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Abel Prize 2017 for Yves Meyer

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The Abel Prize 2017 was awarded to Yves Meyer, mostly for his work concerning wavelets. It is the purpose of the review article to explain the background and application areas of wavelet theory, indicate the connections to Gabor analysis and time-frequency analysis and share some personal experiences.

1 Yves Meyer, biographical background

Yves Francois Meyer was born July 19th, 1939 in Paris, but he grew up in Tunisia. After his studies at the École Normale Supérieure he was a teacher for three years at the school Prytanée Militaire in La Flèche (Loire Valley) and obtained a position in Strasbourg afterwards. During this period he prepared his PhD which he presented in 1966. Formally Jean-Pierre Kahane was his advisor, but he considers himself a "self-made man". From that time on he spentF all of his active time in Paris at different schools, such as Université Paris-Sud, École Polytechnique, Université Paris-Dauphine and École Normale Supérieure de Cachan.

His extensive work has many facets, covering number theory ([38]), harmonic analysis, quasi-crystals, operator theory and of course wavelets, as is nicely described in the article [11] by Ingrid Daubechies, We will focus in our presentation on the last two topics, because they have been the reason for awarding him the Abel Prize 2017. The interview with Yves Meyer published in the EMS Newsletter ([15]) also reveals some interesting background on his personal views on his development as a mathematician, some of his private interests (e.g. in literature) and the achievements which have been important to himself.

I will also add some personal comments to this story, because I had the good luck of meeting Yves Meyer as well as Alex Grossmann in Marseille around the critical period, just after the "discovery of orthonormal wavelets", under the French name of "ondelettes", allowing me to present some (hopefully interesting) background information.

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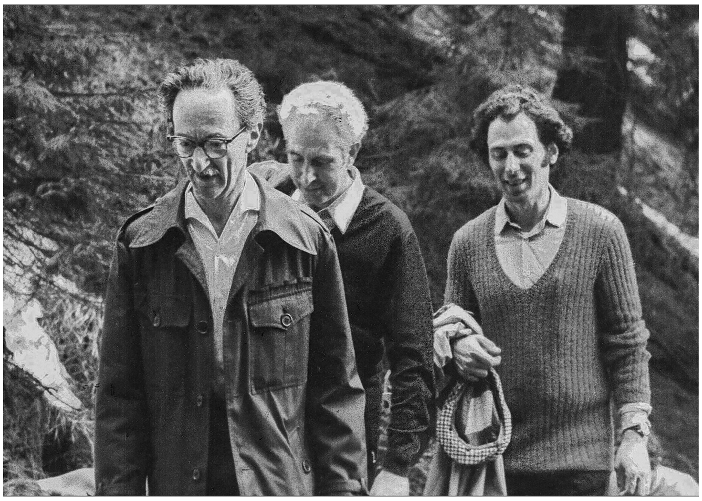


Figure 1: S. Hartmann, J. P. Kahane and Yves Meyer, ca. 1980, in Wisla

2 What are Wavelets?

The first and decisive observation by Y. Meyer in connection with wavelet theory was the *construction* of a particular smooth, real-valued and even functions $\psi \in L^2(\mathbb{R})$ with the property that a collection of certain dilated and shifted versions (atoms of *constant shape*) of ψ form an orthonormal basis (ONB) $(\psi_i)_{i\in I}$ for the Hilbert space $L^2(\mathbb{R})$. The fact that such a function must satisfy $\int_{\mathbb{R}} \psi(t)dt = 0$ implies that it's graph must show both positive and negative parts, it thus looks like a wavelet (or in French: "ondelette"), a *localized wave*.

Before going into more technical details let us discuss some of the immediate consequences, which can be described in a colloquial style:

1. Since clearly every $f \in L^2(\mathbb{R})$ has a unique representation as
$$f = \sum_{i\in I} c_i \psi_i := \sum_{i\in I} \langle f, \psi_i \rangle \psi_i \tag{1}$$
the coefficients c_i provide information about the "energy content within f" at the *scale and position*, corresponding the dilation factor and the center of the function ψ_i. Summing over the coarse scales only provides a *sparse* approximation of f, using a fairly small number of non-zero coefficients;

2. Certain operators (namely the so-called Calderon-Zygmund operators) which behave well with respect to dilations have an interesting, "diagonally concentrated" (infinite) matrix representation with respect to such an ONB, which helps to verify their mapping properties on the classical smoothness spaces;

3. The fact that the dilations applied to ψ allows to create narrow building blocks indicates already that even jumps or strongly transient parts in a function f do not require a huge number of coefficients (in contrast to Fourier series expansions). This fact implies among others that functions which are piecewise smooth with some jumps in between can be well approximated using finite wavelet sums.

