right\}.$$

(A minimizer exists by the compactness above.) Assume $\{e_i'\}_{i=1}^N$ is an ONB for H' that minimizes P_e over all ONBs for H', i.e.,

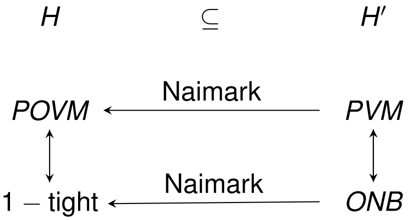
$$P_e(\{e_i'\}_{i=1}^N) = \inf \left\{ P_e(\{y_i\}_{i=1}^N) : \{y_i\}_{i=1}^N \text{ ONB for } H' \right\}.$$

Then

$$P_e(\{e_i\}_{i=1}^N) = P_e(\{e_i'\}_{i=1}^N) = P_e(\{P_H e_i'\}_{i=1}^N),$$

where P_H is the orthogonal projection onto H.

Diagram of Naimark Hilbert space dilation theorem



- The relationship between projective measurements (PVM), positive operator valued measurements (POVM), orthonormal bases (ONB), and 1-tight frames (1-tight).
- A 1-tight frame for H is a projection onto H of an ONB for some $H' \supseteq H$.
- A POVM for H is the projection onto H of a PVM for some Hilbert space $H' \supseteq H$.
- Naimark and Dr. Czaja's theory.

The geometry of finite tight frames (FUNTFs)

- The vertices of platonic solids are FUNTFs.
- However, points that constitute FUNTFs do not have to be equidistributed, e.g., ONBs and Grassmanian frames.
- FUNTFs can be characterized as minimizers of a frame potential function (with Fickus) analogous to Coulomb's Law.
- Frame potential energy optimization has basic applications dealing with classification problems for hyperspectral and multi-spectral (biomedical) image data.

Frame force and potential energy

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

$$P : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R},$$

where $P(a,b) = p(\|a-b\|), \qquad p'(x) = -xf(x)$

- Coulomb force

$$CF(a,b) = (a-b)/\|a-b\|^3, \quad f(x) = 1/x^3$$

- Frame force

$$FF(a,b) = \langle a,b\rangle(a-b), \quad f(x) = 1 - x^2/2$$

- Total potential energy for the frame force

$$TFP(\{x_n\}) = \sum_{m=1}^{N}\sum_{n=1}^{N} |\langle x_m, x_n\rangle|^2$$

Characterization of FUNTFs

Theorem

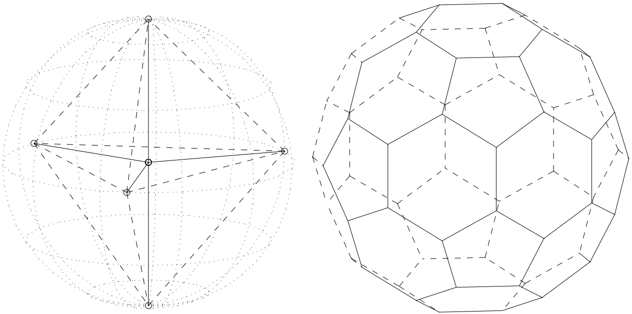
Let $N \leq d$. The minimum value of *TFP*, for the frame force and N variables, is N; and the *minimizers* are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Let $N \geq d$. The minimum value of *TFP*, for the frame force and N variables, is N^2/d; and the *minimizers* are precisely the FUNTFs of N elements for $\mathbb{R}^d$.

Problem

Find FUNTFs analytically, effectively, computationally.

Examples of frames



(a) Non–FUNTF (b) FUNTF

Frame optimization problem in light of Naimark

Problem

H d-dimensional Hilbert space, $\{x_i : \|x_i\| = 1\}_{i=1}^N \subseteq H$, and $\{\rho_i : \rho_i > 0\}_{i=1}^N$ summing to 1.

Assume $N \geq d$. Let H' be an N-dimensional Hilbert space such that H is a linear subspace of H'. The frame optimization problem is to find an orthonormal basis $\{e_i'\}_{i=1}^N \subseteq H'$ that minimizes P_e over all N element orthonormal sets in H'.

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The role of frame force for solution

Find an ONB $\{e_i'\}_{i=1}^N \subseteq H'$ that minimizes P_e over all ONBs for H'. We consider the quantity P_e as a potential

$$V = P_e = \sum_{i=1}^{N} \rho_i (1 - |\langle x_i, e_i' \rangle|^2) = \sum_{i=1}^{N} V_i,$$

where each

$$V_i = \rho_i (1 - \langle x_i, e_i' \rangle^2) = \rho_i \left(1 - \left(1 - \frac{1}{2} \|x_i - e_i'\|^2 \right)^2 \right),$$

and where we use the facts $\|x_i\| = \|e_i'\| = 1$ and

$$\|x_i - e_i'\|^2 = \langle x_i - e_i', x_i - e_i' \rangle = \|x_i\|^2 - 2\langle x_i, e_i' \rangle + \|e_i'\|^2 = 2 - 2\langle x_i, e_i' \rangle.$$

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The role of frame force for solution, continued

$F_i = -\nabla_i V_i$ is a conservative central force field, where ∇_i is an N-dimensional gradient taken by keeping x_i fixed and differentiating with respect to the variable e_i'. Set $x = \|x_i - e_i'\|$, and write

$$V_i(x_i, e_i') = v_i(\|x_i - e_i'\|) = \rho_i \left[1 - \left(1 - \frac{1}{2}x^2\right)^2\right].$$

Taking the derivative with respect to x gives

$$v_i'(x) = -2\rho_i \left(1 - \frac{1}{2}x^2\right)(-x) = 2\rho_i \left(1 - \frac{1}{2}x^2\right) x = -x f_i(x),$$

$$f_i(x) = -2\rho_i \left(1 - \frac{1}{2}x^2\right).$$

Therefore, the corresponding central force can be written as

$$F_i(x_i, e_i') = f_i(\|x_i - e_i'\|)(x_i - e_i') = -2\rho_i \langle x_i, e_i' \rangle (x_i - e_i').$$

F_i is *frame force*!

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Idea for completion of solution

Thus, the frame optimization problem can be viewed as a physical system, where the given vectors (pure states) $\{x_i\}_{i=1}^N$ are fixed points on the unit sphere in H'; and we have a "rigid" orthonormal basis $\{e_i'\}_{i=1}^N$ which moves according to the frame force F_i between each e_i' and x_i.

The problem is to find the equilibrium set $\{\overline{e'}_i\}_{i=1}^n$. These are the points where all the forces F_i balance and produce no net motion. In this situation, the potential V obtains an extreme value, and, in particular, we consider the case in which V is minimized.

This leads to differential equations on $O(N)$ and to the solution of the frame optimization problem.

Problems

• Develop frame-POVM relationship for more complex frames and pairs H, X. Then model realistic quantum measurement problems in these terms.

• Integrate noise reduction capability of frames for quantum measurement.

• Solve quantum detection problems beyond our pure states solution (suggested by Dr. Balu). Do analysis of mixed and entangled states, as well as associated tensor products.

• Generalize Gleason's theorem beyond Busch's generalization, that requires POVMs, in terms of defining *Gleason functions* (giá frame functions) for 1-tight (Parseval) frames instead of ONBs. (Collaboration with Drs. Balu and Koprowski).

Thats all folks!