It was also clear to Yves Meyer from the very beginning that these countable systems of smooth functions are not just an orthonormal basis for $L^2(\mathbb{R})$ but since the ones he constructed are all Schwartz functions, i.e. belong to the space $\mathbf{S}(\mathbb{R})$ of *rapidly decreasing functions* on $\mathbb{R}$, they also belong to all the classical smoothness spaces, including the (inhomogeneous) Besov spaces and the Triebel-Lizorkin spaces (as described in the books of Hans Triebel [51, 52, 53, 54], or the book of Elias Stein [47]). Note that Bessel-potential spaces, and in particular the classical Sobolev spaces $\mathcal{H}^s(\mathbb{R})$ belong to this family of function spaces.

Even more importantly, these ONBs form in addition *unconditional bases* for these spaces. This fact justifies the use of many (even non-linear) procedures, such as the (hard or soft) thresholding procedure, which works as follows: Given a (perhaps very *noisy*) signal f one tries to "clean" or "denoise" it by setting all the small wavelet coefficients to zero before resynthesis. Since for each of the classical function spaces mentioned above there is some solid BK-space over the index set, i.e. a Banach space $(Y, \|\cdot\|_Y)$ of sequences over I with the crucial property that for $\mathbf{c} = (c_i)_{i\in I} \in Y$ and any other sequence $(d_i)_{i\in I}$ with $|d_i| \leq |c_i|$ for all $i \in I$ one finds that $\mathbf{d} \in Y$ and $\|\mathbf{d}\|_Y \leq \|\mathbf{c}\|_Y$ characterizing the membership of a - say tempered distribution - f to the corresponding function space. Typically these BK-space are weighted mixed norm spaces, where the order of summation decides about the type of function space which can be characterized.

The (French) books of Yves Meyer ([44, 43, 44]) and the "Ten Lectures" by Ingrid Daubechies, published with SIAM in 1992, based on her course given at the first wavelet conference in Lowell (main organizer was Beth Ruskai) have been the first books covering the basic principles of the arising field of *wavelet analysis*. The are good sources until now.

Yves Meyer was also promoting the idea of an MRA (a multi-resolution analysis), which is another important aspect of wavelet theory, closely linked with a systematic construction of wavelet ONBs, especially in the multi-dimensional case. The concept of MRA had been introduced by Stephane Mallat ([36, 35]), motivated by precursors in image analysis. It is well described and illustrated in his important book [34]. It is useful e.g. for the transfer of images, where one would like to transfer first (and fast) the coarse information, while subsequently, by orthogonal enrichment of the already transmitted information, details can be filled in (e.g. for teleconferencing applications at low bit rates).

In Chapter II of his book [42] Yves Meyer gives already the detailed description[1] of a multi-resolution for the Hilbert spaces $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$: it consists of an increasing sequence of closed subspaces V_j, indexed by $j \in \mathbb{Z}$, satisfying

1. $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$, and $\bigcup_{j=-\infty}^{\infty} V_j$ is dense in $L^2(\mathbb{R}^d)$,

2. $\forall f \in L^2(\mathbb{R}^d), \forall j \in \mathbb{Z} : f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$,

3. $\forall f \in L^2(\mathbb{R}^d), \forall k \in \mathbb{Z}^d : f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$,

4. $\exists g \in V_0$ such that $g(x-k), k \in \mathbb{Z}^d$ is Riesz basis for V_0;

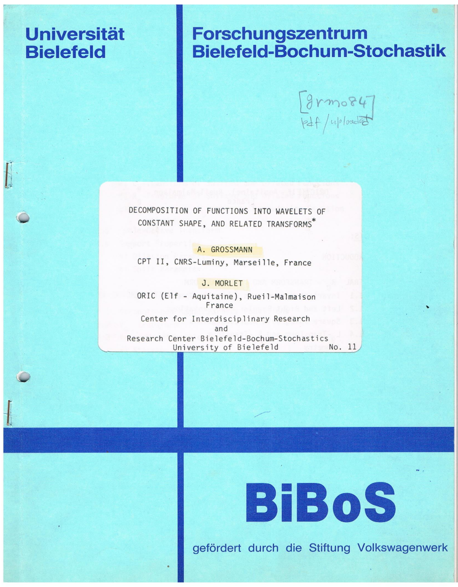
In words: at each level $j \in \mathbb{Z}$ (representing scale) the space V_j is translation invariant, very much like the space of e.g. cubic spline functions (where the cubic B-spline would take the role of the generator g). This function is often called the "father wavelet" (or scaling function) because it is the starting point for the derivation of some "mother wavelet" ψ (in higher dimensions one needs $2^d - 1$ such wavelets), which span in a similar way the orthogonal complement $W_j := V_{j+1} \ominus V_j$. The whole Hilbert space $L^2(\mathbb{R})$ is then an infinite orthogonal sum of all these "incremental spaces" $W_j, j \in \mathbb{Z}$, providing information about $f \in L^2(\mathbb{R})$ at the different scales j (with well defined localizations of their atoms).

3 My Personal Involvement

As a responsible (volunteer) for the Mathematical Library of our Institute of Mathematics in Strudlhofgasse 4 in Vienna I was lucky to immediately identify first the paper by Grossmann-Morlet ([26]) when it appeared in the BIBOS preprint series published in Bielefeld. I decided to visit professor Alex Grossmann in 1985.

When subsequently, in the autumn of 1985, Yves Meyer's paper [39] appeared (also first as a preprint) it was clear that this was an "extra-ordinary" event (at least for me). Hence I got into contact with professor Yves Meyer around Christmas 1985, because I had already an invitation for a Colloqium Talk in Nancy (by George Bohnke) in February 1986.

[1] Under the given Riesz basis assumption one can find another generator g_1 for V_0 which forms an orthonormal bases of translates in (4.) above.



During a relatively short meeting in Paris Yves Meyer disclosed to me that they had (at that time) just two constructions of orthonormal wavelet bases as described in [39] and [32], one together with his student Pierre Gilles Lemarié. But he was also very excited about their findings and immediately realized the great potential for applications, notably in connection with the theory of Calderon-Zygmund operators. These operators, generalizing the Hilbert transform, had been in the center of his research (see [8] with R. Coifman and A. McIntosh, work of his students J. L. Journee and G. David [13] and the books [44, 45]).

When I met Yves Meyer and Alex Grossmann I had already developed my own theory of *modulation spaces* and atomic decompositions (see [17], published in 2002 as [18], see also [19]) and thus it was not hard to realize that there might be something in common with wavelet theory.

The result of this research together with Karlheinz Gröchenig is the so-called *Theory of Coorbit Spaces* (see [20, 21]) which is continuing to develop until now.

During this project K. Gröchenig was also visiting Yves Meyer in Paris, where he solved a problem concerning multi-dimensional wavelets (see [25]) which had been under discussion in Yves Meyer's group at that time. Afterwards Gröchenig also had an extended visit to A. Grossmann in Marseille.

Another influential paper which appeared in 1986 was [12], entitled "Painless Non-Orthogonal Expansions", describing situations where *frame expansions* could be obtained in an easy way, because the frame operator is a simple multiplication operator. According to Yves Meyer (citation from memory) this paper was "painless for me, because I just had to provide some ideas and my co-authors took care of the manuscript". The coauthors have been Alex Grossmann and Ingrid Daubechies (she was a Post-Doc in Marseille at that time).

Although the concept of frames had been already introduced in by Duffin and Schaeffer ([14]) in 1952, it had not been viewed as very important until wavelet frames and Gabor frames (as described in [12]) got into the focus of attention. Without discussing them in detail let us just mention that *frames* are indexed families $(g_j)_{j\in J}$ (typically with a countable index set) which form a *stable set of generators*, in the sense that one can guarantee that every element can be expanded as an infinite, unconditionally convergent series with $\ell^2(J)$-coefficients. For the foundations of frame theory one may recommend [5] or [10].

4 Prehistory of Wavelet Theory

The history of wavelet theory is quite interesting, also from a science historical point of view. While the concrete construction of orthonormal wavelet bases was the starting point of an exciting movement during the last 30 years, ut was not unrelated to many other developments in (Fourier) analysis.

First of all it is commonly agreed that the *idea* to use what is now called "wavelets" goes essentially back to Jean Morlet (1931 - 2007), a geophysicist at Elf-Aquitaine. Together with Alex Grossmann, a theoretical physicist in Marseille the *continuous wavelet transform*, together with the continuous representation of functions (or distributions) was developed. Writing abstractly ψ_g for a generic dilated and shifted version of ψ their representation formula takes the form

$$f = \int_G \langle f, \psi_g\rangle \psi_g dg, \quad f \in L^2(\mathbb{R}), \tag{2}$$

where the integral is over the half plane respectivly over the affine "$ax+b$"-group G with respect to the left invariant (Haar) measure dg. As it turned out this was a rediscovery of what is nowadays called *Calderon's reproducing formula* (see [6]).

Since it is common sense to expect that an integral representation allows (by writing corresponding Riemanian sums) to obtain approximate representation using countable grids (a multiplicative lattice of the form $a^n, n \in \mathbb{Z}$, for some $a > 1$ and an arithmetic grid of translations of the form by $kb, k \in \mathbb{Z}$, for some $b > 0$) it was plausible that one could approximate every $f \in L^2(\mathbb{R})$ by finite wavelet sums. Looking for good examples, the so-called Mexican hat function, the second derivative of the usual Gauss function $g_0(t) := e^{-\pi t^2}$, was proposed as a candidate (see also [3, 4]). Numerically it showed quite good approximation properties, allowing an almost exact reproduction of the functions f, almost like an orthonormal basis. Nowadays we know that it can form a snug[2], i.e. almost *tight frame*.

Hence Alex Grossmann suggested to contact finally Yves Meyer to check whether maybe the deviation was only a question of numerical approximation. However,

[2] This term, used in early papers, was soon discarded.

Yves Meyer, being aware of a 1981 paper [1] by Roger Balian related to Gabor thought (according to what he told me in February 1986) that it is impossible to have such an orthonormal system, starting from a smooth function ψ. As it turned out to his surprise, his conjecture was wrong, because he himself *found a counter-example*. But in fact, to find a way to construct such a "counterexample" was good news, and Yves Meyer immediately recognized the potential of his invention (or discovery, one can discuss this question over a glass of wine). So the first papers appeared in that year: [39, 32, 40, 41].

Working in the "exploding" area of wavelets Yves Meyer became a sedulous prophet of this new branch of mathematical analysis and showed in many cases how the rich structure of "good wavelet systems" can be used to study boundedness of operators. His credo (once formulated in a private conversation) is: *"Function spaces are only good for the description of operators!"*

While the first, now so-called Meyer wavelets had been real-valued functions of exponential decay, but with compactly supported Fourier transform (hence analytic functions) it was Ingrid Daubechies who was able to describe the first construction of compactly supported wavelets (published in 1988, [9]) of prescribed smoothness. As with splines one has to accept larger support size with increased smoothness request. She also provided iterative rules which allows to compute the wavelet coefficients in a numerical efficient way.

It took a while until it was realized, that there have been precursors to his construction. First of all (as pointed out in [10], or [29]) the Haar system (see see [27]) can be viewed as a wavelet system of the lowest order, i.e. consisting of piecewise constant functions (in fact taking the values $1, -1$ and 0), but one can argue that these are well localized, discontinuous step-functions. There was however another one, consisting of continuous functions, published before the construction of the Meyer wavelets, in the work of J. O. Strömberg [48] on Franklin bases for $H^1(\mathbb{R})$, which form a wavelet system of piecewise linear and continuous. But his contribution was not immediately recognized, although he presented (as Peter Jones mentioned in one of his talks) his results at the conference to celebrate Antoni Zygmund's 80th birthday, in front of a group of leading experts in the field.

The original paper of A. Haar ([27]) does not mention dilations at all (see [29]), i.e. it is a re-interpretation introduced in [10] in order to describe the idea of the basic wavelet algorithms through a simple example. On the other hand the good compressive properties of wavelet expansion are only valid for smooth wavelets (as proposed by I. Daubechies or Y. Meyer) and not by the simple Haar wavelets.

It was Yves Meyer who recognized the relevance and the possibilities that opened up with the existence of what is nowadays called an *orthonormal wavelet basis*. Recall that there does not exist any orthonormal (and not even Riesz) basis in the Gaborian case. This fact forced the community to work with Banach frames and redundant representation in that context ([20, 22]).

5 Further Information

In addition to the comments and stories provided above let us mention a couple of further sources.

On Youtube one can view 18 videos recorded at the final event of my semester on the Morlet Chair at the CIRM in Luminy (Marseille). One can access all the presentations, including many contributions by pioneers in wavelet theory, by searching for the title of the event on YouTube, namely "30 Years of Wavelets". Let me particularly point to the contribution by Patrick Flandrin who showed that wavelets is not only a mathematical subject, but also a topic that found widespread applications in engineering. He also indicated that the field of wavelets has very much contributed to an intensive cooperation across scientific disciplines, which I consider another important aspect of wavelet theory.

The relevance of wavelet systems is partially due to the perfect fit between function spaces of Besov-Triebel-Lizorkin type and their characterization through "good wavelet bases". The transition translates the membership of a distribution to one of this function spaces to the membership of its wavelet coefficients in the corresponding Banach lattice of multi-dimensional sequences. This characterization is independent of the concrete wavelet system, as long as it satisfies certain quality criteria (decay and moment conditions). Since thresholding operators are harmless in such such lattices the wavelet expansions allow for these non-linear operations, preserving the smoothness of the original input f.

This in sharp contrast to the situation for $L^p(\mathbb{T})$ and Fourier-series expansions, as has been shown by V. Temlyakov ([50]). There it cannot be assured that one has convergence of the partial sums obtained by letting a sequence of threshold parameters tend to zero. The problem of "conditionality" of Fourier series has been discussed in detail earlier by T. W. Körner ([30, 31]) for continuous functions.

We also want to point out that the connection between wavelet characterizations of the function spaces is based on the Fourier characterization of these function spaces, as described by the pioneers of interpolation theory, Jaak Peetre ([46]) and Hans Triebel (see his books). An important step for these characterizations is the Paley-Littlewood characterization (see [16]) of L^p-spaces using dyadic decompositions on the Fourier transform side. These decompositions have been also been the basis for the *atomic decompositions* and the ϕ-transform approach by M. Frazier and B. Jawerth (in [23, 24]).

Chris Heil and Dave Walnut, who have contributed to the early popularization of wavelets through [28] have also put together a collection of "Fundamental Papers in Wavelet Theory" in [29], making these papers available to the English speaking community. (starting from Haar's paper of 1910, written in German, to the early papers about "ondelettes" of the French school around Yves Meyer).

A short summary of the history of wavelets is given by Albert Cohen ([7]). The paper [33] also reports about Yves Meyer as a Gauss Prize Winner. A popular description of the world of wavelets has been given already in 1996 (since then several new editions, also in different languages) by Hubbard Burke, see [2].

6 Final Comments

As we have seen the theory of wavelets is an interesting and important branch of modern Harmonic Analysis and Yves Meyer was one of the key figures contributing greatly to the development of this field, by showing that it is possible to create orthonormal wavelet bases. But he was not only constructing such bases, but already in his early publications on the subject indicating how they can be used to characterize function spaces (most of the classical ones) and how to use this fact in order to prove boundedness results for certain classes of operators, in particular for Calderon-Zygmund operators.

The theory of *wavelets* has developed greatly and has also found a lot of recognition in the engineering community. For a while the (meanwhile terminated) *Wavelet Digest* had more than 20000 subscribers. In the last 30 years more than 300 PhD theses have been written in the field (according to the Mathematical Genealogy Database, theses with the word "wavelet" in the title), but the numbers have started to decrease in the last few years.

Wavelet theory is a well established set of tools, but it will continue to expand further and find new applications also in the future, also through interesting new generalizations (like shearlets) or new application areas. Wavelets have been useful for a number of real-world applications, e.g. for the JPEG-2000 standard, see [37] and [49].

With Yves Meyer, the Norwegian Abel Prize committee has honoured in 2017 one of the outstanding pioneers of the field of wavelets and the person who has been promoting their use for so many years in so many different branches of mathematics. At the end, he himself claims that there is nothing that cannot be also done without wavelets. But I would like to add: "but by using wavelets things are often easier to understand", and also the idea of multi-scale is certainly here to stay.

In addition, the wavelet movement has been the basis for other, more recent branches of mathematical analysis of high application potential, such as sparse approximation, compressive sensing, or deep learning based on Mallat's idea of a scattering transform.

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